Differential Calculus and Sage

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and David Joyner

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Preface

This is a free and open source differential calculus book. The "free and open source" part means you, as a student, can give digital versions of this book to anyone you want (for free). It means that if you are a teacher, you can (a) give or print or xerox copies for your students, (b) use potions for your own class notes (if they are published then you might need to add some acknowledgement, depending on which parts you copied), and you can xerox even very large portions of it to your hearts content. The "differential calculus" part means it covers derivatives and applications but not integrals. It is heavily based on the first half of a classic text, Granville's "Elements of the Differential and Integral Calculus," quite possibly a book your great grandfather might have used when he was college age. Some material from Sean Mauch's excellent public domain text on Applied Mathematics,

`http://www.its.caltech.edu/~sean/book.html`

was also included.

Calculus has been around for several hundred years and the teaching of it has not changed radically. Of course, like any topic which is taught in school, there are *some* modifications, but not major ones in this case. If $x(t)$ denotes the distance a train has traveled in a straight line at time t then the derivative is the *velocity*. If $q(t)$ denotes the charge on a capacitor at time t in a simple electrical circuit then the derivative is the *current*. If $C(t)$ denotes the concentration of a solvent in a chemical mixture at time t then the derivative is the *reaction rate*. If $P(t)$ denotes the population size of a country at time t then the derivative is the *growth rate*. If $C(x)$ denotes the cost to manufacture x units of a production item (such as a broom, say) then the derivative is the *marginal cost*.

Some of these topics, electrical circuits for example, were not studied in calculus when Granville's book was first written. However, aside from some changes in grammar and terminology (which have been updated in this version), the mathematical *content* of the calculus course taught today is basically the same as that taught a hundred years ago. Terminology has changed, and no one talks about "versines" any more (they were used in navigation tables before the advent of computers), but the *basic techniques* have not. Therefore, to make the book more useful to current students, some modification and rearrangement of the material in Granville's old text is appropriate. Overall, though the rigor and detailed explanations are still at their same high level of quality.

Here is a quote form Granville's original preface:

> The author has tried to write a textbook that is thoroughly modern and teachable, and the capacity and needs of the student pursuing a first course in the Calculus have been kept constantly in mind. The book contains more material than is necessary for the usual course of one hundred lessons given in our colleges and engineering schools; but this gives teachers an opportunity to choose such subjects as best suit the needs of their classes. It is believed that the volume contains all topics from which a selection naturally would be made in preparing students either for elementary work in applied science or for more advanced work in pure mathematics.
>
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> GETTYSBURG COLLEGE
> Gettysburg, Pa.

For further information on William Granville, please see the Wikipedia article at `http://en.wikipedia.org/wiki/William_Anthony_Granville`, which has a short biography and links for further information.

Granville's book "Elements of the Differential and Integral Calculus" fell into the public domain (in the United States - other countries may be different) and then much of it (but not all, at the time of this writing) was scanned into

`http://en.wikisource.org/wiki/Elements_of_the_Differential_and_Integral_Calculus`

primarily by P. J. Hall. This wikisource document uses MathML and LaTeX and some Greek letter fonts.

In keeping with the "free and open source" aspect of this textbook, and the theme of updating to today's much more technologically-aware students, a free and open sourse mathematical software package Sage was used to illustrate examples throughout. You don't need to know Sage to read the book (just ignore the Sage examples if you want) but it certainly won't hurt to learn a little about it. Besides, you might find that with some practice Sage is fun to "play with" and helps you with homework or other mathematical problems in some of your other classes. It is a general purpose mathematical software program and it may very likely be the only mathematical software you will ever need.

This LaTeX'd version is due to the second-named author, who is responsible for formatting, the correction of any typos in the scanned version, significant revision for readability, and some extra material (for example, the Sage examples and graphics). In particular, the existence of this document owes itself primarily to three great open source projects: TeX/LaTeX, Wikipedia, and Sage. All the figures were created using Sage and then edited and converted using the excellent open source image manipulation program GIMP (http://www.gimp.org). The Sage code for each image can be found in the LaTeX source code, available at

http://sage.math.washington.edu/home/wdj/teaching/calc1-sage/.

More information on Sage can be found at the Sage website (located at http://www.sagemath.org) or in the Appendix (Chapter 13) below.

Though the original text of Granville is public domain, the extra material added in this version is licensed under the GNU Free Documentation License (reproduced in an Appendix below), as is Wikipedia.

Acknowledgements: I thank the following readers for careful proofreading and reporting typos: Mario Pernici, Jacob Hicks, Georg Muntingh, and Minh Van Nguyen. I also thank Trevor Lipscombe for excellent stylistic advice on the presentation of the book. However, any remaining errors are solely my responsibility. Please send comments, suggestions, proposed changes, or corrections by email to wdjoyner@gmail.com.

CHAPTER

ONE

Variables and functions

1.1 Variables and constants

A *variable* is a quantity to which an unlimited number of values can be assigned. Variables are denoted by the later letters of the alphabet. Thus, in the equation of a straight line,

$$\frac{x}{a} + \frac{y}{b} = 1$$

x and y may be considered as the variable coordinates of a point moving along the line. A quantity whose value remains unchanged is called a *constant*.

Numerical or absolute constants retain the same values in all problems, as 2, 5, $\sqrt{7}$, π, etc.

Arbitrary constants, or parameters, are constants to which any one of an unlimited set of numerical values may be assigned, and they are supposed to have these assigned values throughout the investigation. They are usually denoted by the earlier letters of the alphabet. Thus, for every pair of values arbitrarily assigned to a and b, the equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

represents some particular straight line.

1.2 Interval of a variable

Very often we confine ourselves to a portion only of the number system. For example, we may restrict our variable so that it shall take on only such values as lie between a and b, where a and b may be included, or either or both excluded. We shall employ the symbol $[a,\ b]$, a being less than b, to represent the numbers $a,\ b$, and all the numbers between them, unless otherwise stated. This symbol $[a,\ b]$ is read the interval from a to b.

1.3 Continuous variation

A variable x is said to vary continuously through an interval $[a,\ b]$, when x starts with the value a and increases until it takes on the value b in such a manner as to assume the value of every number between a and b in the order of their magnitudes. This may be illustrated geometrically as follows:

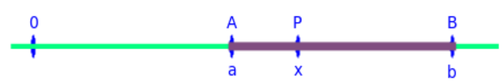

Figure 1.1: Interval from A to B.

The origin being at O, layoff on the straight line the points A and B corresponding to the numbers a and b. Also let the point P correspond to a particular value of the variable x. Evidently the interval $[a,\ b]$ is represented by the segment AB. Now as x varies continuously from a to b inclusive, i.e. through the interval $[a,\ b]$, the point P generates the segment AB.

1.4 Functions

A function f of the real numbers $\mathbb{R}$ is a well-defined rule which associated to each $x \in \mathbb{R}$ a unique value $f(x)$. Usually functions are described algebraically using some formula (such as $f(x) = x^2$, for all real numbers x) but it doesn't have to be so simple. For example,

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is an integer,} \\ 0, & \text{otherwise,} \end{cases}$$

is a function on $\mathbb{R}$ but it is given by a relatively complicated rule. Namely, the rule f tells you to associate to a number x the value 0 unless x is an integer, in which case you are to associate the value x^2. (In particular, $f(x)$ is always an integer, no matter what x is.) This type of rule defining a function of x is sometimes called a *piecewise-defined function*. In this book, we shall usually focus on functions given by simpler symbolic expressions. However, be aware that piecewise-defined functions *do* arise naturally in applications. For example, in electronics, when a 6 volt battery-powered flashlight is powered on or off using a switch, the voltage to the lightbulb is modeled by a piecewise-defined function which has the value 0 when the device is off and 6 when it is switched on.

When two variables are so related that the value of the first variable depends on the value of the second variable, then the first variable is said to be a *function* of the second variable.

Nearly all scientific problems deal with quantities and relations of this sort, and in the experiences of everyday life we are continually meeting conditions illustrating the dependence of one quantity on another. For instance, the weight a man is able to lift depends on his strength, other things being equal. Similarly, the distance a boy can run may be considered as depending on the time. Or, we may say that the area of a square is a function of the length of a side, and the volume of a sphere is a function of its diameter.

1.5 Notation of functions

The symbol $f(x)$ is used to denote a function of x, and is read "f of x". In order to distinguish between different functions, the prefixed letter is changed, as $F(x)$, $\phi(x)$, $f'(x)$, etc.

During any investigation the same functional symbol always indicates the same law of dependence of the function upon the variable. In the simpler cases this law takes the form of a series of analytical operations upon that variable. Hence, in such a case, the same functional symbol will indicate the same operations or series of operations, even though applied to different quantities. Thus, if

$$f(x) = x^2 - 9x + 14,$$

then

$$f(y) = y^2 - 9y + 14.$$

Also

$$f(a) = a^2 - 9a + 14,$$
$$f(b+1) = (b+1)^2 - 9(b+1) + 14 = b^2 - 7b + 6,$$
$$f(0) = 0^2 - 9 \cdot 0 + 14 = 14,$$
$$f(-1) = (-1)^2 - 9(-1) + 14 = 24,$$
$$f(3) = 3^2 - 9 \cdot 3 + 14 = -4,$$
$$f(7) = 7^2 - 9 \cdot 7 + 14 = 0,$$

etc. Similarly, $\phi(x,\ y)$ denotes a function of x and y, and is read "ϕ of x and y". If

$$\phi(x,\ y) = \sin(x+y),$$

then

$$\phi(a,\ b) = \sin(a+b),$$

and

$$\phi\left(\frac{\pi}{2}, 0\right) = \sin\frac{\pi}{2} = 1.$$

Again, if

$$F(x,\ y,\ z) = 2x + 3y - 12z,$$

then

$$F(m,\ -m,\ m) = 2m - 3m - 12m = -13m.$$

and

$$F(3,\ 2,\ 1) = 2 \cdot 3 + 3 \cdot 2 - 12 \cdot 1 = 0.$$

Evidently this system of notation may be extended indefinitely.

You can define a function in Sage in several ways:

Sage

```
sage: x,y = var("x,y")
sage: f = log(sqrt(x))
sage: f(4)
log(4)/2
sage: f(4).simplify_log()
log(2)
sage: f = lambda x: (x^2+1)/2
sage: f(x)
(x^2 + 1)/2
sage: f(1)
```

```
1
sage: f = lambda x,y: x^2+y^2
sage: f(3,4)
25
sage: R.<x> = PolynomialRing(CC,"x")
sage: f = x^2+2
sage: f.roots()
[(1.41421356237309*I, 1), (2.77555756156289e-17 - 1.41421356237309*I, 1)]
```

1.6 Independent and dependent variables

The second variable, to which values may be assigned at pleasure within limits depending on the particular problem, is called the *independent variable*, or *argument*; and the first variable, whose value is determined as soon as the value of the independent variable is fixed, is called the *dependent variable*, or *function*.

Though we shall wait to introduce differentiation later, please keep in mind that you differentiate the *dependent* variable with respect to the *independent* variable.

Example 1.6.1. *In the equation of an upper half-circle of radius* 1,

$$y = \sqrt{1 - x^2},$$

we typically call x *the independent variable and* y *the dependent variable.*

Frequently, when we are considering two related variables, it is in our power to fix upon whichever we please as the independent variable; but having once made the choice, no change of independent variable is allowed without certain precautions and transformations.

One quantity (the dependent variable) may be a function of two or more other quantities (the independent variables, or arguments). For example, the cost of cloth is a function of both the quality and quantity; the area of a triangle is a function of the base and altitude; the volume of a rectangular parallelepiped is a function of its three dimensions.

In the Sage example below, t is the independent variable and f is the dependent variable.

Sage

```
sage: t = var('t')
sage: f = function('f', t)
sage: f = cos
```

```
sage: f(pi/2)
0
sage: (f(-3*pi)-2*f(1))^2
(-2*cos(1) - 1)^2
```

1.7 The domain of a function

The values of the independent variable for which a function $f(x)$ is defined is often referred to as the *domain* of the function, denoted $\mathrm{domain}(f)$.

Consider the functions

$$x^2 - 2x + 5,\ \sin x,\ \arctan x$$

of the independent variable x. Denoting the dependent variable in each case by y, we may write

$$y = x^2 - 2x + 5,\ y = \sin x,\ y = \arctan x.$$

In each case y (the value of the function) is known, or, as we say, defined, for all values of x. We write in this case, $\mathrm{domain}(f) = \mathbb{R}$. This is not by any means true of all functions, as the following examples illustrating the more common exceptions will show.

$$y = \frac{a}{x-b} \tag{1.1}$$

Here the value of y (i.e. the function) is defined for all values of x except $x = b$. When $x = b$ the divisor becomes zero and the value of y cannot be computed from (1.1). We write in this case, $\mathrm{domain}(y) = \mathbb{R} - \{b\}$.

$$y = \sqrt{x}. \tag{1.2}$$

In this case the function is defined only for positive values of x. Negative values of x give imaginary values for y, and these must be excluded here, where we are confining ourselves to real numbers only. We write in this case, $\mathrm{domain}(y) = \{x \in \mathbb{R} \mid x \geq 0\}$.

$$y = \log_a x. \qquad a > 0 \tag{1.3}$$

Here y is defined only for positive values of x. For negative values of x this function does not exist (see 2.7).

$$y = \arcsin x, \ y = \arccos x. \tag{1.4}$$

Since sines, and cosines cannot become greater than $+1$ nor less than -1, it follows that the above functions are defined for all values of x ranging from -1 to $+1$ inclusive, but for no other values.

Sage

```
sage: t = var("t'')
sage: f = function('f', t)
sage: g = function('g', t)
sage: f = sin
sage: g = asin
sage: f(g(t))
t
sage: g(f(t))
t
sage: g(f(0.2))
0.200000000000000
```

1.8 Exercises

1. Given $f(x) = x^3 - 10x^2 + 31x - 30$; show that

$$f(0) = -30, \quad f(y) = y^3 - 10y^2 + 31y - 30,$$
$$f(2) = 0, \quad f(a) = a^3 - 10a^2 + 31a - 30,$$
$$f(3) = f(5), \quad f(yz) = y^3z^3 - 10y^2z^2 + 31yz - 30,$$
$$f(1) > f(-3), \quad f(x-2) = x^3 - 16x^2 + 83x - 140,$$
$$f(-1) = 6f(6).$$

2. If $f(x) = x^3 - 3x + 2$, find $f(0)$, $f(1)$, $f(-1)$, $f\left(-\frac{1}{2}\right)$, $f\left(\frac{4}{3}\right)$.

3. If $f(x) = x^3 - 10x^2 + 31x - 30$, and $\phi(x) = x^4 - 55x^2 - 210x - 216$, show that

   $f(2) = \phi(-2), f(3) = \phi(-3), f(5) = \phi(-4), f(0) + \phi(0) + 246 = 0.$

4. If $F(x) = 2x$, find $F(0)$, $F(-3)$, $F\left(\frac{1}{3}\right)$, $F(-1)$.

5. Given $F(x) = x(x-1)(x+6)\left(x - \frac{1}{2}\right)\left(x + \frac{5}{4}\right)$, show that $F(0) = F(1) = F(-6) = F\left(\frac{1}{2}\right) = F\left(-\frac{5}{4}\right) = 0$.

6. If $f(m_1) = \frac{m_1 - 1}{m_1 + 1}$, show that $\frac{f(m_1) - f(m_2)}{1 + f(m_1)f(m_2)} = \frac{m_1 - m_2}{1 + m_1 m_2}$.

7. If $\phi(x) = a^x$, show that $\phi(y) \cdot \phi(z) = \phi(y + z)$.

8. Given $\phi(x) = \log \frac{1-x}{1+x}$, show that $\phi(x) + \phi(y) = \phi\left(\frac{x+y}{1+xy}\right)$.

9. If $f(\phi) = \cos\phi$, show that $f(\phi) = f(-\phi) = -f(\pi - \phi) = -f(\pi + \phi)$.

10. If $F(\theta) = \tan\theta$, show that $F(2\theta) = \frac{2F(\theta)}{1 - [F(\theta)]^2}$.

    Here's how to use Sage to verify the double angle identity for $\tan$ above:

    Sage

```
sage: theta = var("theta")
sage: tan(2*theta).expand_trig()
2*tan(theta)/(1 - tan(theta)^2)
```

11. Given $\psi(x) = x^{2n} + x^{2m} + 1$, show that $\psi(1) = 3$, $\psi(0) = 1$, and $\psi(a) = \psi(-a)$, for any real number a (Hint: Use the fact that $(-1)^2 = 1$.)

12. If $f(x) = \frac{2x-3}{x+7}$, find $f(\sqrt{2})$.

CHAPTER

TWO

Theory of limits

In this book, a *variable* denotes a quantity which takes values in the real numbers.

2.1 Limit of a variable

If a variable v takes on successively a series of values that approach nearer and nearer to a constant value L in such a manner that $|v - L|$ becomes and remains less than any assigned arbitrarily small positive quantity, then v is said to approach the limit L, or to converge to the limit L. Symbolically this is written $\lim_{v=L}$, or more commonly

$$\lim_{v \to L} .$$

The following familiar examples illustrate what is meant:

1. As the number of sides of a regular inscribed polygon is indefinitely increased, the limit of the area of the polygon is the area of the circle. In this case the variable is always less than its limit.

2. Similarly, the limit of the area of the circumscribed polygon is also the area of the circle, but now the variable is always greater than its limit.

3. Hold a penny exactly 1 meter above the ground and observe its motion as you release it. First it travels 1/2 the distance from the ground (at this stage its distance fallen is $1/2$ meter), then it travels 1/2 that distance from the

ground (at this stage its distance fallen is $1/2 + 1/4$ meter), then it travels 1/2 that distance from the ground (at this stage its distance fallen is $1/2 + 1/4 + 1/8$ meter), and so on. This leads us to the series

$$\frac{1}{2} + \frac{1}{4} = \frac{1}{8} + \cdots + \frac{1}{2^k} + \cdots .$$

Since the penny hits the ground, this infinite sum is 1. (This computational idea goes back to the Greek scholar Archimedes, c. 287 BC c. 212 BC.)

4. Consider the series

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + (\frac{-1}{2})^k + \cdots . \tag{2.1}$$

The sum of any even number $(2n)$ of the first terms of this series is

$$\begin{aligned} S_{2n} &= 1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots + \tfrac{1}{2^{2n-2}} - \tfrac{1}{2^{2n-1}} \\ &= \tfrac{\frac{1}{2^{2n}}-1}{-\frac{1}{2}-1} \\ &= \tfrac{2}{3} - \tfrac{1}{3 \cdot 2^{2n-1}}, \end{aligned} \tag{2.2}$$

by item 6, Ch. 12, §12.1. Similarly, the sum of any odd number $(2n+1)$ of the first terms of the series is

$$\begin{aligned} S_{2n+1} &= 1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \cdots - \tfrac{1}{2^{2n-1}} + \tfrac{1}{2^{2n}} \\ &= \tfrac{-\frac{1}{2^{2n+1}}-1}{-\frac{1}{2}-1} \\ &= \tfrac{2}{3} + \tfrac{1}{3 \cdot 2^{2n}}, \end{aligned} \tag{2.3}$$

again by item 6, Ch. 12, §12.1.

Writing (2.2) and (2.3) in the forms

$$\frac{2}{3} - S_{2n} = \frac{1}{3 \cdot 2^{2n-1}}, \qquad S_{2n+1} - \frac{2}{3} = \frac{1}{3 \cdot 2^{2n}}$$

we have

$$\lim_{n\to\infty} \left(\frac{2}{3} - S_{2n} \right) = \lim_{n\to\infty} \frac{1}{3 \cdot 2^{2n-1}} = 0,$$

and

$$\lim_{n\to\infty} \left(S_{2n+1} - \frac{2}{3} \right) = \lim_{n\to\infty} \frac{1}{3 \cdot 2^{2n}} = 0.$$

Hence, by definition of the limit of a variable, it is seen that both S_{2n} and S_{2n+1} are variables approaching $\frac{2}{3}$ as a limit as the number of terms increases without limit.

Summing up the first two, three, four, etc., terms of (2.1), the sums are found by ((2.2) and ((2.3) to be alternately less and greater than $\frac{2}{3}$, illustrating the case when the variable, in this case the sum of the terms of ((2.1), is alternately less and greater than its limit.

Sage

```
sage: S = lambda n: add([(-1)^i*2^(-i) for i in range(n)])
sage: RR(S(1)); RR(S(2)); RR(S(5)); RR(S(10)); RR(S(20))
1.00000000000000
0.500000000000000
0.687500000000000
0.666015625000000
0.666666030883789
```

You can see from the Sage example that the limit does indeed seem to approach $2/3$.

In the examples shown the variable never reaches its limit. This is not by any means always the case, for from the definition of the limit of a variable it is clear that the essence of the definition is simply that the absolute value of the difference between the variable and its limit shall ultimately become and remain less than any positive number we may choose, however small.

Example 2.1.1. *As an example illustrating the fact that the variable may reach its limit, consider the following. Let a series of regular polygons be inscribed in a circle, the number of sides increasing indefinitely. Choosing anyone of these, construct. the circumscribed polygon whose sides touch the circle at the vertices of the inscribed polygon. Let p_n and P_n be the perimeters of the inscribed and circumscribed polygons of n sides, and C the circumference of the circle, and suppose the values of a variable x to be as follows:*

$$P_n, \ p_{n+1}, \ C, \ P_{n+1}, \ p_{n+2}, \ C, \ P_{n+2}, \quad \text{etc.}$$

Then, evidently,

$$\lim_{x\to\infty} x = C$$

and the limit is reached by the variable, every third value of the variable being C.

2.2 Division by zero excluded

$\frac{0}{0}$ is indeterminate. For the quotient of two numbers is that number which multiplied by the divisor will give the dividend. But any number whatever multiplied by zero gives zero, and the quotient is indeterminate; that is, any number whatever may be considered as the quotient, a result which is of no value.

$\frac{a}{0}$ has no meaning, a being different from zero, for there exists no number such that if it be multiplied by zero, the product will equal a.

Therefore division by zero is not an admissible operation.

Care should be taken not to divide by zero inadvertently. The following fallacy is an illustration. Assume that

$$a = b.$$

Then evidently

$$ab = a^2.$$

Subtracting b^2,

$$ab - b^2 = a^2 - b^2.$$

Factoring,

$$b(a - b) = (a + b)(a - b).$$

Dividing by $a - b$,

$$b = a + b.$$

But $a = b$, therefore $b = 2b$, or, $1 = 2$. The result is absurd, and is caused by the fact that we divided by $a - b = 0$, which is illegal.

2.3 Infinitesimals

Definition 2.3.1. *A variable v whose limit is zero is called an infinitesimal*[1].

This is written

$$\lim_{v=0}, \text{ or}, \lim_{v\to 0},$$

and means that the successive absolute values of v ultimately become and remain less than any positive number however small. Such a variable is said to become "arbitrarily small."

[1]Hence a constant, no matter how small it may be, is not an infinitesimal.

If $\lim v = l$, then $\lim(v - l) = 0$; that is, the difference between a variable and its limit is an infinitesimal.

Conversely, if the difference between a variable and a constant is an infinitesimal, then the variable approaches the constant as a limit.

2.4 The concept of infinity (∞)

If a variable v ultimately becomes and remains greater than any assigned positive number, however large, we say v is "unbounded and positive " (or "increases without limit"), and write

$$\lim_{v=+\infty}, \text{ or, } \lim_{v\to+\infty}, \text{ or, } v \to +\infty.$$

If a variable v ultimately becomes and remains smaller than any assigned negative number, we say "unbounded and negative " (or "v decreases without limit"), and write

$$\lim_{v=-\infty}, \text{ or, } \lim_{v\to-\infty}, \text{ or, } v \to -\infty.$$

If a variable v ultimately becomes and remains in absolute value greater than any assigned positive number, however large, we say v, in absolute value, "increases without limit", or v becomes arbitrarily large[2], and write

$$\lim_{v=\infty}, \text{ or, } \lim_{v\to\infty}, \text{ or, } v \to \infty.$$

Infinity (∞) is not a number; it simply serves to characterize a particular mode of variation of a variable by virtue of which it becomes arbitrarily large.

Here is a Sage example illustrating $\lim_{t=\infty} 1/t = \lim_{t=-\infty} 1/t = 0$.

Sage

```
sage: t = var('t')
sage: limit(1/t, t = Infinity)
```

[2]On account of the notation used and for the sake of uniformity, the expression $v \to +\infty$ is sometimes read "v approaches the limit plus infinity". Similarly, $v \to -\infty$ is read "v approaches the limit minus infinity", and $v \to \infty$ is read "v, in absolute value, approaches the limit infinity". While the above notation is convenient to use in this connection, the student must not forget that infinity is not a limit in the sense in which we defined it in §2.2, for infinity is not a number at all.

```
0
sage: limit(1/t, t = -Infinity)
0
```

2.5 Limiting value of a function

Given a function $f(x)$. If the independent variable x takes on any series of values such that

$$\lim x = a,$$

and at the same time the dependent variable $f(x)$ takes on a series of corresponding values such that

$$\lim f(x) = A,$$

then as a single statement this is written

$$\lim_{x \to a} f(x) = A.$$

Here is an example of a limit using Sage :

Sage

```
sage: limit((x^2+1)/(2+x+3*x^2),x=infinity)
1/3
```

This tells us that $\lim_{x\to\infty} \frac{x^2+1}{2+x+3*x^2} = \frac{1}{3}$.

2.6 Continuous and discontinuous functions

A function $f(x)$ is said to be *continuous* for $x = a$ if the limiting value of the function when x approaches the limit a in any manner is the value assigned to the function for $x = a$. In symbols, if

$$\lim_{x \to a} f(x) = f(a),$$

then $f(x)$ is continuous for $x = a$. Roughly speaking, a function $y = f(x)$ is continuous if you can draw its graph by hand without lifting your pencil off the paper. In other words, the graph of a continuous function can have no “breaks.”

Example 2.6.1. *The piecewise constant function*

$$u(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

is not continuous since its graph has a “break” at $x = 0$ where it “steps up” from 0 to 1. This function models “on-off” switches in electrical engineering and is called the unit step function or the Heaviside function (after the brilliant engineer Oliver Heaviside, 1850 1925).

The function is said to be *discontinuous* for $x = a$ if this condition is not satisfied. For example, if

$$\lim_{x \to a} f(x) = \infty,$$

the function is discontinuous for $x = a$.

Sage

```
sage: t = var('t')
sage: P1 = plot(1/t, (t, -5, -0.1))
sage: P2 = plot(1/t, (t, 0.1, 5))
sage: show(P1+P2, aspect_ratio=1)
sage: limit(1/t,t=0,dir="plus")
+Infinity
sage: limit(1/t,t=0,dir="minus")
-Infinity
```

The graph in Figure 2.1 suggests that $\lim_{x \to 0+} 1/x = +\infty$ and $\lim_{x \to 0-} 1/x = -\infty$, as the above Sage computation confirms.

The attention of the student is now called to the following cases which occur frequently.

CASE I. As an example illustrating a simple case of a function continuous for a particular value of the variable, consider the function

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

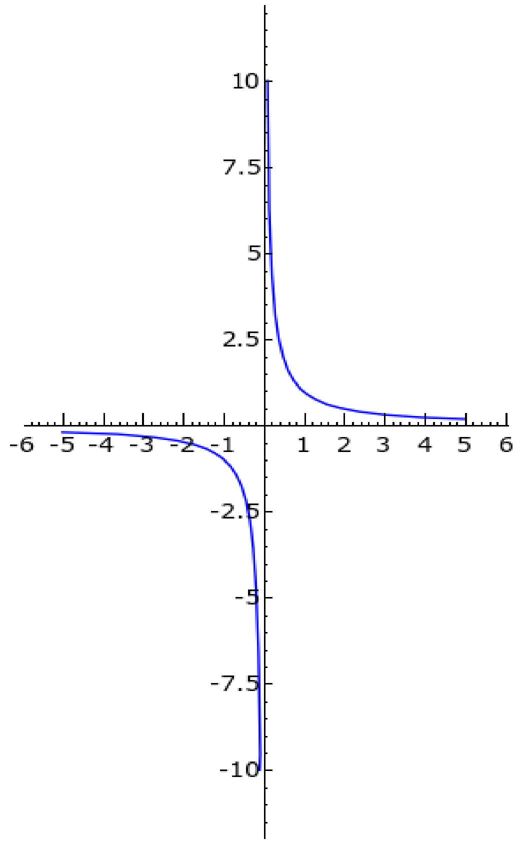

Figure 2.1: The limits $\lim_{x\to 0+} 1/x = +\infty$, $\lim_{x\to 0-} 1/x = -\infty$.

For $x = 1$, $f(x) = f(1) = 3$. Moreover, if x approaches the limit 1 in any manner, the function $f(x)$ approaches 3 as a limit. Hence the function is continuous for $x = 1$.

Sage

```
sage: x = var('x')
sage: limit((x^2-4)/(x-2), x = 1)
3
```

CASE II. The definition of a continuous function assumes that the function is already defined for $x = a$. If this is not the case, however, it is sometimes possible to assign such a value to the function for $x = a$ that the condition of continuity shall be satisfied. The following theorem covers these cases.

Theorem 2.6.1. *If $f(x)$ is not defined for $x = a$, and if*

$$\lim_{x\to a} f(x) = B,$$

then $f(x)$ will be continuous for $x = a$, if B is assumed as the value of $f(x)$ for $x = a$.

Thus the function

$$\frac{x^2-4}{x-2}$$

is not defined for $x = 2$ (since then there would be division by zero). But for every other value of x,

$$\frac{x^2-4}{x+2} = x+2;$$

and

$$\lim_{x\to 2}(x+2) = 4$$

therefore $\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$. Although the function is not defined for $x = 2$, if we assign it the value 4 for $x = 2$, it then becomes continuous for this value.

Sage

```
sage: x = var('x')
sage: limit((x^2-4)/(x-2), x = 2)
4
```

A function $f(x)$ is said to be *continuous in an interval* when it is continuous for all values of x in this interval[3].

[3] In this book we shall deal only with functions which are in general continuous, that is, continuous for all values of x, with the possible exception of certain isolated values, our results in

2.7 Continuity and discontinuity of functions illustrated by their graphs

1. Consider the function x^2, and let

$$y = x^2 \tag{2.4}$$

If we assume values for x and calculate the corresponding values of y, we can plot a series of points. Drawing a smooth line free-hand through these points: a good representation of the general behavior of the function may be obtained. This picture or image of the function is called its *graph*. It is evidently the locus of all points satisfying equation (2.4).

It is very easy to create the above plot in Sage , as the example below shows:

Sage

```
sage: P = plot(x^2,-2,2)
sage: show(P)
```

Such a series or assemblage of points is also called a *curve*. Evidently we may assume values of x so near together as to bring the values of y (and therefore the points of the curve) as near together as we please. In other words, there are no breaks in the curve, and the function x^2 is continuous for all values of x.

2. The graph of the continuous function $\sin x$, plotted by drawing the locus of $y = \sin\ x$,

It is seen that no break in the curve occurs anywhere.

3. The continuous function $exp(x) = e^x$ is of very frequent occurrence in the Calculus. If we plot its graph from

general being understood as valid only for such values of x for which the function in question is actually continuous. Unless special attention is called thereto, we shall as a rule pay no attention to the possibilities of such exceptional values of x for which the function is discontinuous. The definition of a continuous function f(x) is sometimes roughly (but imperfectly) summed up in the statement that a small change in x shall produce a small change in $f(x)$. We shall not consider functions having an infinite number of oscillations in a limited region.

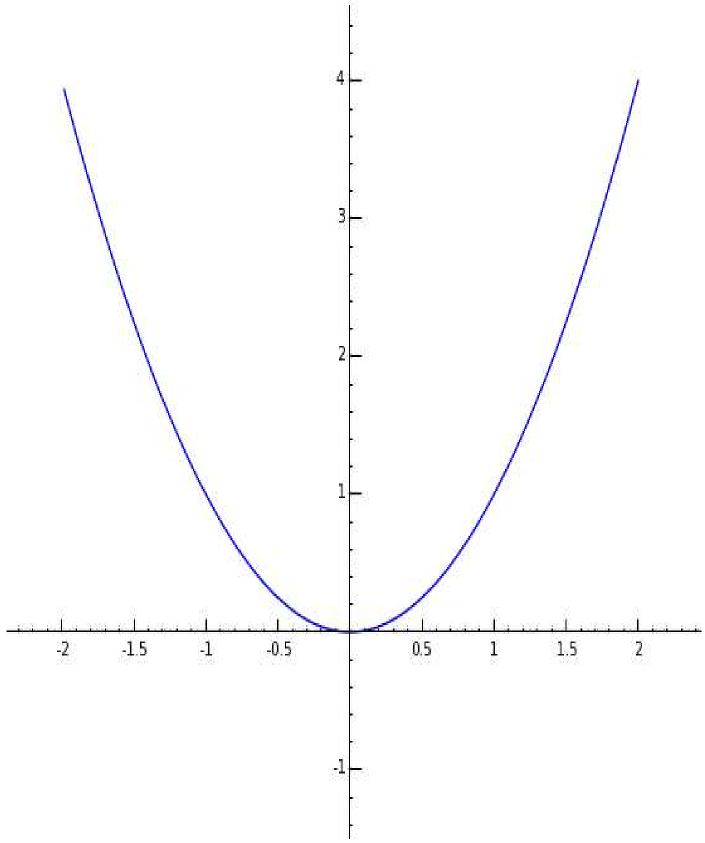

Figure 2.2: The parabola $y = x^2$.

$$y = e^x, \qquad (e = 2.718\cdots),$$

we get a smooth curve as shown.

From this it is clearly seen that,

(a) when $x = 0$, $\lim_{x\to 0} y (= \lim_{x\to 0} e^x) = 1$;

(b) when $x > 0$, $y (= e^x)$ is positive and increases as we pass towards the right from the origin;

(c) when $x < 0$, $y (= e^x)$ is still positive and decreases as we pass towards the left from the origin.

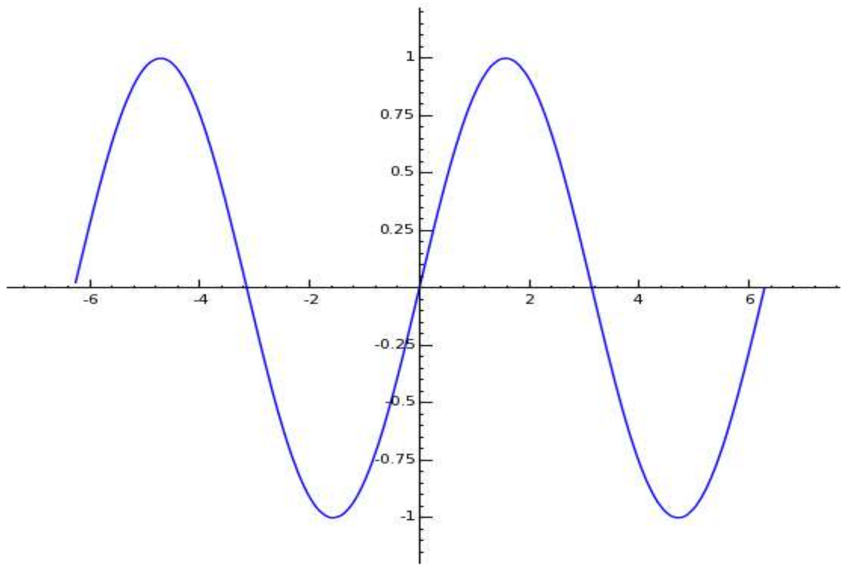

Figure 2.3: The sine function.

4. The function $\ln x = \log_e x$ is closely related to the last one discussed. In fact, if we plot its graph from

$$y = \log_e x,$$

it will be seen that its graph is the reflection of the graph of $y = e^x$ about the diagonal (the $x = y$ line). (This is because they are "inverses" of each other: $\log_e(e^x) = x$ and $e^{\log_e x} = x$.)

Here we see the following facts pictured:

(a) For $x = 1$, $\log_e x = \log_e 1 = 0$.

(b) For $x > 1$, $\log_e x$ is positive and increases as x increases.

(c) For $1 > x > 0$, $\log_e x$ is negative and increases in absolute value as x, that is, $\lim_{x\to 0} \log x = -\infty$.

(d) For $x \leq 0$, $\log_e x$ is not defined; hence the entire graph lies to the right of OY.

5. Consider the function $\frac{1}{x}$, and set

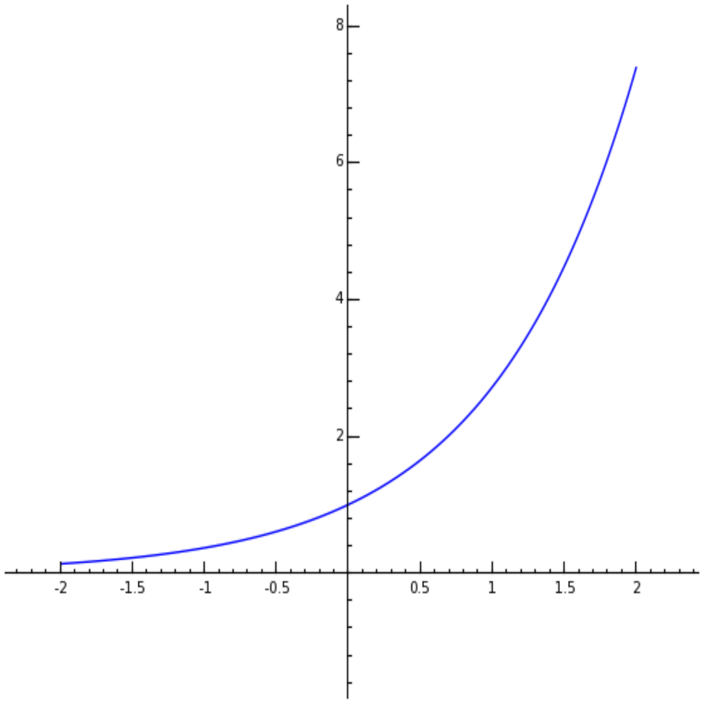

Figure 2.4: The exponential function.

$$y = \frac{1}{x}$$

If the graph of this function be plotted, it will be seen that as x approaches the value zero from the left (negatively), the points of the curve ultimately drop down an infinitely great distance, and as x approaches the value zero from the right, the curve extends upward infinitely far.

The curve then does not form a continuous branch from one side to the other of the axis of y, showing graphically that the function is discontinuous for $x = 0$, but continuous for all other values of x.

6. From the graph (see Figure 2.7) of

$$y = \frac{2x}{1 - x^2}$$

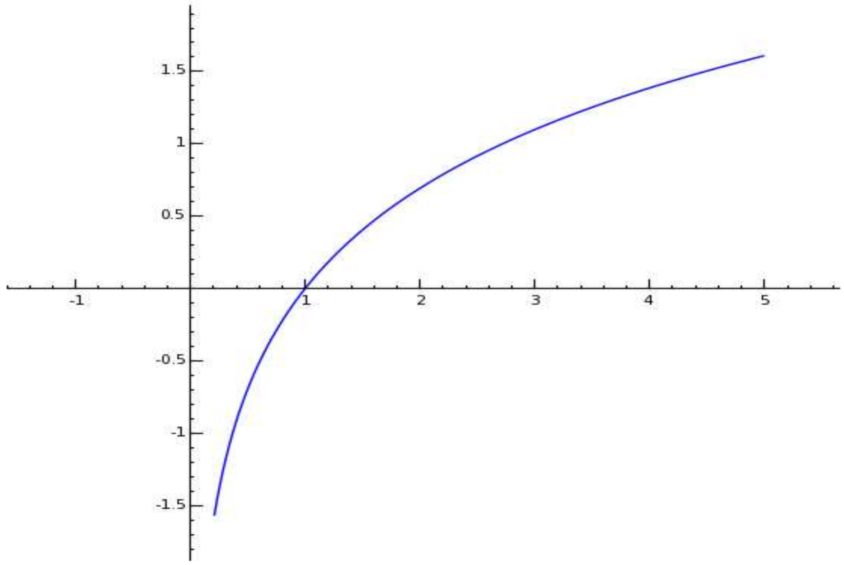

Figure 2.5: The natural logarithm.

it is seen that the function $\frac{2x}{1-x^2}$ is discontinuous for the two values $x = \pm 1$, but continuous for all other values of x.

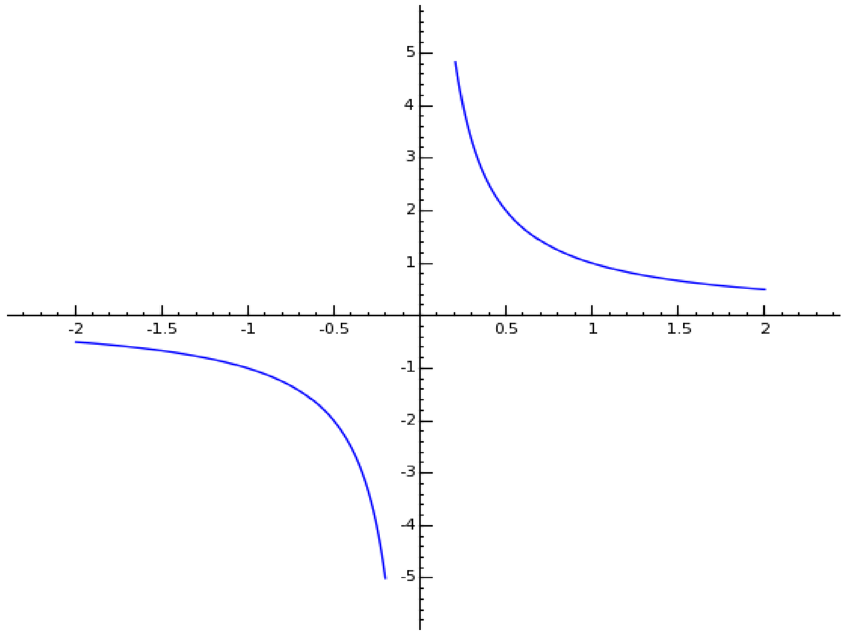

Figure 2.6: The function $y = 1/x$.

7. The graph of

$$y = \tan\ x$$

shows that the function $\tan x$ is discontinuous for infinitely many values of the independent variable x, namely, $x = \frac{n\pi}{2}$, where n denotes any odd positive or negative integer.

8. The function $\arctan\ x$ has infinitely many values for a given value of x, the graph of equation

$$y = \arctan\ x$$

consisting of infinitely many branches.

If, however, we confine ourselves to any single branch, the function is continuous. For instance, if we say that y shall be the smallest angle (in radians) whose tangent is x, that is, y shall take on only values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, then we are limited to the branch passing through the origin, and the condition for continuity is satisfied.

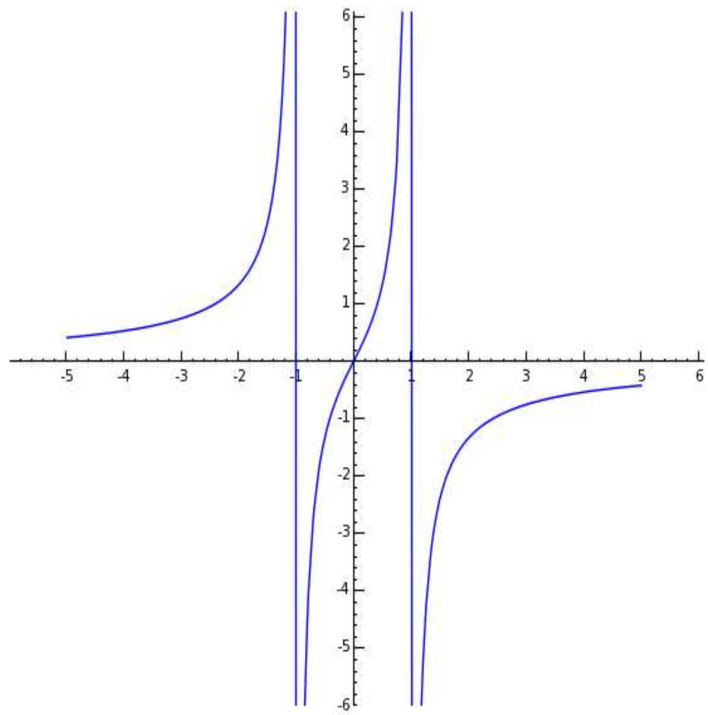

Figure 2.7: The function $y = 2x/(1 - x^2)$.

9. Similarly, $\arctan \frac{1}{x}$, is found to be a many-valued function. Confining ourselves to one branch of the graph of

$$y = \arctan \frac{1}{x},$$

we see that as x approaches zero from the left, y approaches the limit $-\frac{\pi}{2}$, and as x approaches zero from the right, y approaches the limit $+\frac{\pi}{2}$. Hence the function is discontinuous when $x = 0$. Its value for $x = 0$ can be assigned at pleasure.

10. As was previously mentioned, a *piecewise-defined function* is one which is defined by different rules on different non-overlapping intervals. For example,

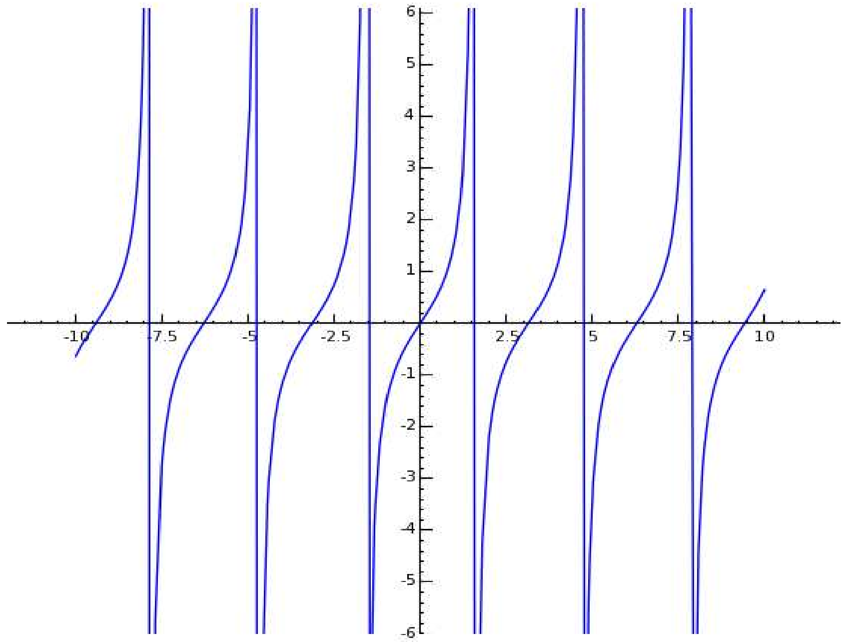

Figure 2.8: The tangent function.

$$f(x) = \begin{cases} -1, & x < -\pi/2, \\ \sin(x), & \pi/2 \le x \le \pi/2, \\ 1, & \pi/2 < x. \end{cases}$$

is a continuous piecewise-defined function.

For example,

$$f(x) = \begin{cases} -1, & x < -2, \\ 3, & -2 \le x \le 3, \\ 2, & 3 < x. \end{cases}$$

is a discontinuous piecewise-defined function, with jump discontinuities at $x = -2$ and $x = 3$.

Sage

```
sage: f = piecewise([[(-5,-2),-1],[(-2,3),3],[(3,5),2]])
```

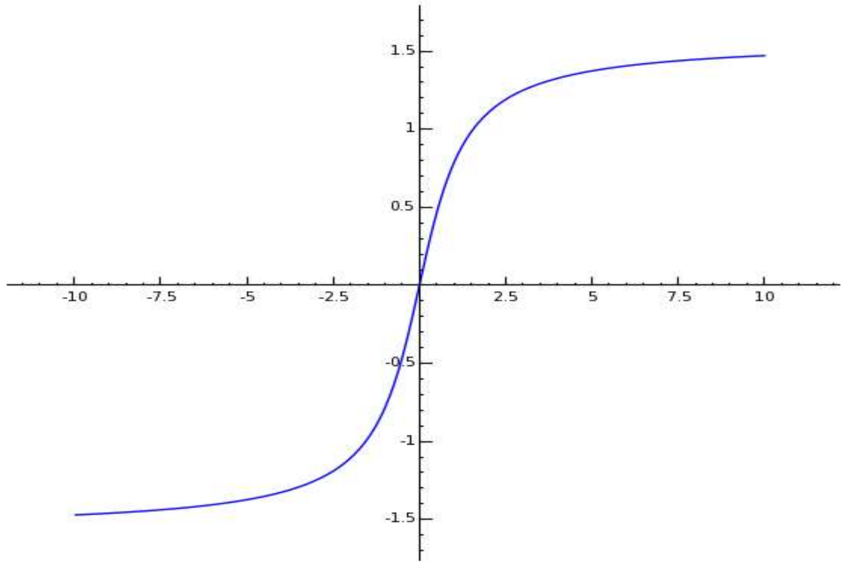

Figure 2.9: The arctangent (or inverse tangent) function.

```
sage: f
Piecewise defined function with 3 parts,
 [[(-5, -2), -1], [(-2, 3), 3], [(3, 5), 2]]
```

Functions exist which are discontinuous for every value of the independent variable within a certain range. In the ordinary applications of the Calculus, however, we deal with functions which are discontinuous (if at all) only for certain isolated values of the independent variable; such functions are therefore in general continuous, and are the only ones considered in this book.

2.8 Fundamental theorems on limits

In problems involving limits the use of one or more of the following theorems is usually implied. It is assumed that the limit of each variable exists and is finite.

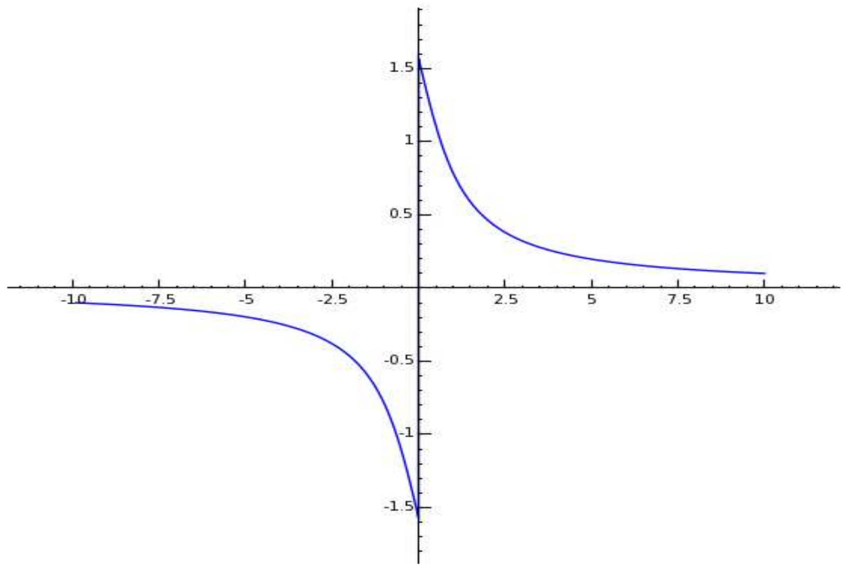

Figure 2.10: The function $y = \arctan(1/x)$.

Theorem 2.8.1. The limit of the algebraic sum of a finite number of variables is equal to the algebraic sum of the limits of the several variables.

In particular,

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Theorem 2.8.2. The limit of the product of a finite number of variables is equal to the product of the limits of the several variables.

In particular,

$$\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x).$$

Here is a Sage example illustrating these facts in a special case.

Sage

```
sage: t = var('t')
sage: f = exp
sage: g = sin
sage: a = var('a')
sage: L1 = limit(f(t)+g(t), t = a)
```

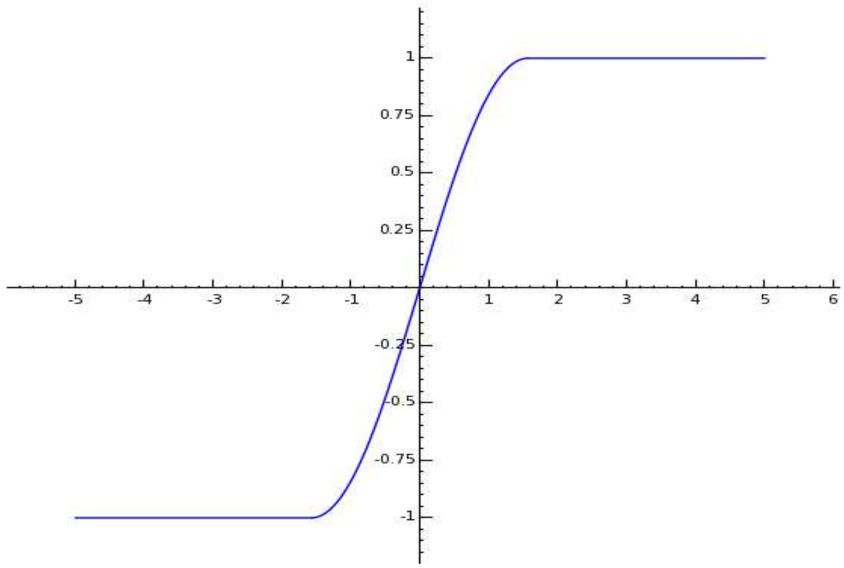

Figure 2.11: A piecewise-defined function.

```
sage: L2 = limit(f(t),t=a)+lim(g(t), t = a)
sage: bool(L1 == L2)
True
sage: L1; L2
sin(a) + e^a
sin(a) + e^a
sage: L1 = limit(f(t)*g(t), t = a)
sage: L2 = limit(f(t),t=a)*lim(g(t), t = a)
sage: bool(L1 == L2)
True
sage: L1; L2
e^a*sin(a)
e^a*sin(a)
```

Theorem 2.8.3. The limit of the quotient of two variables is equal to the quotient of the limits of the separate variables, provided the limit of the denominator is not zero.

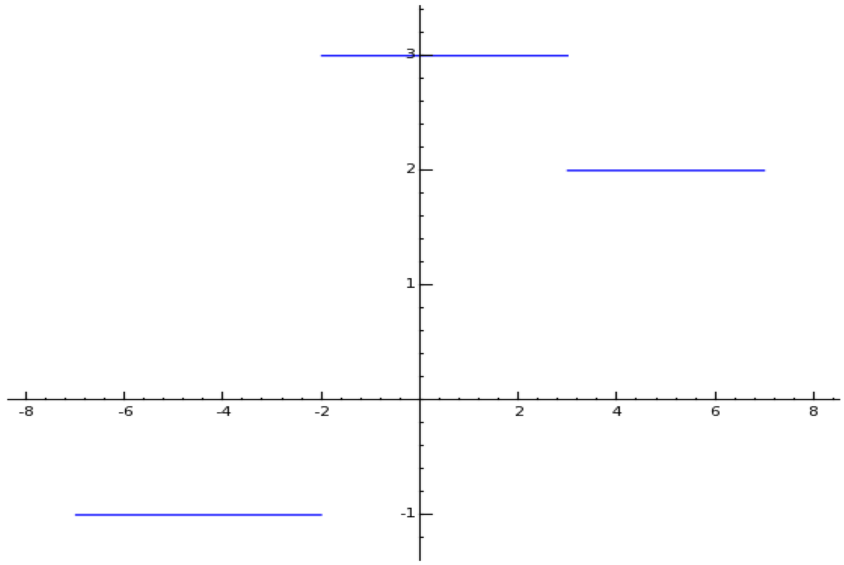

Figure 2.12: Another piecewise defined function.

In particular,

$$\lim_{x\to a}[f(x)/g(x)] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)},$$

provided $\lim_{x\to a} g(x) \neq 0$.

Before proving these theorems it is necessary to establish the following properties of infinitesimals (Definition 2.3.1).

1. The sum of a finite number of infinitesimals is an infinitesimal. To prove this we must show that the absolute value of this sum can be made less than any small positive quantity (as ϵ) that may be assigned (§2.3). That this is possible is evident, for, the limit of each infinitesimal being zero, each one can be made less than, in absolute value, $\frac{\epsilon}{n}$ (n being the number of infinitesimals), and therefore the absolute value of their sum can be made less than ϵ.

2. The product of a constant $c \neq 0$ and an infinitesimal is an infinitesimal. For the absolute value of the product can always be made less than any small

positive quantity (as ϵ) by making the absolute value of the infinitesimal less than $\frac{\epsilon}{|c|}$.

3. If v is a variable which approaches a limit L different from zero, then the quotient of an infinitesimal by v is also an infinitesimal. For if $v \to L$, and k is any number in absolute value less than L, then, by definition of a limit, v will ultimately become and remain in absolute value greater than k. Hence the quotient $\frac{\epsilon}{v}$, where ϵ is an infinitesimal, will ultimately become and remain in absolute value less than $\frac{\epsilon}{k}$, and is therefore, by the previous item, an infinitesimal.

4. The product of any finite number of infinitesimals is an infinitesimal. For the absolute value of the product may be made less than any small positive quantity that can be assigned. If the given product contains n factors, then since each infinitesimal may be assumed less than the $n-th$ root of ϵ, the product can be made less than ϵ itself.

Proof of Theorem 2.8.1. Let v_1, v_2, v_3, $\dots$ be the variables, and L_1, L_2, L_3, $\dots$ their respective limits. We may then write

$$v_1 - L_1 = \epsilon_1, \ \ v_2 - L_2 = \epsilon_2, \ \ v_3 - L_3 = \epsilon_3,$$

where ϵ_1, ϵ_2, ϵ_3, $\dots$ are infinitesimals (i.e. variables having zero for a limit). Adding

$$(v_1 + v_2 + v_3 + \dots) - (L_1 + L_2 + L_3 + ...) = (\epsilon_1 + \epsilon_2 + \epsilon_3 + \dots).$$

Since the right-hand member is an infinitesimal by item (1) above (§2.8), we have, from the converse theorem (§2.3),

$$\lim(v_1 + v_2 + v_3 + \dots) = L_1 + L_2 + L_3 + \dots,$$

or,

$$\lim(v_1 + v_2 + v_3 + \dots) = \lim v_1 + \lim v_2 + \lim v_3 + \dots,$$

which was to be proved. □

Proof of Theorem 2.8.2. Let v_1 and v_2 be the variables, L_1 and L_2 their respective limits, and ϵ_1 and ϵ_2 infinitesimals; then

$$v_1 = L_1 + \epsilon_1$$

and $v_2 = L_2 + \epsilon_2$. Multiplying,

$$\begin{aligned} v_1 v_2 &= (L_1 + \epsilon_1)(L_2 + \epsilon_2) \\ &= L_1 L_2 + L_1 \epsilon_2 + L_2 \epsilon_1 + \epsilon_1 \epsilon_2 \end{aligned}$$

or,

$$v_1 v_2 - L_1 L_2 = L_1 \epsilon_2 + L_2 \epsilon_1 + \epsilon_1 \epsilon_2.$$

Since the right-hand member is an infinitesimal by items (1) and (2) above, (§2.8), we have, as before,

$$\lim(v_1 v_2) = L_1 L_2 = \lim v_1 \cdot \lim v_2,$$

which was to be proved. □

Proof of Theorem 2.8.3. Using the same notation as before,

$$\frac{v_1}{v_2} = \frac{L_1 + \epsilon_1}{L_2 + \epsilon_2} = \frac{L_1}{L_2} + \left(\frac{L_1 + \epsilon_1}{L_2 + \epsilon_2} - \frac{L_1}{L_2} \right),$$

or,

$$\frac{v_1}{v_2} - \frac{L_1}{L_2} = \frac{L_2 \epsilon_1 - L_1 \epsilon_2}{L_2 (L_2 + \epsilon_2)}.$$

Here again the right-hand member is an infinitesimal by item (3) above, (§2.8), if $L_2 \neq 0$; hence

$$\lim \left(\frac{v_1}{v_2} \right) = \frac{L_1}{L_2} = \frac{\lim\ v_1}{\lim\ v_2},$$

which was to be proved. □

It is evident that if any of the variables be replaced by constants, our reasoning still holds, and the above theorems are true.

2.9 Special limiting values

The following examples are of special importance in the study of the Calculus. In the following examples $a > 0$ and $c \neq 0$.

Eqn number	Written in the form of limits	Abbreviated form often used
(1)	$\lim_{x\to 0} \frac{c}{x} = \infty$	$\frac{c}{0} = \infty$
(2)	$\lim_{x\to\infty} cx = \infty$	$c \cdot \infty = \infty$
(3)	$\lim_{x\to\infty} \frac{x}{c} = \infty$	$\frac{\infty}{c} = \infty$
(4)	$\lim_{x\to\infty} \frac{c}{x} = 0$	$\frac{c}{\infty} = 0$
(5)	$\lim_{x\to-\infty} a^x, = +\infty$, when $a < 1$	$a^{-\infty} = +\infty$
(6)	$\lim_{x\to+\infty} a^x = 0$, when $a < 1$	$a^{+\infty} = 0$
(7)	$\lim_{x\to-\infty} a^x = 0$, when $a > 1$	$a^{-\infty} = 0$
(8)	$\lim_{x\to+\infty} a^x = +\infty$, when $a > 1$	$a^{+\infty} = +\infty$
(9)	$\lim_{x\to 0} \log_a\ x = +\infty$, when $a < 1$	$\log_a\ 0 = +\infty$
(10)	$\lim_{x\to+\infty} \log_a\ x = -\infty$, when $a < 1$	$\log_a(+\infty) = -\infty$
(11)	$\lim_{x\to 0} \log_a\ x = -\infty$, when $a > 1$	$\log_a\ 0 = -\infty$
(12)	$\lim_{x\to+\infty} \log_a\ x = +\infty$, when $a > 1$	$\log_a(+\infty) = +\infty$

The expressions in the last column are not to be considered as expressing numerical equalities (∞ not being a number); they are merely symbolical equations implying the relations indicated in the first column, and should be so understood.

2.10 Show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$

To motivate the limit computation of this section, using Sage we compute a number of values of the function $\frac{\sin x}{x}$, as x gets closer and closer to 0:

x	0.5000	0.2500	0.1250	0.06250	0.03125
$\frac{\sin(x)}{x}$	0.9589	0.9896	0.9974	0.9994	0.9998

Indeed, if we refer to the table in §12.4, it will be seen that for all angles less than 10^o the angle in radians and the sine of that angle are equal to three decimal places. To compute the table of values above using Sage , simply use the following commands.

Sage

```
sage: f = lambda x: sin(x)/x
sage: R = RealField(15)
sage: L = [1/2^i for i in range(1,6)]; L
[1/2, 1/4, 1/8, 1/16, 1/32]
sage: [R(x) for x in L]
[0.5000, 0.2500, 0.1250, 0.06250, 0.03125]
sage: [R(f(x)) for x in L]
[0.9589, 0.9896, 0.9974, 0.9994, 0.9998]
```

From this we may well suspect that $\lim_{x\to 0}\frac{\sin\ x}{x}=1$.

Let O be the center of a circle whose radius is unity.

Let $\mathrm{arc}\ AM=\mathrm{arc}\ AM'=x$, and let MT and $M'T$ be tangents drawn to the circle at M and M' (see Figure 2.13).

Using the geometry in Figure 2.13), we find that

$$MPM' < MAM' < MTM';$$

or $2\sin\ x < 2x < 2\tan\ x$. Dividing through by $2\sin\ x$, we get

$$1 < \frac{x}{\sin\ x} < \frac{1}{\cos\ x}.$$

If now x approaches the limit zero,

$$\lim_{x\to 0}\frac{x}{\sin\ x}$$

must lie between the constant 1 and $\lim_{x\to 0}\frac{1}{\cos\ x}$, which is also 1. Therefore $\lim_{x\to 0}\frac{x}{\sin\ x}=1$, or, $\lim_{x\to 0}\frac{\sin\ x}{x}=1$ Theorem 2.8.3. $\square$

It is interesting to note the behavior of this function from its graph, the locus of equation

$$y=\frac{\sin\ x}{x}$$

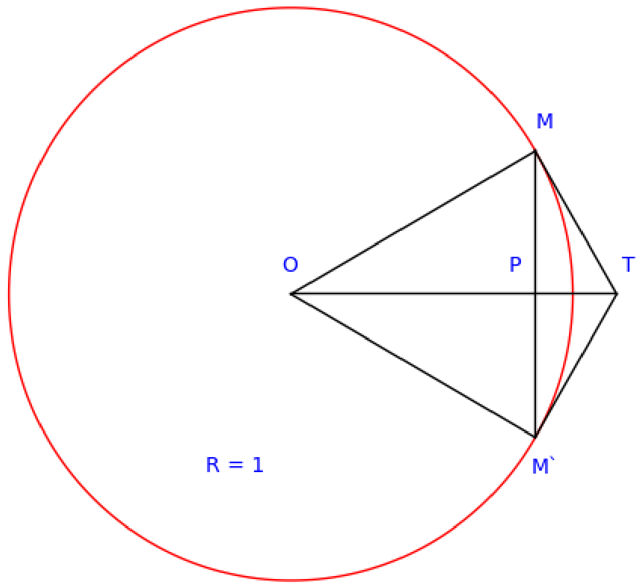

Figure 2.13: Comparing x and $\sin(x)$ on the unit circle.

Although the function is not defined for $x = 0$, yet it is not discontinuous when $x = 0$ if we define $\frac{\sin 0}{0} = 1$ (see Case II in §2.6).

Finally, we show how to use the Sage command `limit` to compute the limit above.

Sage

```
sage: limit(sin(x)/x,x=0)
1
```

2.11 The number e

One of the most important limits in the Calculus is

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = 2.71828 \cdots = e$$

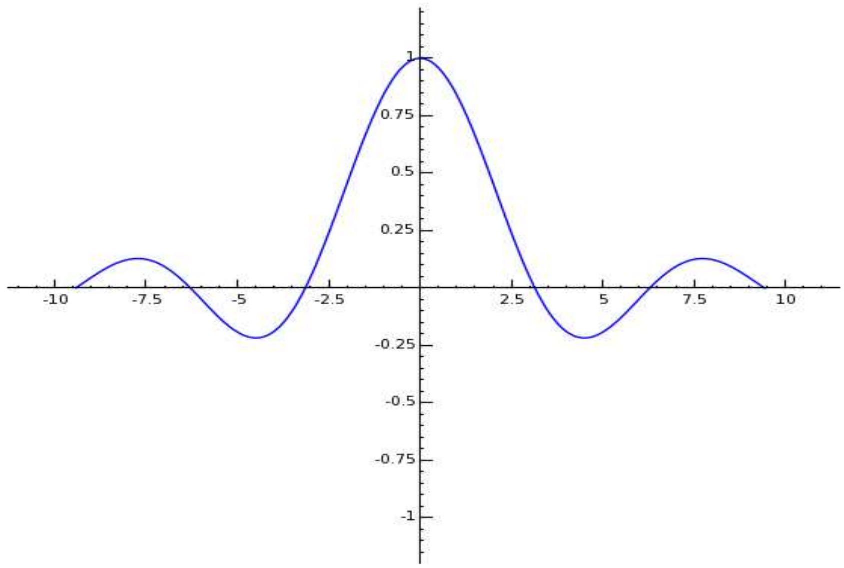

Figure 2.14: The function $\frac{\sin(x)}{x}$.

To prove rigorously that such a limit e exists, is beyond the scope of this book. For the present we shall content ourselves by plotting the locus of the equation

$$y = (1+x)^{\frac{1}{x}}$$

and show graphically that, as $x \doteq 0$, the function $(1+x)^{\frac{1}{x}} (= y)$ takes on values in the near neighborhood of $2.718\ldots$, and therefore $e = 2.718\ldots$, approximately.

x	-.1	-.001	.001	.01	.1	1	5	10
$y = (1+x)^{1/x}$	2.8680	2.7195	2.7169	2.7048	2.5937	2.0000	1.4310	1.0096

As $x \to 0-$ from the left, y decreases and approaches e as a limit. As $x \to 0+$ from the right, y increases and also approaches e as a limit.

As $x \to \infty$, y approaches the limit 1; and as $x \to -1+$ from the right, y increases without limit.

Natural logarithms are those which have the number e for base. These logarithms play a very important rle in mathematics. When the base is not indicated explicitly, the base e is always understood in what follows in this book. Thus $\log_e\ v$ is written simply $\log\ v$ or $\ln\ v$.

Natural logarithms possess the following characteristic property: If $x \to 0$ in any way whatever,

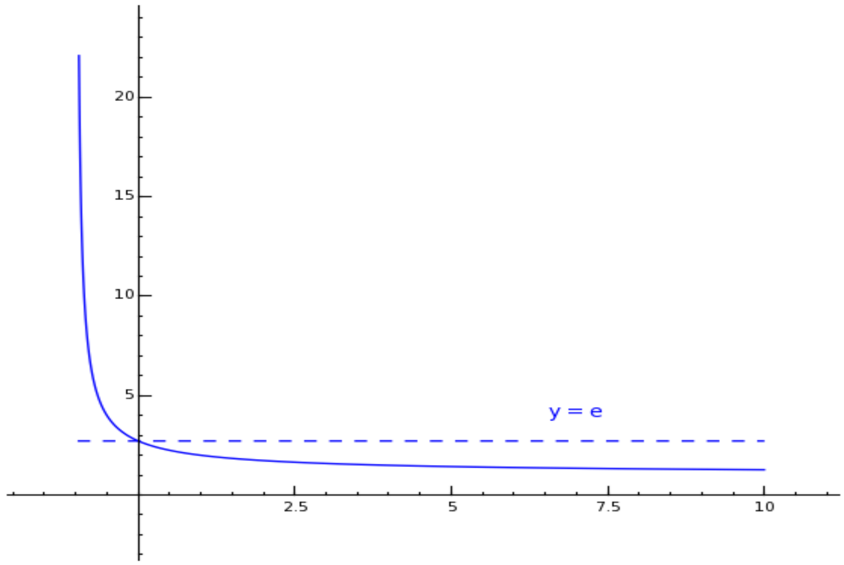

Figure 2.15: The function $(1+x)^{1/x}$.

$$\lim \frac{\log(1+x)}{x} = \lim \log(1+x)^{\frac{1}{x}} = \log\ e = \ln e = 1.$$

2.12 Expressions assuming the form $\frac{\infty}{\infty}$

As ∞ is not a number, the expression $\infty \ \div \ \infty$ is indeterminate. To evaluate a fraction assuming this form, the numerator and denominator being algebraic functions, we shall find useful the following

RULE. Divide both numerator and denominator by the highest power of the variable occurring in either. Then substitute the value of the variable.

Example 2.12.1. Evaluate

Solution. Substituting directly, we get

$$\lim_{x\to\infty} \frac{2x^3 - 3x^2 + 4}{5x - x^2 - 7x^3} = \frac{\infty}{\infty}$$

which is indeterminate. Hence, following the above rule, we divide both numerator and denominator by x^3, Then

$$\lim_{x\to\infty} \frac{2x^3 - 3x^2 + 4}{5x - x^2 - 7x^3} = \lim_{x\to\infty} \frac{2 - \frac{3}{x} + \frac{4}{x^3}}{\frac{5}{x^2} - \frac{1}{x} - 7} = -\frac{2}{7}.$$

2.13 Exercises

Prove the following:

1. $\lim_{x\to\infty} \left(\frac{x+1}{x}\right) = 1.$

   Solution:

$$\begin{aligned} \lim_{x\to\infty} \left(\tfrac{x+1}{x}\right) &= \lim_{x\to\infty} \left(1 + \tfrac{1}{x}\right) \\ &= \lim_{x\to\infty}(1) + \lim x \to \infty \left(\tfrac{1}{x}\right) \\ &= 1 + 0 = 1, \end{aligned}$$

   by Theorem 2.8.1

2. $\lim_{x\to\infty} \left(\frac{x^2+2x}{5-3x^2}\right) = -\frac{1}{3}.$

   Solution:

$$\lim_{x\to\infty} \left(\frac{x^2 + 2x}{5 - 3x^2}\right) = \lim_{x\to\infty} \left(\frac{1 + \frac{2}{x}}{\frac{5}{x^2} - 3}\right)$$

   [Dividing both numerator and denominator by x^2.]

$$= \frac{\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)}{\lim_{x\to\infty} \left(\frac{5}{x^2} - 3\right)}$$

   by Theorem 2.8.3

$$= \frac{\lim_{x\to\infty}(1) + \lim_{x\to\infty} \left(\frac{2}{x}\right)}{\lim_{x\to\infty} \left(\frac{5}{x^2}\right) - \lim_{x\to\infty}(3)} = \frac{1+0}{0-3} = -\frac{1}{3},$$

   by Theorem 2.8.1.

3. $\lim_{x\to 1} \frac{x^2-2x+5}{x^2+7} = \frac{1}{2}.$

4. $\lim_{x\to 0} \frac{3x^3+6x^2}{2x^4-15x^2} = -\frac{2}{5}.$

5. $\lim_{x\to -2} \frac{x^2+1}{x+3} = 5$.

6. $\lim_{h\to 0}(3ax^2 - 2hx + 5h^2) = 3ax^2$.

7. $\lim_{x\to\infty}(ax^2 + bx + c) = \infty$.

8. $\lim_{k\to 0} \frac{(x-k)^2-2kx^3}{x(x+k)} = 1$.

9. $\lim_{x\to\infty} \frac{x^2+1}{3x^2+2x-1} = \frac{1}{3}$.

10. $\lim_{x\to\infty} \frac{3+2x}{x^2-5x} = 0$.

11. $\lim_{\alpha\to\frac{\pi}{2}} \frac{\cos(\alpha-a)}{\cos(2\alpha-a)} = -\tan\alpha$.

12. $\lim_{x\to\infty} \frac{ax^2+bx+c}{dx^2+ex+f} = \frac{a}{d}$.

13. $\lim_{z\to 0} \frac{a}{2}(e^{\frac{z}{a}} + e^{-\frac{z}{a}}) = a$.

14. $\lim_{x\to 0} \frac{2x^3+3x^2}{x^3} = \infty$.

15. $\lim_{x\to\infty} \frac{5x^2-2x}{x} = \infty$.

16. $\lim_{y\to\infty} \frac{y}{y+1} = 1$.

17. $\lim_{n\to\infty} \frac{n(n+1)}{(n+2)(n+3)} = 1$.

18. $\lim_{s\to 1} \frac{s^3-1}{s-1} = 3$.

19. $\lim_{h\to 0} \frac{(x+h)^n-x^n}{h} = nx^{n-1}$.

20. $\lim_{h=0} \left[\cos(\theta+h)\frac{\sin h}{h}\right] = \cos\theta$.

21. $\lim_{x\to\infty} \frac{4x^2-x}{4-3x^2} = -\frac{4}{3}$.

22. $\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \frac{1}{2}$.

    Here is an example of the above limit using Sage :

Sage

```
sage: theta = var("theta")
sage: limit((1 - cos(theta))/(theta^2),theta=0)
1/2
```

This implies that, for small values of θ, $\cos(\theta) \cong 1 - \frac{1}{2}\theta^2$. (This is an approximation accurate to several decimal places for $|\theta| < 1/4$.)

23. $\lim_{x\to a} \frac{1}{x-a} = -\infty$, if x is increasing as it approaches the value a.

24. $\lim_{x\to a} \frac{1}{x-a} = +\infty$, if x is decreasing as it approaches the value a.

CHAPTER

THREE

Differentiation

3.1 Introduction

In this chapter, we investigate the ways in which a function can change in value as the independent variable changes. For instance, if $f(t)$ is a function of t (time), we want to quantify what it means to talk about the "rate of change" of $f(t)$. A fundamental problem of differential calculus is to establish a mathematically precise measure of this change in the function.

It was while investigating problems of this sort that Newton[1] was led to the discovery of the fundamental principles of calculus. Today, Gottfried Leibniz (1646-1716) is generally credited with independently discovering calculus around the same time[2].

[1] Sir Isaac Newton (1642-1727), an Englishman, was a man of the most extraordinary genius. He developed the science of calculus under the name of "Fluxions." Although Newton had discovered and made use of the new theory as early as 1670, his first published work in which it occurs is dated 1687, having the title **Philosophiae Naturalis Principia Mathematica**. This was Newton's principal work. Laplace said of it, "It will always remain preeminent above all other productions of the human mind." See frontispiece.

[2] However, see `http://en.wikipedia.org/wiki/Newton_v._Leibniz_calculus_controversy` and the footnote in §3.9 below.

3.2 Increments

The *increment* of a variable in changing from one numerical value to another is the difference found by subtracting the first value from the second. An increment of x is denoted by the symbol Δx, read "delta x" and typically to be regarded as "a small change in x." (The student is warned against reading this symbol as "delta times x.") Evidently this increment may be either positive or negative. according as the variable in changing is increasing or decreasing in value. Similarly,

- Δy denotes an increment of y,
- $\Delta\phi$ denotes an increment of ϕ,
- $\Delta f(x)$ denotes an increment $f(x)$, etc.

If in $y = f(x)$ the independent variable x, takes on an increment Δx, then Δy is always understood to denote the For example, if $\Delta x = x_1 - x_0$ then

$$\Delta y = y_1 - y_0 = f(x_1) - f(x_0) = f(x_0 + \Delta) - f(x_0).$$

Example 3.2.1. For instance, consider the function

$$y = x^2.$$

Assuming $x = 10$ for the initial value of x fixes $y = 100$ as the initial value of y. Suppose x increases to $x = 12$, that is, $\Delta x = 2$; then y increases to $y = 144$, and $\Delta y = 44$. Suppose x decreases to $x = 9$, that is, $\Delta x = -1$; then y increases to $y = 81$, and $\Delta y = -19$.

Sage

```
sage: x = var("x")
sage: f(x) = x^2; y = f(x)
sage: Deltax = 2; x0 = 10
sage: Deltay = f(x0 + Deltax) - f(x0)
sage: Deltay
44
```

3.3 Comparison of increments

Consider the function

$$y = x^2.$$

Assuming a fixed initial value for x, let x take on an increment Δx. Then y will take on a corresponding increment Δy, and we have

$$y + \Delta y = (x + \Delta x)^2,$$

or,

$$y + \Delta y = x^2 + 2x \cdot \Delta x + (\Delta x)^2.$$

Subtracting $y = x^2$ from this,

$$\Delta y = 2x \cdot \Delta x + (\Delta x)^2, \tag{3.1}$$

we get the increment Δy in terms of x and Δx. To find the ratio of the increments, divide (3.1) by Δx, giving

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x.$$

If the initial value of x is 4, it is evident that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 8.$$

Let us carefully note the behavior of the ratio of the increments of x and y as the increment of x diminishes.

Initial value of x	New value of x	Increment $\Delta\, x$	Initial value of y	New value of y	Increment $\Delta\, y$	$\frac{\Delta y}{\Delta x}$
4	5.0	1.0	16	25.	9.	9.
4	4.8	0.8	16	23.04	7.04	8.8
4	4.6	0.6	16	21.16	5.16	8.6
4	4.4	0.4	16	19.36	3.36	8.4
4	4.2	0.2	16	17.64	1.64	8.2
4	4.1	0.1	16	16.81	0.81	8.1
4	4.01	0.01	16	16.0801	0.0801	8.01

It is apparent that as Δx decreases, Δy also diminishes, but their ratio takes on the successive values 9, 8.8, 8.6, 8.4, 8.2, 8.1, 8.01; illustrating the fact that $\frac{\Delta y}{\Delta x}$ can be brought as near to 8 in value as we please by making Δx small enough. Therefore[3],

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 8.$$

3.4 Derivative of a function of one variable

The fundamental definition of the Differential Calculus is:

Definition 3.4.1. The *derivative*[4] of a function is the limit of the ratio of the increment of the function to the increment of the independent variable, when the latter increment varies and approaches the limit zero.

When the limit of this ratio exists, the function is said to be *differentiable*, or to *possess a derivative*.

The above definition may be given in a more compact form symbolically as follows: Given the function

$$y = f(x), \tag{3.2}$$

and consider x to have a fixed value. Let x take on an increment Δx; then the function y takes on an increment Δy, the new value of the function being

$$y + \Delta y = f(x + \Delta x). \tag{3.3}$$

To find the increment of the function, subtract (3.2) from (3.3), giving

$$\Delta y = f(x + \Delta x) - f(x).$$

Dividing by the increment of the variable, Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{3.4}$$

[3]The student should guard against the common error of concluding that because the numerator and denominator of a fraction are each approaching zero as a limit, the limit of the value of the fraction (or ratio) is zero. The limit of the ratio may take on any numerical value. In the above example the limit is 8.

[4]Also called the differential coefficient or the derived function.

The limit of this ratio when $\Delta\, x$ approaches the limit zero is, from our definition, the derivative and is denoted by the symbol $\frac{dy}{dx}$. Therefore

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

defines the *derivative of* y *[or* $f(x)$*] with respect to* x. From (3.3), we also get

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The process of finding the derivative of a function is called *differentiation*.

It should be carefully noted that the derivative is the limit of the ratio, not the ratio of the limits. The latter ratio would assume the form $\frac{0}{0}$, which is indeterminate (§2.2).

3.5 Symbols for derivatives

Since $\Delta\, y$ and $\Delta\, x$ are always finite and have definite values, the expression

$$\frac{\Delta y}{\Delta x}$$

is really a fraction. The symbol

$$\frac{dy}{dx},$$

however, is to be regarded not as a fraction but as the limiting value of a fraction. In many cases it will be seen that this symbol does possess fractional properties, and later on we shall show how meanings may be attached to dy and dx, but for the present the symbol $\frac{dy}{dx}$ is to be considered as a whole.

Since the derivative of a function of x is in general also a function of x, the symbol $f'(x)$ is also used to denote the derivative of $f(x)$.

Hence, if $y = f(x)$, we may write $\frac{dy}{dx} = f'(x)$, which is read "the derivative of y with respect to x equals f prime of x." The symbol

$$\frac{d}{dx}$$

when considered by itself is called the *differentiating operator*, and indicates that any function written after it is to be differentiated with respect to x. Thus

- $\frac{dy}{dx}$ or $\frac{d}{dx}y$ indicates the derivative of y with respect to x;
- $\frac{d}{dx}f(x)$ indicates the derivative of $f(x)$ with respect to x;
- $\frac{d}{dx}(2x^2+5)$ indicates the derivative of $2x^2+5$ with respect to x;
- y' is an abbreviated form of $\frac{dy}{dx}$.

The symbol D_x is used by some writers instead of $\frac{d}{dx}$. If then

$$y = f(x),$$

we may write the identities

$$y' = \frac{dy}{dx} = \frac{d}{dx}y = D_x f(x) = f'(x).$$

3.6 Differentiable functions

From the theory of limits (Chapter 2), it is clear that if the derivative of a function exists for a certain value of the independent variable, the function itself must be continuous for that value of the variable.

However, the converse is not always true. Functions have been constructed that are continuous and yet possess no derivative. But in this book we only consider functions $f(x)$ that possess a derivative for all values of the independent variable, save at most for some isolated (discrete) values of x.

3.7 General rule for differentiation

From the definition of a derivative it is seen that the process of differentiating a function $y = f(x)$ consists in taking the following distinct steps:

General rule for differentiating[5]:

- FIRST STEP. In the function replace x by $x + \Delta\, x$, giving a new value of the function, $y + \Delta\, y$.
- SECOND STEP. Subtract the given value of the function from the new value in order to find $\Delta\, y$ (the increment of the function).

[5] Also called the Four-step Rule.

- THIRD STEP. Divide the remainder $\Delta\, y$ (the increment of the function) by $\Delta\, x$ (the increment of the independent variable).

- FOURTH STEP. Find the limit of this quotient, when $\Delta\, x$ (the increment of the independent variable) varies and approaches the limit zero. This is the derivative required.

The student should become thoroughly familiar with this rule by applying the process to a large number of examples. Three such examples will now be worked out in detail.

Example 3.7.1. Differentiate $3x^2 + 5$.
Solution. Applying the successive steps in the General Rule, we get, after placing

$$y = 3x^2 + 5,$$

First step.

$$y + \Delta\, y = 3(x + \Delta\, x)2 + 5 = 3x^2 + 6x \cdot \Delta x + 3(\Delta x)^2 + 5.$$

Second step.

$$\begin{aligned} y + \Delta\, y &= 3x^2 + 6x \cdot \Delta x + 3(\Delta x)^2 + 5 \\ y &= 3x^2 + 5 \\ \Delta\, y &= 6x \cdot \Delta x + 3(\Delta x)^2. \end{aligned}$$

Third step. $\frac{\Delta y}{\Delta x} = 6x + 3 \cdot \Delta x$.
Fourth step. $\frac{dy}{dx} = 6x$. We may also write this

$$\frac{d}{dx}(3x^2 + 5) = 6x.$$

Here's how to use Sage to verify this (for simplicity, we set $h = \Delta x$):

Sage

```
sage: x = var("x")
sage: h = var("h")
sage: f(x) = 3*xˆ2 + 5
sage: Deltay = f(x+h)-f(x)
sage: (Deltay/h).expand()
6*x + 3*h
```

```
sage: limit((f(x+h)-f(x))/h,h=0)
6*x
sage: diff(f(x),x)
6*x
```

Example 3.7.2. Differentiate $x^3 - 2x + 7$.

Solution. Place $y = x^3 - 2x + 7$.

First step.

$$\begin{aligned} y + \Delta y &= (x + \Delta x)3 - 2(x + \Delta x) + 7 \\ &= x^3 + 3x^2 \cdot \Delta x + 3x \cdot (\Delta x)^2 + (\Delta x)^3 - 2x - 2 \cdot \Delta x + 7 \end{aligned}$$

Second step.

$$\begin{aligned} y + \Delta y &= x^3 + 3x^2 \cdot \Delta x + 3x \cdot (\Delta x)^2 + (\Delta x)^3 - 2x - 2 \cdot \Delta x + 7 \\ y &= x^3 - 2x + 7 \\ \Delta y &= 3x^2 \cdot \Delta x + 3x \cdot (\Delta x)^2 + (\Delta x)^3 - 2 \cdot \Delta x \end{aligned}$$

Third step. $\frac{\Delta y}{\Delta x} = 3x^2 + 3x \cdot \Delta x + (\Delta x)^2 - 2$.

Fourth step. $\frac{dy}{dx} = 3x^2 - 2$. Or,

$$\frac{d}{dx}(x^3 - 2x + 7) = 3x^2 - 2.$$

Example 3.7.3. Differentiate $\frac{c}{x^2}$.

Solution. Place $y = \frac{c}{x^2}$.

First step. $y + \Delta y = \frac{c}{(x+\Delta x)^2}$.

Second step.

$$\begin{aligned} y + \Delta y &= \tfrac{c}{(x+\Delta x)^2} \\ y &= \tfrac{c}{x^2} \\ \Delta y &= \tfrac{c}{(x+\Delta x)^2} - \tfrac{c}{x^2} = \tfrac{-c \cdot \Delta x(2x+\Delta x)}{x^2(x+\Delta x)^2}. \end{aligned}$$

Third step. $\frac{\Delta y}{\Delta x} = -c \cdot \frac{2x+\Delta x}{x^2(x+\Delta x)^2}$.

Fourth step. $\frac{dy}{dx} = -c \cdot \frac{2x}{x^2(x)^2} = -\frac{2c}{x^3}$. Or, $\frac{d}{dx}\left(\frac{c}{x^2}\right) = \frac{-2c}{x^3}$.

Example 3.7.4. ("Cubic splines") Differentiate $f(x)$, where

$$f(x) = \begin{cases} -2x^3 + 3x^2, & 0 < x < 1, \\ 0, & x \leq 0 \text{ or } x \geq 1. \end{cases}$$

(The polynomial $-2x^3 + 3x^2$ smoothly connects the line $y = 0$ for $x < 0$ to the line $y = 1$ for $x > 1$. Such "cubic splines" are used in industry to design roads, buildings, car bodies, ship hulls, and so on.)

The function is given in parts, so the problem must be solved case-by-case. First, assume $0 < x < 1$.

$0 < x < 1$: In this case, the derivative can be computed ass in the examples above to show

$$f'(x) = -6x^2 + 6x, \qquad 0 < x < 1.$$

This is not the final answer though! You must also deal with the cases $x > 1$, $x < 0$ and $x = 0, 1$ (as limits).

$x > 1$ or $x < 0$: Here $f'(x) = 0$.

$x = 0$: Note that for "small" h (by which we really mean $|h| < 1$),

$$\frac{f(0+h) - f(0)}{h} = \begin{cases} -2h^2 + 3h, & 0 < h < 1, \\ 0, & h \leq 0. \end{cases}$$

Taking the limit as $h \to 0$ gives

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = 0.$$

Is $f'(x)$ continuous at $x = 0$? Note

$$\lim_{x \to 0-} f'(x) = 0$$

and

$$\lim_{x \to 0+} f'(x) = \lim_{x \to 0+} -6x^2 + 6x = 0.$$

Therefore, the slope of the graph $y = f(x)$ is zero as you approach 0 form the left or from the right. This tells us $f'(x)$ is continuously differentiable at both ends.

$x = 1$: This is similar to the case $x = 0$ and left to the reader.

3.8 Exercises

Use the General Rule, §3.7 in differentiating the following functions:

1. $y = 3x^2$

   Ans: $\frac{dy}{dx} = 6x$

2. $y = x^2 + 2$

   Ans: $\frac{dy}{dx} = 2x$

3. $y = 5 - 4x$

   Ans: $\frac{dy}{dx} = -4$

4. $s = 2t^2 - 4$

   Ans: $\frac{ds}{dt} = 4t$

5. $y = \frac{1}{x}$

   Ans:$\frac{dy}{dx} = -\frac{1}{x^2}$

6. $y = \frac{x+2}{x}$

   Ans: $\frac{dy}{dx} = -\frac{-2}{x^2}$

7. $y = x^3$

   Ans: $\frac{dy}{dx} = 3x^2$

8. $y = 2x^2 - 3$

   Ans: $\frac{dy}{dx} = 4x$

9. $y = 1 - 2x^3$

   Ans: $\frac{dy}{dx} = -6x^2$

10. $\rho = a\theta^2$

    Ans: $\frac{d\rho}{d\theta} = 2a\theta$

11. $y = \frac{2}{x^2}$

    Ans: $\frac{dy}{dx} = -\frac{4}{x^3}$

12. $y = \frac{3}{x^2-1}$

    Ans: $\frac{dy}{dx} = -\frac{6x}{(x^2-1)^2}$

    Here's how to use Sage to verify this:

    Sage

```
sage: y = 3/(x^2-1)
sage: diff(y,x)
-6*x/(x^4 - 2*x^2 + 1)
```

13. $y = 7x^2 + x$
14. $s = at^2 - 2bt$
15. $r = 8t + 3t^2$
16. $y = \frac{3}{x^2}$
17. $s = -\frac{a}{2t+3}$
18. $y = bx^3 - cx$
19. $\rho = 3\theta^3 - 2\theta^2$
20. $y = \frac{3}{4}x^2 - \frac{1}{2}x$
21. $y = \frac{x^2-5}{x}$
22. $\rho = \frac{\theta^2}{1+\theta}$
23. $y = \frac{1}{2}x^2 + 2x$
24. $z = 4x - 3x^2$
25. $\rho = 3\theta + \theta^2$
26. $y = \frac{ax+b}{x^2}$
27. $z = \frac{x^3+2}{x}$

28. $y = x^2 - 3x + 6$

    Ans: $y' = 2x - 3$

29. $s = 2t^2 + 5t - 8$

    Ans: $s' = 4t + 5$ Here's how to use Sage to verify this (for simplicity, we set $h = \Delta t$):

Sage

```
sage: h = var("h")
sage: t = var("t")
sage: s(t) = 2*t^2 + 5*t - 8
sage: Deltas = s(t+h)-s(t)
sage: (Deltas/h).expand()
4*t + 2*h + 5
sage: limit((s(t+h)-s(t))/h,h=0)
4*t + 5
sage: diff(s(t),t)
4*t + 5
```

30. $\rho = 5\theta^3 - 2\theta + 6$

    Ans: $\rho' = 15\theta^2 - 2$

31. $y = ax^2 + bx + c$

    Ans: $y' = 2ax + b$

3.9 Applications of the derivative to geometry

We consider a theorem which is fundamental in all differential calculus to geometry.

Let

$$y = f(x) \tag{3.5}$$

be the equation of a curve AB.

Now differentiate (3.5) by the General Rule and interpret each step geometrically.

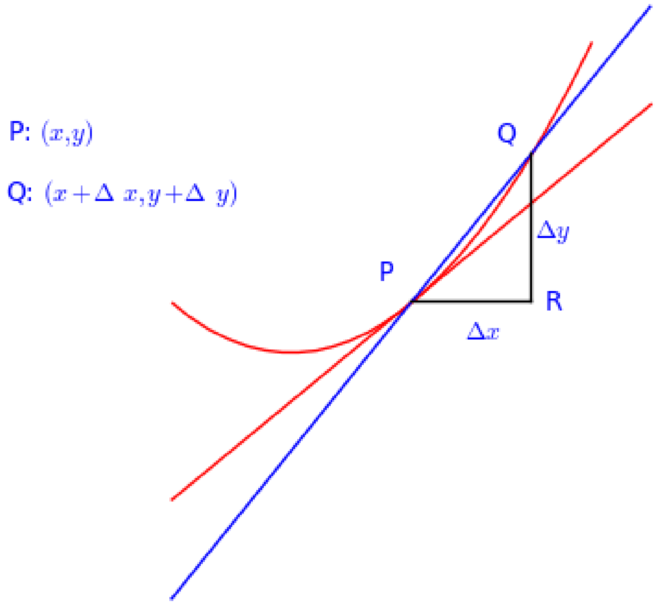

Figure 3.1: The geometry of derivatives.

- FIRST STEP. $y + \Delta y = f(x + \Delta x) = NQ$
- SECOND STEP.

$$\begin{aligned} y + \Delta y &= f(x + \Delta x) = NQ \\ y &= f(x) = MP = NR \\ \Delta y &= f(x + \Delta x) - f(x) = RQ. \end{aligned}$$

- THIRD STEP.

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{RQ}{MN} = \frac{RQ}{PR} \\ &= \tan RPQ = \tan \phi \\ &= \text{slope of secant line } PQ. \end{aligned}$$

- FOURTH STEP.

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \\ &= \frac{dy}{dx} = \text{value of the derivative at } P. \end{aligned}$$

But when we let $\Delta x \to 0$, the point Q will move along the curve and approach nearer and nearer to P, the secant will turn about P and approach the tangent as a limiting position, and we have also

$$\begin{aligned} \lim_{\Delta x \to 0} \tfrac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \tan \phi = \tan \tau \\ &= \text{slope of the tangent at } P. \end{aligned}$$

Hence , $\frac{dy}{dx}$ = slope of the tangent line PT. Therefore

Theorem 3.9.1. *The value of the derivative at any point of a curve is equal to the slope of the line drawn tangent to the curve at that point.*

It was this tangent problem that led Leibnitz[6] to the discovery of the Differential Calculus.

Example 3.9.1. Find the slopes of the tangents to the parabola $y = x^2$ at the vertex, and at the point where $x = \frac{1}{2}$.

Solution. Differentiating by General Rule, (§3.7), we get

$$y' = \frac{dy}{dx} = 2x = \text{slope of tangent line at any point on curve.}$$

To find slope of tangent at vertex, substitute $x = 0$ in $y' = 2x$, giving

$$\frac{dy}{dx} = 0.$$

Therefore the tangent at vertex has the slope zero; that is, it is parallel to the axis of x and in this case coincides with it.

To find slope of tangent at the point P, where $x = \frac{1}{2}$, substitute in $y' = 2x$, giving

$$\frac{dy}{dx} = 1;$$

that is, the tangent at the point P makes an angle of 45^o with the axis of x.

[6]Gottfried Wilhelm Leibnitz (1646-1716) was a native of Leipzig. His remarkable abilities were shown by original investigations in several branches of learning. He was first to publish his discoveries in Calculus in a short essay appearing in the periodical Acta Eruditorum at Leipzig in 1684. It is known, however, that manuscripts on Fluxions written by Newton were already in existence, and from these some claim Leibnitz got the new ideas. The decision of modern times seems to be that both Newton and Leibnitz invented the Calculus independently of each other. The notation used today was introduced by Leibnitz. See frontispiece.

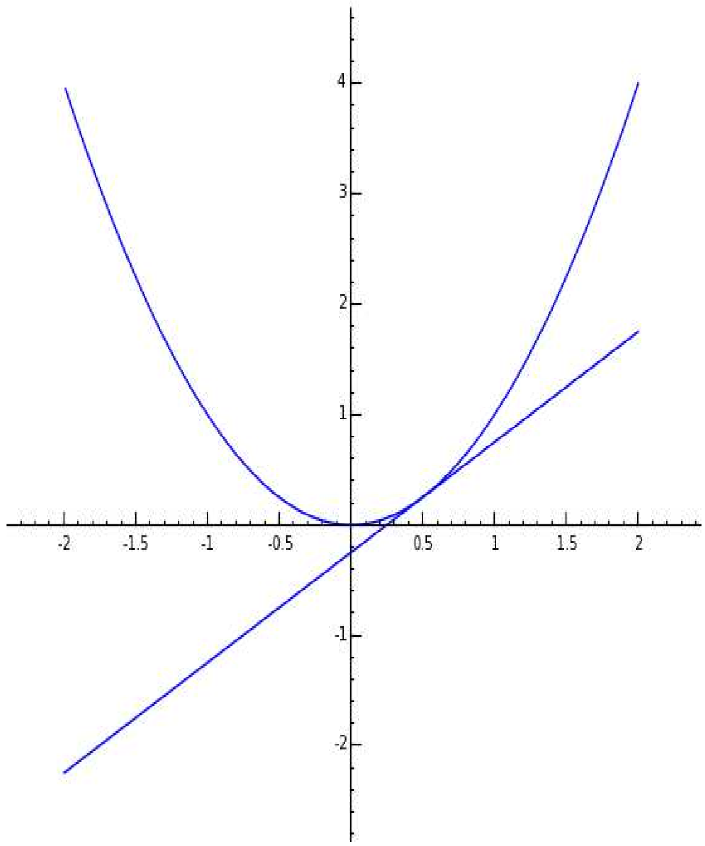

Figure 3.2: The geometry of the derivative of $y = x^2$.

3.10 Exercises

Find by differentiation the slopes of the tangents to the following curves at the points indicated. Verify each result by drawing the curve and its tangent.

1. $y = x^2 - 4$, where $x = 2$. (Ans. 4.)

2. $y = 6 - 3x^2$ where $x = 1$. (Ans. -6.)

3. $y = x^3$, where $x = -1$. (Ans. -3.)

4. $y = \frac{2}{x}$, where $x = -1$. (Ans. $-\frac{1}{2}$.)

5. $y = x - x^2$, where $x = 0$. (Ans. 1.)

6. $y = \frac{1}{x-1}$, where $x = 3$. (Ans. $-\frac{1}{4}$.)

7. $y = \frac{1}{2}x^2$, where $x = 4$. (Ans. 4.)

8. $y = x^2 - 2x + 3$, where $x = 1$. (Ans. 0.)

9. $y = 9 - x^2$, where $x = -3$. (Ans. 6.)

10. Find the slope of the tangent to the curve $y = 2x^3 - 6x + 5$, (a) at the point where $x = 1$; (b) at the point where $x = 0$.

    (Ans. (a) 0; (b) -6.)

11. (a) Find the slopes of the tangents to the two curves $y = 3x^2 - 1$ and $y = 2x^2 + 3$ at their points of intersection. (b) At what angle do they intersect?

    (Ans. (a) ± 12, ± 8; (b) $\arctan \frac{4}{97}$.)

    Here's how to use Sage to verify these:

    Sage

```
sage: solve(3*x^2 - 1 == 2*x^2 + 3,x)
[x == -2, x == 2]
sage: g(x) = diff(3*x^2 - 1,x)
sage: h(x) = diff(2*x^2 + 3,x)
sage: g(2); g(-2)
12
-12
sage: h(2); h(-2)
8
-8
sage: atan(12)-atan(8)
atan(12) - atan(8)
sage: atan(12.0)-atan(8.0)
0.0412137626583202
sage: RR(atan(4/97))
0.0412137626583202
```

12. The curves on a railway track are often made parabolic in form. Suppose that a track has the form of the parabola $y = x^2$ (see Figure 3.2 in §3.9), the directions of the positive x-axis and positive y-axis being east and north respectively, and the unit of measurement 1 mile. If the train is going east when passing through the origin, in what direction will it be going

    (a) when $\frac{1}{2}$ mi. east of the y-axis?
    (Ans. Northeast.)

    (b) when $\frac{1}{2}$ mi. west of the y-axis?
    (Ans. Southeast.)

    (c) when $\frac{\sqrt{3}}{2}$ mi. east of the y-axis?
    (Ans. N. 30^oE.)

    (d) when $\frac{1}{12}$ mi. north of the x-axis?
    (Ans. E. 30^oS., or E. 30^oN.)

13. A street-car track has the form of the cubic $y = x^3$. Assume the same directions and unit as in the last example. If a car is going west when passing through the origin, in what direction will it be going

    (a) when $\frac{1}{\sqrt{3}}$ mi. east of the y-axis? (Ans. Southwest.)

    (b) when $\frac{1}{\sqrt{3}}$ mi. west of the y-axis? (Ans. Southwest.)

    (c) when $\frac{1}{2}$ mi. north of the x-axis? (Ans. S. 27^o $43'$ W.)

    (d) when 2 mi. south of the x-axis?

    (e) when equidistant from the x-axis and the y-axis?

CHAPTER

FOUR

Rules for differentiating standard elementary forms

4.1 Importance of General Rule

The General Rule for Differentiation, given in §3.7 of the last chapter, is fundamental, being a step-by-step procedural implementation of the very definition of a derivative. It should be stressed that the student should be thoroughly familiar with this procedure. However, the process of applying the rule to examples in general is often either too tedious or too difficult. Consequently, special rules have been derived from the General Rule for differentiating certain standard forms of frequently occurring expressionss in order to facilitate process.

It's convenient to express these special rules by means of formulas, a list of which follows. The student should not only memorize each formula when deduced, but should be able to state the corresponding rule in words. (The extra time it takes you to memorize the formulas will probably be repaid in the time saved doing homework and exam problems correctly.) In these formulas u, v, and w denote differentiable functions of x.

Formulas for differentiation

$$\frac{dc}{dx} = 0 \tag{4.1}$$

$$\frac{dx}{dx} = 1 \tag{4.2}$$

$$\frac{d}{dx}(u+v-w) = \frac{du}{dx} \ + \ \frac{dv}{dx} \ - \ \frac{dw}{dx} \tag{4.3}$$

$$\frac{d}{dx}(cv) = c\frac{dv}{dx} \tag{4.4}$$

Product rule:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} \ + \ v\frac{du}{dx} \tag{4.5}$$

Power rule:

$$\frac{d}{dx}\left(v^n\right) = nv^{n-1}\frac{dv}{dx} \tag{4.6}$$

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \tag{4.7}$$

Quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} \ - \ u\frac{dv}{dx}}{v^2} \tag{4.8}$$

$$\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{\frac{du}{dx}}{c} \tag{4.9}$$

$$\frac{d}{dx}\left(\log_a v\right) = \log_a \ e \ \cdot \ \frac{\frac{dv}{dx}}{v} \tag{4.10}$$

$$\frac{d}{dx}\left(\ln v\right) = \frac{\frac{dv}{dx}}{v} \tag{4.11}$$

Note: Often $\log_e$, $e = 2.71828...$ the base of the natural log, is denoted $\ln$ (or sometimes just $\log$).

$$\frac{d}{dx}\left(a^v\right) = a^v \ln a \cdot \frac{dv}{dx} \tag{4.12}$$

$$\frac{d}{dx}\left(e^v\right) = e^v\frac{dv}{dx} \tag{4.13}$$

$$\frac{d}{dx}\left(u^{v}\right) = vu^{v-1}\frac{du}{dx} \ + \ \log \ u \ \cdot \ u^{v}\frac{dv}{dx} \tag{4.14}$$

$$\frac{d}{dx}(\sin \ v) = \cos \ v\frac{dv}{dx} \tag{4.15}$$

$$\frac{d}{dx}(\cos \ v) = -\sin \ v\frac{dv}{dx} \tag{4.16}$$

$$\frac{d}{dx}(\cos \ v) = -\sin \ v\frac{dv}{dx} \tag{4.17}$$

$$\frac{d}{dx}(\cot \ x) = -\csc^{2} v\frac{dv}{dx} \tag{4.18}$$

$$\frac{d}{dx}(\sec \ v) = \sec \ v \ \tan \ v\frac{dv}{dx} \tag{4.19}$$

$$\frac{d}{dx}(\csc \ v) = -\csc \ v \ \cot \ v\frac{dv}{dx} \tag{4.20}$$

$$\frac{d}{dx}(\arcsin \ v) = \frac{\frac{dv}{dx}}{\sqrt{1-v^{2}}} \tag{4.21}$$

$$\frac{d}{dx}(\arccos \ v) = -\frac{\frac{dv}{dx}}{\sqrt{1-v^{2}}} \tag{4.22}$$

$$\frac{d}{dx}(\arctan \ v) = \frac{\frac{dv}{dx}}{1+v^{2}} \tag{4.23}$$

$$\frac{d}{dx}(\text{arccot}\, v) = -\frac{\frac{dv}{dx}}{1+v^{2}} \tag{4.24}$$

Note: Sometimes $\arcsin$, $\arccos$, and so on, are denoted asin, acos, and so on.

$$\frac{d}{dx}(\text{arcsec}\, v) = \frac{\frac{dv}{dx}}{v\sqrt{v^{2}-1}} \tag{4.25}$$

$$\frac{d}{dx}(\text{arccsc}\, v) = -\frac{\frac{dv}{dx}}{v\sqrt{v^{2}-1}} \tag{4.26}$$

Chain rule:

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}, \tag{4.27}$$

where y is a function of v, v a function of x.

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \tag{4.28}$$

where y is a function of x.

Here's how to see some of these using Sage :

Sage

```
sage: t = var("t")
sage: diff(acos(t),t)
-1/sqrt(1 - t^2)
sage: v = var("v")
sage: diff(acsc(v),v)
-1/(sqrt(1 - 1/v^2)*v^2)
sage: x = var("x")
sage: u = function("u",x)
sage: v = function("v",x)
sage: diff(u(x)*v(x),x)
u(x)*diff(v(x), x, 1) + v(x)*diff(u(x), x, 1)
```

These tell us that $\frac{d\arccos t}{dt} = -\frac{1}{\sqrt{1-t^2}}$ and $\frac{d\mathrm{arccsc}\, v}{dv} = -\frac{1}{v\sqrt{v^2-1}}$.

Here are some more examples using Sage :

Sage

```
sage: x = var("x")
sage: u = function('u', x)
sage: v = function('v', x)
sage: diff(u/v,x)
diff(u(x), x, 1)/v(x) - u(x)*diff(v(x), x, 1)/v(x)^2
sage: diff(sin(v),x)
cos(v(x))*diff(v(x), x, 1)
sage: diff(arcsin(v),x)
diff(v(x), x, 1)/sqrt(1 - v(x)^2)
```

The last Sage computation verifies that $\frac{d}{dx}(\arcsin\ v) = \frac{\frac{dv}{dx}}{\sqrt{1-v^2}}$.

4.2 Differentiation of a constant

The simplest type of function is one that is known to have the same value for all values of the independent variable, i.e., a constant function. Let

$$y = c$$

denote a constant function. As x takes on an increment Δx, the function does not change in value, that is, $\Delta y = 0$, and so

$$\frac{\Delta y}{\Delta x} = 0.$$

But

$$\lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \right) = \frac{dy}{dx} = 0.$$

Therefore, $\frac{dc}{dx} = 0$ (equation (4.1) above). *The derivative of a constant is zero.*

4.3 Differentiation of a variable with respect to itself

Let $y = x$.

Following the General Rule, §3.7, we have

- FIRST STEP. $y + \Delta y = x + \Delta x$.
- SECOND STEP. $\Delta y = \Delta x$
- THIRD STEP. $\frac{\Delta y}{\Delta x} = 1$.
- FOURTH STEP. $\frac{dy}{dx} = 1$.

Therefore, $\frac{dy}{dx} = 1$ (equation (4.2) above). The derivative of a variable with respect to itself is unity.

4.4 Differentiation of a sum

Let $y = u + v - w$. By the General Rule,

- FIRST STEP. $y + \Delta y = u + \Delta u + v + \Delta v - w - \Delta w$.
- SECOND STEP. $\Delta y = \Delta u + \Delta v - \Delta w$.
- THIRD STEP. $\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x} - \frac{\Delta w}{\Delta x}$.
- FOURTH STEP. $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$. [Applying Theorem 2.8.1]

Therefore, $\frac{d}{dx}(u + v - w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$ (equation (4.3) above). Similarly, for the algebraic sum of any finite number of functions.

The derivative of the algebraic sum of a finite number of functions is equal to the same algebraic sum of their derivatives.

4.5 Differentiation of the product of a constant and a function

Let $y = cv$. By the General Rule,

- FIRST STEP. $y + \Delta y = c(v + \Delta v) = cv + c\Delta v$.
- SECOND STEP. $\Delta y = c \cdot \Delta v$.
- THIRD STEP. $\frac{\Delta y}{\Delta x} = c\frac{\Delta v}{\Delta x}$.
- FOURTH STEP. $\frac{dy}{dx} = c\frac{dv}{dx}$. [Applying Theorem 2.8.2]

Therefore, $\frac{d}{dx}(cv) = c\frac{dv}{dx}$ (equation (4.4) above).

The derivative of the product of a constant and a function is equal to the product of the constant and the derivative of the function.

4.6 Differentiation of the product of two functions

Let $y = uv$. By the General Rule,

- FIRST STEP. $y + \Delta y = (u + \Delta u)(v + \Delta v)$. Multiplying out this becomes

$$y + \Delta y = uv + u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v.$$

- SECOND STEP. $\Delta y = u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v$.
- THIRD STEP. $\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$.
- FOURTH STEP. $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$. [Applying Theorem 2.8.1], since when $\Delta x \to 0$, $\Delta u \to 0$, and $\left(\Delta u\frac{\Delta v}{\Delta x}\right) \to 0$.]

Therefore, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ (equation (4.5) above).

Product rule: The derivative of the product of two functions is equal to the first function times the derivative of the second, plus the second function times the derivative of the first.

Here's how to use Sage to verify this rule in a special case:

Sage

```
sage: t = var("t")
sage: f = cos(t)
sage: g =  exp(2*t)
sage: diff(f*g,t)
2*e^(2*t)*cos(t) - e^(2*t)*sin(t)
sage: diff(f,t)*g+f*diff(g,t)
2*e^(2*t)*cos(t) - e^(2*t)*sin(t)
```

This simply computes $\frac{d}{dt}(e^{2t}\cos(t)$ in two ways (one: directly, the second: using the product rule) and checks that they are the same.

4.7 Differentiation of the product of any finite number of functions

Now in dividing both sides of equation (4.5) by uv, this formula assumes the form

$$\frac{\frac{d}{dx}(uv)}{uv} = \frac{\frac{du}{dx}}{u} + \frac{\frac{dv}{dx}}{v}.$$

If then we have the product of n functions $y = v_1 v_2 \cdots v_n$, we may write

$$\begin{aligned}
\frac{\frac{d}{dx}(v_1 v_2 \cdots v_n)}{v_1 v_2 \cdots v_n} &= \frac{\frac{dv_1}{dx}}{v_1} + \frac{\frac{d}{dx}(v_2 v_3 \cdots v_n)}{v_2 v_3 \cdots v_n} \\
&= \frac{\frac{dv_1}{dx}}{v_1} + \frac{\frac{dv_2}{dx}}{v_2} + \frac{\frac{d}{dx}(v_3 v_4 \cdots v_n)}{v_3 v_4 \cdots v_n} \\
&= \frac{\frac{dv_1}{dx}}{v_1} + \frac{\frac{dv_2}{dx}}{v_2} + \frac{\frac{dv_3}{dx}}{v_3} + \cdots + \frac{\frac{dv_n}{dx}}{v_n} \frac{d}{dx}(v_1 v_2 \cdots v_n) \\
&= (v_2 v_3 \cdots v_n)\frac{dv_1}{dx} + (v_1 v_3 \cdots v_n)\frac{dv_2}{dx} + \cdots + (v_1 v_2 \cdots v_{n-1})\frac{dv_n}{dx}.
\end{aligned}$$

The derivative of the product of a finite number of functions is equal to the sum of all the products that can be formed by multiplying the derivative of each function by all the other functions.

4.8 Differentiation of a function with a constant exponent

If the n factors in the above result are each equal to v, we get

$$\frac{\frac{d}{dx}(v^n)}{v^n} = n\frac{\frac{dv}{dx}}{v}.$$

Therefore, $\frac{d}{dx}(v^n) = nv^{n-1}\frac{dv}{dx}$, (equation (4.6) above).

When $v = x$ this becomes $\frac{d}{dx}(x^n) = nx^{n-1}$ (equation (4.7) above).

We have so far proven equation (4.6) only for the case when n is a positive integer. In §4.15, however, it will be shown that this formula holds true for any value of n, and we shall make use of this general result now.

The derivative of a function with a constant exponent is equal to the product of the exponent, the function with the exponent diminished by unity, and the derivative of the function.

Sage

```
sage: t = var('t')
sage: f = function('f', t)
sage: g = function('g', t)
sage: (f(t)*g(t)).diff(t)       # product rule for 2 functions
f(t)*diff(g(t), t, 1) + g(t)*diff(f(t), t, 1)
sage: h = function('h', t)
sage: (f(t)*g(t)*h(t)).diff(t)  # product rule for 3 functions
f(t)*g(t)*diff(h(t), t, 1) + f(t)*h(t)*diff(g(t), t, 1)
```

```
+ g(t)*h(t)*diff(f(t), t, 1)
```

4.9 Differentiation of a quotient

Let $y = \frac{u}{v} v \neq 0$. By the General Rule,

- FIRST STEP. $y + \Delta y = \frac{u+\Delta u}{v+\Delta v}$.
- SECOND STEP. $\Delta y = \frac{u+\Delta u}{v\Delta v} - \frac{u}{v} = \frac{v\cdot\Delta u - u\cdot\Delta v}{v(v+\Delta v)}$.
- THIRD STEP. $\frac{\Delta y}{\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{v(v+\Delta v)}$.
- FOURTH STEP. $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$. [Applying Theorems 2.8.2 and 2.8.3]

Therefore, $\frac{d}{dx}\left(\frac{u}{v}\right) \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (equation (4.8) above).

The derivative of a fraction is equal to the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Sage

```
sage: t = var('t')
sage: f = function('f', t)
sage: g = function('g', t)
sage: (f(t)/g(t)).diff(t)
diff(f(t), t, 1)/g(t) - f(t)*diff(g(t), t, 1)/g(t)^2
sage: (1/f(t)).diff(t)
-diff(f(t), t, 1)/f(t)^2
```

When the denominator is constant, set $v = c$ in (4.8), giving (4.9) $\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{\frac{du}{dx}}{c}$. [Since $\frac{dv}{dx} = \frac{dc}{dx} = 0$.] We may also get (4.9) from (4.4) as follows:

$$\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{1}{c}\frac{du}{dx} = \frac{\frac{du}{dx}}{c}.$$

The derivative of the quotient of a function by a constant is equal to the derivative of the function divided by the constant.

All explicit algebraic functions of one independent variable may be differentiated by following the rules we have deduced so far.

4.10 Examples

Differentiate the following[1]:

1. $y = x^3$.

   Solution. $\frac{dy}{dx} = \frac{d}{dx}(x^3) = 3x^2$. (By (4.7), n = 3.)

2. $y = ax^4 - bx^2$.

   Solution.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(ax^4 - bx^2) \\ &= \frac{d}{dx}(ax^4) - \frac{d}{dx}(bx^2) \quad \text{by } (4.3) \\ &= a\frac{d}{dx}(x^4) - b\frac{d}{dx}(x^2) \quad \text{by } (4.4) \\ &= 4ax^3 - 2bx \quad \text{by } (4.7). \end{aligned}$$

3. $y = x^{\frac{4}{3}} + 5$.

   Solution.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^{\frac{4}{3}}) + \frac{d}{dx}(5) \quad \text{by } (4.3) \\ &= \frac{4}{3}x^{\frac{1}{3}} \quad \text{by } (4.7) \text{ and } (4.1) \end{aligned}$$

4. $y = \frac{3x^3}{\sqrt[5]{x^2}} - \frac{7x}{\sqrt[3]{x^4}} + 8\sqrt[7]{x^3}$

   Solution.

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(3x^{\frac{13}{5}}\right) + \frac{d}{dx}\left(7x^{-\frac{1}{3}}\right) + \frac{d}{dx}\left(8x^{\frac{3}{7}}\right) \quad \text{by } (4.3) \\ &= \frac{39}{5}x^{\frac{8}{5}} + \frac{7}{3}x^{-\frac{4}{3}} + \frac{24}{7}x^{-\frac{4}{7}} \quad \text{by } (4.4) \text{ and } (4.7). \end{aligned}$$

[1] To the student: Though the answers are included below for all of the problems, it may be that your computation differs from the solution given. You should then try to show algebraically that your form is that same as that given.

5. $y = (x^2 - 3)^5$.

   Solution.

$$\begin{aligned} \tfrac{dy}{dx} &= 5(x^2-3)^4 \tfrac{d}{dx}(x^2-3) \quad \text{by (4.6)}, \mathrm{v = x^2 - 3 \ and \ n = 5} \\ 5(x^2-3)^4 \cdot 2x &= 10x(x^2-3)^4. \end{aligned}$$

   We might have expanded this function by the Binomial Theorem (see §12.1) and then applied (4.3), etc., but the above process is much simpler.

6. $y = \sqrt{a^2 - x^2}$.

   Solution.

$$\begin{aligned} \tfrac{dy}{dx} &= \tfrac{d}{dx}(a^2-x^2)^{\frac{1}{2}} \\ &= \tfrac{1}{2}(a^2-x^2)^{-\frac{1}{2}} \tfrac{d}{dx}(a^2-x^2), \quad \text{by (4.6)} \ (v = a^2 - x^2, \text{ and } n = 5) \\ &= \tfrac{1}{2}(a^2-x^2)^{-\frac{1}{2}}(-2x) = -\tfrac{x}{\sqrt{a^2-x^2}}. \end{aligned}$$

7. $y = (3x^2 + 2)\sqrt{1 + 5x^2}$.

   Solution.

$$\begin{aligned} \tfrac{dy}{dx} &= (3x^2+2)\tfrac{d}{dx}(1+5x^2)^{\frac{1}{2}} + (1+5x^2)^{\frac{1}{2}} \tfrac{d}{dx}(3x^2+2) \\ &\quad (\text{by (4.5)}, u = 3x^2+2, \text{ and } v = (1+5x^2)^{\frac{1}{2}}) \\ &= (3x^2+2)\tfrac{1}{2}(1+5x^2)^{-\frac{1}{2}} \tfrac{d}{dx}(1+5x^2) + (1+5x^2)^{\frac{1}{2}} 6x \quad \text{by (4.6), etc.} \\ &= (3x^2+2)(1+5x^2)^{-\frac{1}{2}} 5x + 6x(1+5x^2)^{\frac{1}{2}} \\ &= \tfrac{5x(3x^2+2)}{\sqrt{1+5x^2}} + 6x\sqrt{1+5x^2} \\ &= \tfrac{45x^3+16x}{\sqrt{1+5x^2}}. \end{aligned}$$

8. $y = \frac{a^2+x^2}{\sqrt{a^2-x^2}}$.

   Solution. By (4.8), we have

$$\begin{aligned} \tfrac{dy}{dx} &= \tfrac{(a^2-x^2)^{\frac{1}{2}} \frac{d}{dx}(a^2-x^2) - (a^2+x^2)\frac{d}{dx}(a^2-x^2)^{\frac{1}{2}}}{a^2-x^2} \\ &= \tfrac{2x(a^2-x^2)+x(a^2+x^2)}{(a^2-x^2)^{-\frac{3}{2}}} \\ &\quad (\text{multiplying both numerator and denominator by } (a^2-x^2)^{\frac{1}{2}}) \\ &= \tfrac{\frac{3}{x}^2 x - x^3}{(a^2-x^2)^{\frac{3}{2}}}. \end{aligned}$$

9. $5x^4+3x^2-6$. (Ans. $\frac{dy}{dx}=20x^3+6x$)

10. $y=3cx^2-8dx+5e$. (Ans. $\frac{dy}{dx}=6cx-8d$)

11. $y=x^{a+b}$. (Ans. $\frac{dy}{dx}=(a+b)x^{a+b-1}$)

12. $y=x^n+nx+n$. (Ans. $\frac{dy}{dx}=nx^{n-1}+n$)

13. $f(x)=\frac{2}{3}x^3-\frac{3}{2}x^2+5$. (Ans. $f'(x)=2x^2-3x$)

14. $f(x)=(a+b)x^2+cx+d$. (Ans. $f'(x)=2(a+b)x+c$)

15. $\frac{d}{dx}(a+bx+cx^2)=b+2cx$.

16. $\frac{d}{dy}(5y^m-3y+6)=5my^{m-1}-3$.

17. $\frac{d}{dx}(2x^{-2}+3x^{-3})=-4x^{-3}-9x^{-4}$.

18. $\frac{d}{ds}(3s^{-4}-s)=-12s^{-5}-1$.

19. $\frac{d}{dx}(4x^{\frac{1}{2}}+x^2)=2x^{-\frac{1}{2}}+2x$.

20. $\frac{d}{dy}(y^{-2}-4y^{-\frac{1}{2}})=-2y^{-3}+2y^{-\frac{3}{2}}$.

21. $\frac{d}{dx}(2x^3+5)=6x^2$.

22. $\frac{d}{dt}(3t^5-2t^2)=15t^4-4t$.

23. $\frac{d}{d\theta}(a\theta^4+b\theta)=4a\theta^3+b$.

24. $\frac{d}{d\alpha}(5-2\alpha^{\frac{3}{2}})=-3\alpha^{\frac{1}{2}}$.

25. $\frac{d}{dt}(9t^{\frac{5}{3}}+t^{-1})=15t^{\frac{2}{3}}-t^{-2}$.

26. $\frac{d}{dx}(2x^{12}-x^9)=24x^{11}-9x^8$.

27. $r=c\theta^3+d\theta^2+e\theta$. (Ans. $r'=3c\theta^2+2d\theta+e$)

28. $y=6x^{\frac{7}{2}}+4x^{\frac{5}{2}}+2x^{\frac{3}{2}}$. (Ans. $y'=21x^{\frac{5}{2}}+10x^{\frac{3}{2}}+3x^{\frac{1}{2}}$)

29. $y=\sqrt{3x}+\sqrt{3}x+\frac{1}{x}$. (Ans. $y'=\frac{3}{2\sqrt{3x}}+\frac{1}{3\sqrt[3]{x^2}}-\frac{1}{x^2}$)

30. $y=\frac{a+bx+cx^2}{x}$. (Ans. $y'=c-\frac{a}{x^2}$)

31. $y = \frac{(x-1)^3}{x^{\frac{1}{3}}}$. (Ans. $y' = \frac{8}{3}x^{\frac{5}{3}} - 5x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} + \frac{1}{3}x^{-\frac{4}{3}}$)

32. $y = (2x3 + x^2 - 5)^3$. (Ans. $y' = 6x(3x+1)(2x^3 + x^2 - 5)^2$)

33. $y = (2x^3 + x^2 - 5)^3$. (Ans. $y' = 6x(3x+1)(2x^3 + x^2 - 5)^2$)

34. $f(x) = (a + bx^2)^{\frac{5}{4}}$. (Ans. $f'(x) = \frac{5bx}{2}(a + bx^2)^{\frac{1}{4}}$)

35. $f(x) = (1 + 4x^3)(1 + 2x^2)$. (Ans. $f'(x) = 4x(1 + 3x + 10x^3)$)

36. $f(x) = (a + x)\sqrt{a - x}$. (Ans. $f'(x) = \frac{a-3x}{2\sqrt{a-x}}$)

37. $f(x) = (a+x)^m(b+x)^n$. (Ans. $f'(x) = (a+x)^m(b+x)^n\left[\frac{m}{a+x} + \frac{n}{b+x}\right]$)

38. $y = \frac{1}{x^n}$. (Ans. $\frac{y}{x} = -\frac{n}{x^{n+1}}$)

39. $y = x(a^2 + x^2)\sqrt{a^2 - x^2}$. (Ans. $\frac{dy}{dx} = \frac{a^4+a^2x^2-4x^4}{\sqrt{a^2-x^2}}$)

40. Differentiate the following functions:

$(a)\ \frac{d}{dx}(2x^3 - 4x + 6)$	$(e)\ \frac{d}{dt}(b + at^2)^{\frac{1}{2}}$	$(i)\ \frac{d}{dx}(x^{\frac{2}{3}} - a^{\frac{2}{3}})$
$(b)\ \frac{d}{dt}(at^7 + bt^5 - 9)$	$(f)\ \frac{d}{dx}(x^2 - a^2)^{\frac{3}{2}}$	$(j)\ \frac{d}{dt}(5 + 2t)^{\frac{9}{2}}$
$(c)\ \frac{d}{d\theta}(3\theta^{\frac{3}{2}} - 2\theta^{\frac{1}{2}} + 6\theta)$	$(g)\ \frac{d}{d\phi}(4 - \phi^{\frac{2}{5}})$	$(k)\ \frac{d}{ds}\sqrt{a + b\sqrt{s}}$
$(d)\ \frac{d}{dx}(2x^3 + x)^{\frac{5}{3}}$	$(h)\ \frac{d}{dt}\sqrt{1 + 9t^2}$	$(l)\ \frac{d}{dx}(2x^{\frac{1}{3}} + 2x^{\frac{5}{3}})$

41. $y = \frac{2x^4}{b^2-x^2}$. (Ans. $\frac{dy}{dx} = \frac{8b^2x^3-4x^5}{(b^2-x^2)^2}$)

42. $y = \frac{a-x}{a+x}$. (Ans. $\frac{dy}{dx} = -\frac{2a}{(a+x)^2}$)

43. $s = \frac{t^3}{(1+t)^2}$. (Ans. $\frac{ds}{dt} = \frac{3t^2+t^3}{(1+t)^3}$)

44. $f(s) = \frac{(s+4)^2}{s+3}$. (Ans. $f'(s) = \frac{(s+2)(s+4)}{(s+3)^2}$)

45. $f(\theta) = \frac{\theta}{\sqrt{a-b\theta^2}}$. (Ans. $f'(\theta) = \frac{a}{(a-b\theta^2)^{\frac{3}{2}}}$)

46. $F(r) = \sqrt{\frac{1+r}{1-r}}$. (Ans. $F'(r) = \sqrt{1}(1-r)\sqrt{1-r^2}$)

47. $\psi(y) = \left(\frac{y}{1-y}\right)^m$. (Ans. $\psi'(y) = \frac{my^{m-1}}{(1-y)^{m+1}}$)

48. $\phi(x) = \frac{2x^2-1}{x\sqrt{1+x^2}}$. (Ans. $\phi'(x) = \frac{1+4x^2}{x^2(1+x^2)^{\frac{3}{2}}}$)

49. $y = \sqrt{2px}$. (Ans. $y' = \frac{p}{y}$)

50. $y = \frac{b}{a}\sqrt{a^2 - x^2}$. (Ans. $y' = -\frac{b^2x}{a^2y}$)

51. $y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$. (Ans. $y' = -\sqrt[3]{\frac{y}{x}}$)

52. $r = \sqrt{a\phi} + c\sqrt{\phi^3}$. (Ans. $r' = \frac{\sqrt{a}+3c\phi}{2\sqrt{\phi}}$)

53. $u = \frac{v^c+v^d}{cd}$. (Ans. $u' = \frac{v^{c-1}}{d} + \frac{v^{d-1}}{c}$)

54. $p = \frac{(q+1)^{\frac{3}{2}}}{\sqrt{q-1}}$. (Ans. $p' = \frac{(q-2)\sqrt{q+1}}{(q-1)^{\frac{3}{2}}}$)

55. Differentiate the following functions:

$$
\begin{array}{lll}
(a)\ \frac{d}{dx}\left(\frac{a^2-x^2}{a^2+x^2}\right) & (d)\ \frac{d}{dy}\left(\frac{ay^2}{b+y^3}\right) & (g)\ \frac{d}{dx}\frac{x^2}{\sqrt{1-x^2}} \\
(b)\ \frac{d}{dx}\left(\frac{x^3}{1+x^4}\right) & (e)\ \frac{d}{ds}\left(\frac{a^2-s^2}{\sqrt{a^2+s^2}}\right) & (h)\ \frac{d}{dx}\frac{1+x^2}{(1-x^2)^{\frac{3}{2}}} \\
(c)\ \frac{d}{dx}\left(\frac{1+x}{\sqrt{1-x}}\right) & (f)\ \frac{d}{dx}\frac{\sqrt{4-2x^3}}{x} & (i)\ \frac{d}{dt}\sqrt{\frac{1+t^2}{1-t^2}}
\end{array}
$$

4.11 Differentiation of a function of a function

It sometimes happens that y, instead of being defined directly as a function of x, is given as a function of some other variable, say v, and that v is defined as a function of x. In that case y is a function of x through v and is called a *composite function*. The process of substituting one function into another is sometimes called *composition*.

For example, if $y = \frac{2v}{1-v^2}$, and $v = 1 - x^2$, then y is a composite function. By eliminating v we may express y directly as a function of x, but in general this is not the best plan when we wish to find $\frac{dy}{dx}$.

Sage

```
sage: t = var('t')
sage: f = function('f', t)
sage: g = lambda v: 2*v/(1-v^2)
sage: g(f(t)).diff(t)  # this gives the general form, for any f
2*diff(f(t), t, 1)/(1 - f(t)^2) + 4*f(t)^2*diff(f(t), t, 1)/(1 - f(t)^2)^2
sage: f = lambda x: 1-x^2
```

```
sage: g(f(t)).diff(t)  # this gives the specific answer in this case
-4*t/(1 - (1 - t^2)^2) - 8*t*(1 - t^2)^2/(1 - (1 - t^2)^2)^2
```

If $y = f(v)$ and $v = g(x)$, then y is a function of x through v. Hence, when we let x take on an increment Δx, v will take on an increment Δv and y will also take on a corresponding increment Δy. Keeping this in mind, let us apply the General Rule simultaneously to the two functions $y = f(v)$ and $v = g(x)$.

- FIRST STEP. $y + \Delta y = f(v + \Delta v)$, $v + \Delta v = g(x + \Delta x)$.
- SECOND STEP.
$$\begin{aligned} y + \Delta y &= f(v + \Delta v), & v + \Delta v &= g(x + \Delta x) \\ y &= f(v), & v &= g(x) \\ \Delta y &= f(v + \Delta v) - f(v), & \Delta v &= g(x + \Delta x) - g(x) \end{aligned}$$
- THIRD STEP. $\frac{\Delta y}{\Delta v} = \frac{f(v+\Delta v)-f(v)}{\Delta v}$, $\frac{\Delta v}{\Delta x} = \frac{g(x+\Delta x)-g(x)}{\Delta x}$.

  The left-hand members show one form of the ratio of the increment of each function to the increment of the corresponding variable, and the right-hand members exhibit the same ratios in another form. Before passing to the limit let us form a product of these two ratios, choosing the left-hand forms for this purpose.

  This gives $\frac{\Delta y}{\Delta v} \cdot \frac{\Delta v}{\Delta x}$, which equals $\frac{\Delta y}{\Delta x}$. Write this
$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta v} \cdot \frac{\Delta v}{\Delta x}.$$
- FOURTH STEP. Passing to the limit,
$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}, \tag{4.29}$$
  by Theorem 2.8.2.This may also be written
$$\frac{dy}{dx} = f'(v) \cdot g'(x),$$
  or
$$\frac{dy}{dx} = g'(x)f'(g(x)). \tag{4.30}$$

The above formula is sometimes referred to as the *chain rule* for differentiation. If $y = f(v)$ and $v = g(x)$, the derivative of y with respect to x equals the product of the derivative of y with respect to v and the derivative of v with respect to x.

4.12 Differentiation of inverse functions

Let $y = f(x)$ be a given function of x.

It is often possible in the case of functions considered in this book to solve this equation for x, giving

$$x = \phi(y);$$

that is, to consider y as the independent and x as the dependent variable. In that case $f(x)$ and $\phi(y)$ are said to be *inverse functions* (and one often writes $\phi = f^{-1}$).

When we wish to distinguish between the two it is customary to call the first one given the *direct function* and the second one the *inverse function*. Thus, in the examples which follow, if the second members in the first column are taken as the direct functions, then the corresponding members in the second column will be respectively their inverse functions.

Example 4.12.1.

- $y = x^2 + 1$, $x = \pm\sqrt{y-1}$.
- $y = a^x$, $x = \log_a y$.
- $y = \sin\ x$, $x = \arcsin\ y$.

The plot of the inverse function $\phi(y)$ is related to the plot of the function $f(x)$ in a simple manner. The plot of $f(x)$ over an interval (a, b) in which f is increasing is the same as the plot of $\phi(y)$ over $(f(a), f(b))$. The plot of $y = f(x)$ is the "mirror image" of the plot of $y = \phi(x)$, reflected about the "diagonal line" $y = x$.

Example 4.12.2. If $f(x) = x^2$, for $x > 0$, and $\phi(y) = \sqrt{y}$, then the graphs are

Now flip this graph about the 45^o line:

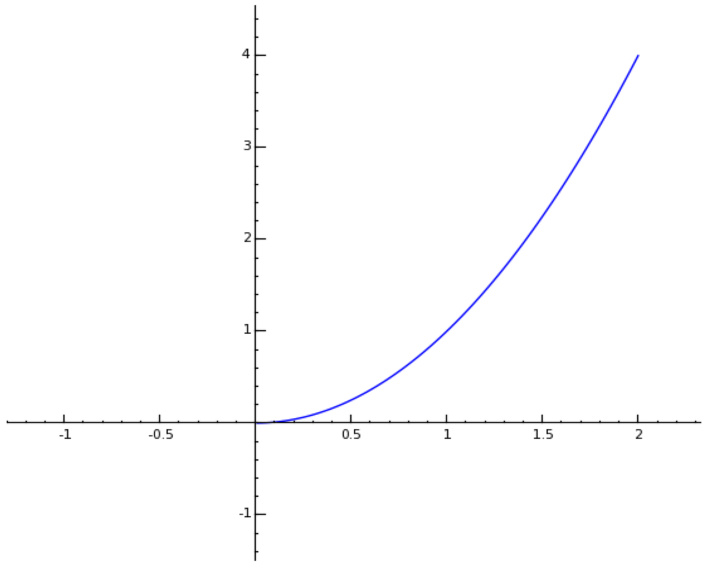

Figure 4.1: The function $f(x) = x^2$.

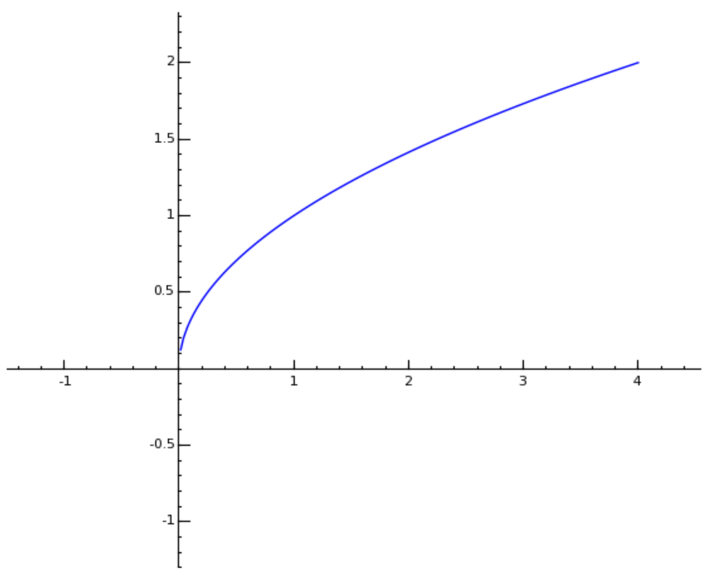

Figure 4.2: The function $\phi(y) = f^{-1}(y) = \sqrt{y}$.

The graph of inverse trig functions, for example, $\tan(x)$ and $\arctan(x)$, are related in the same way.

Let us now differentiate the inverse functions

$$y = f(x) \text{ and } x = \phi(y)$$

simultaneously by the General Rule.

- FIRST STEP. $y + \Delta y = f(x + \Delta x)$, $x + \Delta x = \phi(y + \Delta y)$
- SECOND STEP.

$$\begin{aligned} y + \Delta y &= f(x + \Delta x), \\ y &= f(x), \\ \Delta y &= f(x + \Delta x) - f(x), \end{aligned} \qquad \begin{aligned} x + \Delta x &= \phi(y + \Delta y) \\ x &= \phi(y) \\ \Delta x &= \phi(y + \Delta y) - \phi(y) \end{aligned}$$

- THIRD STEP.

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \qquad \frac{\Delta x}{\Delta y} = \frac{\phi(y + \Delta y) - \phi(y)}{\Delta y}.$$

Taking the product of the left-hand forms of these ratios, we get $\frac{\Delta y}{\Delta x} \cdot \frac{\Delta x}{\Delta y} = 1$, or, $\frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}}$.

- FOURTH STEP. Passing to the limit,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \tag{4.31}$$

or,

$$f'(x) = \frac{1}{\phi'(y)}.$$

The derivative of the inverse function is equal to the reciprocal of the derivative of the direct function.

4.13 Differentiation of a logarithm

Let[2] $y = log_a v$.

Differentiating by the General Rule (§3.7), considering v as the independent variable, we have

- FIRST STEP. $y + \Delta y = \log_a(v + \Delta v)$.
- SECOND STEP.

$$\begin{aligned}\Delta y &= \log_a(v + \Delta v) - \log_a v \\ &= \log_a \left(\tfrac{v+\Delta v}{v}\right) \\ &= \log_a \left(1 + \tfrac{\Delta v}{v}\right).\end{aligned}$$

  by item (8), §12.1.

- THIRD STEP.

$$\begin{aligned}\tfrac{\Delta y}{\Delta x} &= \tfrac{1}{\Delta v} \log_a \left(1 + \tfrac{\Delta v}{v}\right) \\ &= \log_a \left(1 + \tfrac{\Delta v}{v}\right)^{\frac{1}{\Delta v}} \\ &= \tfrac{1}{v} \log_a \left(1 + \tfrac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}.\end{aligned}$$

  [Dividing the logarithm by v and at the same time multiplying the exponent of the parenthesis by v changes the form of the expression but not its value (see item (9), §12.1.]

- FOURTH STEP. $\frac{dy}{dv} = \frac{1}{v} \log_a e$. [When $\Delta v \to 0$ $\frac{\Delta v}{v} \to 0$. Therefore $\lim_{\Delta v \to 0} \left(1 + \frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}} = e$, from §2.11, placing $x = \frac{\Delta v}{v}$.]

Hence

$$\frac{dy}{dv} = \frac{d}{dv}(\log_a v) = \log_a e \cdot \frac{1}{v}. \tag{4.32}$$

Since v is a function of x and it is required to differentiate $\log_a v$ with respect to x, we must use formula (4.29), for differentiating a function of a function, namely,

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

[2]The student must not forget that this function is defined only for positive values of the base a and the variable v.

Substituting the value of $\frac{dy}{dv}$ from (4.32), we get

$$\frac{dy}{dx} = \log_a e \cdot \frac{1}{v} \cdot \frac{dv}{dx}.$$

Therefore, $\frac{d}{dx}(\log_a x) = \log_s e \cdot \frac{\frac{dv}{dx}}{v}$ (equation (4.10) above). When $a = e$, $\log_a e = log_e e = 1$, and (4.10) becomes $\frac{d}{dx}(\log v) = \frac{\frac{dv}{dx}}{v}$ (equation (4.11) above).

Sage

```
sage: t = var('t')
sage: f = function('f', t)
sage: log(f(t)).diff(t)
diff(f(t), t, 1)/f(t)
sage: f = 1-t^2
sage: log(f(t)).diff(t)
-2*t/(1 - t^2)
```

4.14 Differentiation of the simple exponential function

Let $y = a^v$, $\quad a > 0$. Taking the logarithm of both sides to the base e, we get $\log y = v \log a$, or $v = \frac{\log y}{\log a} = \frac{1}{\log a} \cdot \log y$. Differentiate with respect to y by formula (4.11),

$$\frac{dv}{dy} = \frac{1}{\log a} \cdot \frac{1}{y};$$

and from (4.31),relating to inverse functions, we get $\frac{dy}{dv} = \log a \cdot y$, or,

$$\frac{dy}{dv} = \log a \cdot a^v.$$

Since v is a function of x and it is required to differentiate a^v with respect to x, we must use formula (4.29), for differentiating a function of a function, namely,

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

Substituting the value of $\frac{dy}{dx}$ from above, we get

$$\frac{dy}{dx} = \log a \cdot a^v \cdot \frac{dv}{dx}.$$

Therefore, $\frac{d}{dx}(a^v) = \log a \cdot a^v \cdot \frac{dv}{dx}$ (equation (4.12) in §4.1 above).

Sage

```
sage: t = var('t')
sage: f = function('f', t)
sage: (3^f(t)).diff(t)
log(3)*3^f(t)*diff(f(t), t, 1)
sage: f = t^7
sage: (3^f(t)).diff(t)
7*log(3)*t^6*3^t^7
```

When $a = e$, $\log a = \log e = 1$, and (4.12) becomes $\frac{d}{dx}(e^v) = e^v \frac{dv}{dx}$ (equation (4.13) in §4.1 above) .

The derivative of a constant with a variable exponent is equal to the product of the natural logarithm of the constant, the constant with the variable exponent, and the derivative of the exponent.

4.15 Differentiation of the general exponential function

Let[3] $y = u^v$. Taking the logarithm of both sides to the base e, $\log_e y = v \log_e u$, or, $y = e^{v \log u}$.

Differentiating by formula (4.13),

$$\begin{aligned} \frac{dy}{dx} &= e^{v \log u} \frac{d}{dx}(v \log u) \\ &= e^{v \log u} \left(\frac{v}{u} \frac{du}{dx} + \log u \frac{dv}{dx} \right) \quad \text{by } (4.5) \\ &= u^v \left(\frac{v}{u} \frac{du}{dx} + \log u \frac{dv}{dx} \right) \end{aligned}$$

Therefore, $\frac{d}{dx}(u^v) = vu^{v-1} \frac{du}{dx} + \log u \cdot u^v \frac{dv}{dx}$ (equation (4.14) in §4.1 above).

[3]Here u can assume only positive values.

The derivative of a function with a variable exponent is equal to the sum of the two results obtained by first differentiating by (4.6), regarding the exponent as constant, and again differentiating by (4.12), regarding the function as constant.

Let $v = n$, any constant; then (4.14) reduces to

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

But this is the form differentiated in §4.8; therefore (4.6) holds true for any value of n.

Example 4.15.1. Differentiate $y = \log(x^2 + a)$.

Solution. By (4.11) (with $v = x^2 + a$), we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(x^2+a)}{x^2+a}\\ &= \frac{2x}{x^2+a}.\end{aligned}$$

Example 4.15.2. Differentiate $y = \log\sqrt{1-x^2}$.

Solution. By (4.11) and (4.6),

$$\begin{aligned}\frac{dy}{dx} &= \frac{\frac{d}{dx}(1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}}\\ &= \frac{\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)}{(1-x^2)^{\frac{1}{2}}}\\ &= \frac{x}{x^2-1}.\end{aligned}$$

Example 4.15.3. Differentiate $y = a^{3x^2}$.

Solution. By (4.12),

$$\begin{aligned}\frac{dy}{dx} &= \log a \cdot a^{3x^2}\frac{d}{dx}(3x^2)\\ &= 6x\log a \cdot a^{3x^2}.\end{aligned}$$

Sage

```
sage: t,a = var('t,a')
sage: f = 3*t^2
sage: (a^f(t)).diff(t)
6*a^(3*t^2)*log(a)*t
```

Example 4.15.4. Differentiate $y = be^{c^2+x^2}$.
Solution. By (4.4) and (4.13),

$$\begin{aligned} \frac{dy}{dx} &= b\frac{d}{dx}\left(e^{c^2+x^2}\right) \\ &= be^{c^2+x^2}\frac{d}{dx}(c^2+x^2) \\ &= 2bxe^{c^2+x^2}. \end{aligned}$$

Example 4.15.5. Differentiate $y = x^{e^x}$.
Solution. By (4.14),

$$\begin{aligned} \frac{dy}{dx} &= e^x x^{e^x-1}\frac{d}{dx}(x) + x^{e^x}\log x\frac{d}{dx}(e^x) \\ &= e^x x^{e^x-1} + x^{e^x}\log x \cdot e^x \\ &= e^x x^{e^x}\left(\frac{1}{x} + \log x\right) \end{aligned}$$

4.16 Logarithmic differentiation

Instead of applying (4.10) and (4.11) at once in differentiating logarithmic functions, we may sometimes simplify the work by first making use of one of the formulas 7-10 in §12.1. Thus above Illustrative Example 4.15.2 may be solved as follows:

Example 4.16.1. Differentiate $y = \log\sqrt{1-x^2}$.
Solution. By using 10, in §12.1, we may write this in a form free from radicals as follows: $y = \frac{1}{2}\log(1-x^2)$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}\frac{\frac{d}{dx}(1-x^2)}{1-x^2} \text{ by } (4.11) \\ &= \frac{1}{2}\cdot\frac{-2}{1-x^2} = \frac{x}{x^2-1}. \end{aligned}$$

Example 4.16.2. Differentiate $y = \log\sqrt{\frac{1+x^2}{1-x^2}}$.
Solution. Simplifying by means of (10) and (8), in §12.1,

$$\begin{aligned} y &= \frac{1}{2}[\log(1+x^2) - \log(1-x^2)] \\ \frac{dy}{dx} &= \frac{1}{2}\left[\frac{\frac{d}{dx}(1+x^2)}{1+x^2} - \frac{\frac{d}{dx}(1-x^2)}{1-x^2}\right] \text{ by } (4.11), \text{ etc.} \\ &= \frac{x}{1+x^2} + \frac{x}{1-x^2} = \frac{2x}{1-x^4}. \end{aligned}$$

In differentiating an exponential function, especially a variable with a variable exponent, the best plan is first to take the logarithm of the function and then differentiate. Thus Example 4.15.5 is solved more elegantly as follows:

Example 4.16.3. Differentiate $y = x^{e^x}$.

Solution. Taking the logarithm of both sides, $\log y = e^x \log x$, by Formula 9 in §12.1. Now differentiate both sides with respect to x:

$$\begin{aligned} \frac{\frac{dy}{dx}}{y} &= e^x \tfrac{d}{dx}(\log x) + \log x \tfrac{d}{dx}(e^x) \text{ by (4.10) and (4.5)} \\ &= e^x \cdot \tfrac{1}{x} + \log x \cdot e^x, \end{aligned}$$

or,

$$\frac{dy}{dx} = e^x \cdot y \left(\frac{1}{x} \log x \right) = e^x x^{e^x} \left(\frac{1}{x} + \log x \right).$$

Example 4.16.4. Differentiate $y = (4x^2 - 7)^{2+\sqrt{x^2-5}}$.

Solution. Taking the logarithm of both sides,

$$\log y = (2 + \sqrt{x^2 - 5}) \log(4x^2 - 7).$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = (2 + \sqrt{x^2 - 5}) \frac{8x}{4x^2 - 7} + \log(4x^2 - 7) \cdot \frac{x}{\sqrt{x^2 - 5}}.$$

$$\frac{dy}{dx} = x(4x^2 - 7)^{2+\sqrt{x^2-5}} \left[\frac{8(2 + \sqrt{x^2 - 5})}{4x^2 - 7} + \frac{\log(4x^2 - 7)}{\sqrt{x^2 - 5}} \right].$$

In the case of a function consisting of a number of factors it is sometimes convenient to take the logarithm before differentiating. Thus,

Example 4.16.5. Differentiate $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$.

Solution. Taking the logarithm of both sides,

$$\log y = \frac{1}{2}[\log(x - 1) + \log(x - 2) - \log(x - 3) - \log(x - 4)].$$

Differentiating both sides with respect to x,

$$\begin{aligned} \tfrac{1}{y}\tfrac{dy}{dx} &= \tfrac{1}{2}\left[\tfrac{1}{x-1} + \tfrac{1}{x-2} - \tfrac{1}{x-3} - \tfrac{1}{x-4}\right] \\ &= -\tfrac{2x^2-10x+11}{(x-1)(x-2)(x-3)(x-4)}, \end{aligned}$$

or,

$$\frac{dy}{dx} = -\frac{2x^2 - 10x - 11}{(x - 1)^{\frac{1}{2}}(x - 2)^{\frac{1}{2}}(x - 3)^{\frac{3}{2}}(x - 4)^{\frac{3}{2}}}.$$

4.17 Examples

Differentiate the following[4]:

1. $y = \log(x + a)$ Ans: $\frac{dy}{dx} = \frac{1}{x+a}$
2. $y = \log(ax + b)$ Ans: $\frac{dy}{dx} = \frac{a}{ax+b}$
3. $y = \log\frac{1+x^2}{1-x^2}$ Ans: $\frac{dy}{dx} = \frac{4x}{1-x^4}$
4. $y = \log(x^2 + x)$ Ans: $y' = \frac{2x+1}{x^2+x}$
5. $y = \log(x^3 - 2x + 5)$ Ans: $y' = \frac{3x^2-2}{x^3-2x+5}$
6. $y = \log_a(2x + x^3)$ Ans: $y' = \log_a e \cdot \frac{2+3x^2}{2x+x^3}$
7. $y = x\log x$ Ans: $y' = \log x + 1$
8. $f(x) = \log(x^3)$ Ans: $f'(x) = \frac{3}{x}$
9. $f(x) = \log^3 x$ Ans: $f'(x) = \frac{3\log^2 x}{x}$
   (Hint: $\log^3 x = (\log x)^3$. Use first (4.6), $v = \log x$, $n = 3$; and then (4.11).)
10. $f(x) = \log\frac{a+x}{a-x}$ Ans: $f'(x) = \frac{2a}{a^2-x^2}$
11. $f(x) = \log(x+\sqrt{1+x^2})$ Ans: $f'(x) = \frac{1}{\sqrt{1+x^2}}$
12. $\frac{d}{dx}e^{ax} = ae^{ax}$
13. $\frac{d}{dx}e^{4x+5} = 4e^{4x+5}$
14. $\frac{d}{dx}a^{3x} = 3a^{3x}\log a$
15. $\frac{d}{dt}\log(3 - 2t^2) = \frac{4t}{2t^2-3}$
16. $\frac{d}{dy}\log\frac{1+y}{1-y} = \frac{2}{1-y^2}$
17. $\frac{d}{dx}e^{b^2+x^2} = 2xe^{b^2+x^2}$

[4]Though the answers are given below, it may be that your computation differs from the solution given. You should then try to show algebraically that your form is that same as that given.

18. $\frac{d}{d\theta}a^{\log a} = \frac{1}{\theta}a^{\log\theta}\log a$

19. $\frac{d}{ds}b^{s^2} = 2x\log b \cdot b^{s^2}$

20. $\frac{d}{dv}ae^{\sqrt{v}} = \frac{ae^{\sqrt{v}}}{2\sqrt{v}}$

21. $\frac{d}{dx}a^{e^x} = \log a \cdot a^{e^x} \cdot e^x$

22. $y = 7^{x^2+2x}$ Ans: $y' = 2\log 7 \cdot (x+1)7^{x^2+2x}$

23. $y = c^{a^2-x^2}$ Ans: $y' = -2x\log c \cdot c^{a^2-x^2}$

24. $y = \log\frac{e^x}{1+e^x}$ Ans: $\frac{dy}{dx} = \frac{1}{1+e^x}$

25. $\frac{d}{dx}\left[e^x(1-x^2\right] = e^x(1-2x-x^2)$

26. $\frac{d}{dx}\left(\frac{e^x-1}{e^x+1}\right) = \frac{2e^x}{(e^x+1)^2}$

27. $\frac{d}{dx}\left(x^2e^{ax}\right) = xe^{ax}(ax+2)$

28. $y = \frac{a}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}})$ Ans: $\frac{dy}{dx} = \frac{1}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$

29. $y = \frac{e^x-e^{-x}}{e^x+e^{-x}}$ Ans: $\frac{dy}{dx} = \frac{4}{(e^x+e^{-x}))^2}$

30. $y = x^n a^x$ Ans: $y' = a^x x^{n-1}(n + x\log a)$

31. $y = x^x$ Ans: $y' = x^x(\log x + 1)$

32. $y = x^{\frac{1}{x}}$ Ans: $y' = \frac{x^{\frac{1}{x}}(1-\log x)}{x^2}$

33. $y = x^{\log x}$ Ans: $y' = \log(x^2)\cdot x^{\log x - 1}$

34. $f(y) = \log y \cdot e^y$ Ans: $f'(y) = e^y\left(\log y + \frac{1}{y}\right)$

35. $f(s) = \frac{\log s}{e^s}$ Ans: $f'(s) = \frac{1-s\log s}{se^s}$

36. $f(x) = \log(\log x)$ Ans: $f'(x) = \frac{1}{x\log x}$

37. $F(x) = \log^4(\log x)$ Ans: $F'(x) = \frac{4\log^3(\log x)}{x\log x}$

38. $\phi(x) = \log(\log^4 x)$ Ans: $\phi'(x) = \frac{4}{x\log x}$

39. $\psi(y) = \log\sqrt{\frac{1+y}{1-y}}$ Ans: $\psi'(y) = \frac{1}{1-y^2}$

40. $f(x) = \log\frac{\sqrt{x^2+1}-x}{\sqrt{x^1+1}+x}$ Ans: $f'(x) = -\frac{2}{\sqrt{1+x^2}}$

41. $y = x^{\frac{1}{\log x}}$ Ans: $\frac{dy}{dx} = 0$

42. $y = e^{x^x}$ Ans: $\frac{dy}{dx} = e^{x^x}(1+\log x)x^x$

43. $y = \frac{c^x}{x^x}$ Ans: $\frac{dy}{dx} = \left(\frac{c}{x}\right)^x\left(\log\frac{c}{x} - 1\right)$

44. $y = \left(\frac{x}{n}\right)^{nx}$ Ans: $\frac{dy}{dx} = n\left(\frac{x}{n}\right)^{nx}\left(1+\log\frac{x}{n}\right)$

45. $w = v^{e^v}$ Ans: $\frac{dw}{dv} = v^{e^v}e^v\left(\frac{1+v\log v}{v}\right)$

46. $z = \left(\frac{a}{t}\right)^t$ Ans: $\frac{dz}{dt} = \left(\frac{a}{t}\right)^t(\log a - \log t - 1)$

47. $y = x^{x^n}$ Ans: $\frac{dy}{dx} = x^{x^n+n-1}(n\log x + 1)$

48. $y = x^{x^x}$ Ans: $\frac{dy}{dx} = x^{x^x}x^x\left(\log x + \log^2 x + \frac{1}{x}\right)$

49. $y = a^{\frac{1}{\sqrt{a^2-x^2}}}$ Ans: $\frac{dy}{dx} = \frac{xy\log a}{(a^2-x^2)^{\frac{3}{2}}}$

50. Compute the following derivatives:

(a) $\frac{d}{dx}x^2\log x$	(f) $\frac{d}{dx}e^x\log x$	(k) $\frac{d}{dx}\log(a^x+b^x)$
(b) $\frac{d}{dx}(e^{2x}-1)^4$	(g) $\frac{d}{dx}x^3 3^x$	(l) $\frac{d}{dx}\log_1 0(x^2+5x)$
(c) $\frac{d}{dx}\log\frac{3x+1}{x+3}$	(h) $\frac{d}{dx}\frac{1}{x\log x}$	(m) $\frac{d}{dx}\frac{2+x^2}{e^{3x}}$
(d) $\frac{d}{dx}\log\frac{1-x^2}{\sqrt{1+x}}$	(i) $\frac{d}{dx}\log x^3\sqrt{1+x^2}$	(n) $\frac{d}{dx}(x^2+a^2)e^{x^2+a^2}$
(e) $\frac{d}{dx}x^{\sqrt{x}}$	(j) $\frac{d}{dx}\left(\frac{1}{x}\right)^x$	(o) $\frac{d}{dx}(x^2+4)^x$.

51. $y = \frac{(x+1)^2}{(x+2)^3(x+3)^4}$ Ans: $\frac{dy}{dx} = -\frac{(x+1)(5x^2+14x+5)}{(x+2)^4(x+3)^5}$

52. $y = \frac{((x-1)^{\frac{5}{2}}}{(x-2)^{\frac{3}{4}}(x-3)^{\frac{7}{3}}}$ Ans: $\frac{dy}{dx} = -\frac{(x-1)^{\frac{3}{2}}(7x^2+30x-97)}{12(x-2)^{\frac{7}{4}}(x-3)^{\frac{10}{3}}}$

53. $y = x\sqrt{1-x}(1+x)$ Ans: $\frac{dy}{dx} = \frac{2+x-5x^2}{2\sqrt{1-x}}$

54. $y = \frac{x(1+x^2)}{\sqrt{1-x^2}}$ Ans: $\frac{dy}{dx} = \frac{1+3x^2-2x^4}{(1-x^2}^{\frac{3}{2}}$

55. $y = x^5(a+3x)^3(a-2x)^2$ Ans: $\frac{dy}{dx} = 5x^4(a+3x)^2(a-2x)(a^2+2ax-12x^2)$

4.18 Differentiation of $\sin v$

Let $y = \sin v$. By the General Rule for Differentiation in §3.7, considering v as the independent variable, we have

- FIRST STEP. $y + \Delta y = \sin(v + \Delta v)$.
- SECOND STEP.

$$\Delta y = \sin(v + \Delta v) - \sin v = 2\cos\left(v + \frac{\Delta v}{2}\right) \cdot \sin\frac{\Delta v}{2}.$$

- THIRD STEP.

$$\frac{\Delta y}{\Delta v} = \cos\left(v + \frac{\Delta v}{2}\right)\left(\frac{\sin\frac{\Delta v}{2}}{\frac{\Delta v}{2}}\right).$$

- FOURTH STEP. $\frac{dy}{dx} = \cos v$.

  (Since $\lim_{\Delta v \to 0}\left(\frac{\sin\frac{\Delta v}{2}}{\frac{\Delta v}{2}}\right) = 1$, by §2.10, and $\lim_{\Delta v \to 0}\cos\left(v + \frac{\Delta v}{2}\right) = \cos v$.)

Since v is a function of x and it is required to differentiate $\sin v$ with respect to x, we must use formula (A), §4.11, for differentiating a function of a function, namely,

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}.$$

Substituting value $\frac{dy}{dx}$ from Fourth Step, we get $\frac{dy}{dx} = \cos v \frac{dv}{dx}$. Therefore,

$$\frac{d}{dx}(\sin v) = \cos v \frac{dv}{dx}$$

(equation (4.15) in §4.1 above).

The statement of the corresponding rules will now be left to the student.

4.19 Differentiation of $\cos v$

Let $y = \cos v$. By item 29, §12.1, this may be written

$$y = \sin\left(\frac{\pi}{2} - v\right).$$

Differentiating by formula (4.15),

$$\begin{aligned}\frac{dy}{dx} &= \cos\left(\tfrac{\pi}{2} - v\right)\tfrac{d}{dx}\left(\tfrac{\pi}{2} - v\right) \\ &= \cos\left(\tfrac{\pi}{2} - v\right)\left(-\tfrac{dv}{dx}\right) \\ &= -\sin x\tfrac{dv}{dx}.\end{aligned}$$

(Since $\cos\left(\frac{\pi}{2}\right) = \sin v$, by 29, §12.1.) Therefore,

$$\frac{d}{dx}(\cos v) = -\sin v\frac{dv}{dx},$$

(equation (4.16) in §4.1 above).

Sage

```
sage: t = var('t')
sage: f = function('f', t)
sage: cos(f(t)).diff(t)
-sin(f(t))*diff(f(t), t, 1)
```

4.20 Differentiation of $\tan v$

Let $y = \tan v$. By item 27, §12.1, this may be written

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos v\frac{d}{dx}(\sin v) - \sin v\frac{d}{dx}(\cos v)}{\cos^2 v} \\ &= \frac{\cos^2 v\frac{dv}{dx} + \sin^2 v\frac{dv}{dx}}{\cos^2 v} \\ &= \frac{\frac{dv}{dx}}{\cos^2 v} = \sec^2 v\frac{dv}{dx}.\end{aligned}$$

Therefore,

$$\frac{d}{dx}(\tan x) = \sec^2 v\frac{dv}{dx},$$

(equation (4.17) in §4.1 above).

4.21 Differentiation of $\cot v$

Let $y = \cot v$. By item 26, §12.1, this may be written $y = \frac{1}{\tan v}$. Differentiating by formula (4.8),

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\frac{d}{dx}(\tan v)}{\tan^2 v} \\ &= -\frac{\sec^2 \frac{dv}{dx}}{\tan^2 v} \\ &= -\csc^2 v \frac{dv}{dx}. \end{aligned}$$

Therefore,

$$\frac{d}{dx}(\cot v) = -\csc^2 v \frac{dv}{dx}$$

(equation (4.17) in §4.1 above).

4.22 Differentiation of $\sec v$

Let $y = \sec v$. By item 26, §12.1, this may be written $y = \frac{1}{\cos v}$. Differentiating by formula (4.8),

$$\begin{aligned} \frac{dy}{dx} &= -\frac{\frac{d}{dx}(\cos v)}{\cos^2 v} \\ &= \frac{\sin v \frac{dv}{dx}}{\cos^2 v} \\ &= \frac{1}{\cos v} \frac{\sin v}{\cos v} \frac{dv}{dx} \\ &= \sec v \tan v \frac{dv}{dx}. \end{aligned}$$

Therefore,

$$\frac{d}{dx}(\sec v) = \sec v \tan v \frac{dv}{dx}$$

(equation (4.19) in §4.1 above).

4.23 Differentiation of $\csc v$

Let $y = \csc\ v$. By item 26, §12.1, this may be written

$$y = \frac{1}{\sin v}.$$

Differentiating by formula (4.8),

$$
\begin{aligned}
\frac{dy}{dx} &= -\frac{\frac{d}{dx}(\sin v)}{\sin^2 v} \\
&= -\frac{\cos v \frac{dv}{dx}}{\sin^2 v} \\
&= -\csc v \cot v \frac{dv}{dx}.
\end{aligned}
$$

Therefore,

$$
\frac{d}{dx}(\csc v) = -\csc v \cot v \frac{dv}{dx}
$$

(equation (4.20) in §4.1 above).

Sage

```
sage: t = var('t')
sage: f = function('f', t)
sage: csc(f(t)).diff(t)
-cot(f(t))*csc(f(t))*diff(f(t), t, 1)
sage: f = tan
sage: csc(f(t)).diff(t)
-sec(t)^2*cot(tan(t))*csc(tan(t))
sage: f = arccos
sage: csc(f(t)).diff(t)
t/(1 - t^2)^(3/2)
sage: f = arccsc
sage: csc(f(t)).diff(t)
1
```

4.24 Exercises

In the derivation of our formulas so far it has been necessary to apply the General Rule, §3.7, (i.e. the four steps), only for the following:

4.3	$\frac{d}{dx}(u+v-w) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$	Algebraic sum.
4.5	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$.	Product.
4.8	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.	Quotient.
4.10	$\frac{d}{dx}(\log_a v) = \log_a e \frac{\frac{dv}{dx}}{v}$.	Logarithm.
4.15	$\frac{d}{dx}(\sin v) = \cos v\frac{dv}{dx}$	Sine.
4.27	$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$.	Function of a function.
4.28	$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.	Inverse functions.

These are very basic formulas for us. Not only do all the other formulas we have verified so far depend on them, but those formulas we'll verify later depend on them as well.

Examples/exercises:

Differentiate the following:

1. $y = \sin(ax^2)$.

$$\frac{dy}{dx} = \cos ax^2 \frac{d}{dx}(ax^2), \quad \text{by 4.15 } (v = ax^2).$$

2. $y = \tan\sqrt{1-x}$.

$$\begin{aligned} \frac{dy}{dx} &= \sec^2\sqrt{1-x}\frac{d}{dx}(1-x)^{\frac{1}{2}}, \quad \text{by 4.17 })v = \sqrt{1-x}) \\ &= \sec^2\sqrt{1-x} \cdot \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) \\ &= -\frac{\sec^2\sqrt{1-x}}{2\sqrt{1-x}}. \end{aligned}$$

3. $y = \cos^3 x$.

   This may also be written, $y = (\cos x)^3$.

$$\begin{aligned} \frac{dy}{dx} &= 3(\cos x)^2\frac{d}{dx}(\cos x) \quad \text{by 4.6 } (v = \cos x \text{ and } n = 3) \\ &= 3\cos^2 x(-\sin x) \quad \text{by 4.16} \\ &= -3\sin x\cos^2 x. \end{aligned}$$

4. $y = \sin nx \sin^n x$.

$$\begin{aligned}\frac{dy}{dx} &= \sin nx \frac{d}{dx}(\sin x)^n + \sin^n x \frac{d}{dx}(\sin nx) \quad \text{by } (4.5)(v = \sin nx \text{ and } v = \sin^n x)\\ &= \sin nx \cdot n(\sin x)^{n-1}\frac{d}{dx}(\sin x) + \sin^n x \cos nx \frac{d}{dx}(nx) \quad \text{by } 4.6 \text{ and } 4.15\\ &= n \sin nx \cdot \sin^{n-1} x \cos x + n \sin^n x \cos nx\\ &= n \sin^{n-1} x(\sin nx \cos x + \cos nx\ sinx)\\ &= n \sin^{n-1} x \sin(n+1)x.\end{aligned}$$

5. $y = \sec ax$ Ans: $\frac{dy}{dx} = a \sec ax \tan ax$
6. $y = \tan(ax + b)$ Ans: $\frac{dy}{dx} = a \sec^2(ax + b)$
7. $s = \cos 3ax$ Ans: $\frac{ds}{dx} = -3a \sin 3ax$
8. $s = \cot(2t^2 + 3)$ Ans: $\frac{ds}{dt} = -4t \csc^2(2t^2 + 3)$
9. $f(y) = \sin 2y \cos y$ Ans: $f'(y) = 2 \cos 2y \cos y - \sin 2y \sin y$
10. $F(x) = \cot^2 5x$ Ans: $F'(x) = -10 \cot 5x \csc^2 5x$
11. $F(\theta) = \tan\theta - \theta$ Ans: $F'(\theta) = \tan^2\theta$
12. $f(\phi) = \phi \sin\phi + \cos\phi$ Ans: $f'(\phi) = \phi \cos\phi$
13. $f(t) = \sin^3 t \cos t$ Ans: $f'(t) = \sin^2 t(3 \cos t - \sin^2 t)$
14. $r = a \cos 2\theta$ Ans: $\frac{dr}{d\theta} = -2a \sin 2\theta$
15. $\frac{d}{dx} \sin^2 x = \sin 2x$
16. $\frac{d}{dx} \cos^3(x^2) = -6x \cos^2(x^2) \sin(x^2)$
17. $\frac{d}{dt} \csc \frac{t^2}{2} = -t \csc \frac{t^2}{2} \cot \frac{t^2}{2}$
18. $\frac{d}{ds} a\sqrt{\cos 2s} = -\frac{a \sin 2s}{\sqrt{\cos 2s}}$
19. $\frac{d}{d\theta} a(1 - \cos\theta) = a \sin\theta$
20. $\frac{d}{dx}(\log \cos x) = -\tan x$
21. $\frac{d}{dx}(\log \tan x) = \frac{2}{\sin 2x}$
22. $\frac{d}{dx}(\log \sin^2 x) = 2 \cot x$

23. $\frac{d}{dt}\cos\frac{a}{t} = \frac{a}{t^2}\sin\frac{a}{t}$

24. $\frac{d}{d\theta}\sin\frac{1}{\theta^2} = -\frac{2}{\theta^3}\cos\frac{1}{\theta^2}$

25. $\frac{d}{dx}e^{\sin x} = e^{\sin x}\cos x$

26. $\frac{d}{dx}\sin(\log x) = \frac{\cos(\log x)}{x}$

27. $\frac{d}{dx}\tan(\log x) = \frac{\sec^2(\log x)}{x}$

28. $\frac{d}{dx}a\sin^3\frac{\theta}{3} = a\sin^2\frac{\theta}{3}\cos\frac{\theta}{3}$

29. $\frac{d}{d\alpha}\sin(\cos\alpha) = -\sin\alpha\cos(\cos\alpha)$

30. $\frac{d}{dx}\frac{\tan x - 1}{\sec x} = \sin x + \cos x$

31. $y = \log\sqrt{\frac{1+\sin x}{1-\sin x}}$ Ans: $\frac{dy}{dx} = \frac{1}{\cos x}$

32. $y = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$ Ans: $\frac{dy}{dx} = \frac{1}{\cos x}$

33. $f(x) = \sin(x+a)\cos(x-a)$ Ans: $f'(x) = \cos 2x$

34. $y = a^{\tan nx}$ Ans: $y' = na^{\tan nx}\sec^2 nx\log a$

35. $y = e^{\cos x}\sin x$ Ans: $y' = e^{\cos x}(\cos x - \sin^2 x)$

36. $y = e^x\log\sin x$ Ans: $y' = e^x(\cot x + \log\sin x)$

37. Compute the following derivatives:

(a) $\frac{d}{dx}\sin 5x^2$	(f) $\frac{d}{dx}\csc(\log x)$	(k) $\frac{d}{dt}e^{a-b\cos t}$
(b) $\frac{d}{dx}\cos(a-bx)$	(g) $\frac{d}{dx}\sin^3 2x$	(l) $\frac{d}{dt}\sin\frac{t}{3}\cos^2\frac{t}{3}$
(c) $\frac{d}{dx}\tan\frac{ax}{b}$	(h) $\frac{d}{dx}\cos^2(\log x)$	(m) $\frac{d}{d\theta}\cot\frac{b}{\theta^2}$
(d) $\frac{d}{dx}\cot\sqrt{ax}$	(i) $\frac{d}{dx}\tan^2\sqrt{1-x^2}$	(n) $\frac{d}{d\phi}\sqrt{1+\cos^2\phi}$
(e) $\frac{d}{dx}\sec e^{3x}$	(j) $\frac{d}{dx}\log(\sin^2 ax)$	(o) $\frac{d}{ds}\log\sqrt{1-2\sin^2 s}$

38. $\frac{d}{dx}(x^n e^{\sin x}) = x^{n-1}e^{\sin x}(n + x\cos x)$

39. $\frac{d}{dx}(e^{ax}\cos mx) = e^{ax}(a\cos mx - m\sin mx)$

40. $f(\theta) = \frac{1+\cos\theta}{1-\cos\theta}$ Ans: $f'(\theta) = -\frac{2\sin\theta}{(1-\cos\theta)^2}$

41. $f(\phi) = \frac{e^{a\phi}(a\sin\phi - \cos\phi)}{a^2+1}$ Ans: $f'(\phi) = e^{a\phi}\sin\phi$

42. $f(s) = (s\cot s)^2$ Ans: $f'(s) = 2s\cot s(\cot s - s\csc^2 s)$

43. $r = \frac{1}{3}\tan^3\theta - \tan\theta + \theta$ Ans: $\frac{dr}{d\theta} = \tan^4\theta$

44. $y = x^{\sin x}$ Ans: $\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \log x\cos x\right)$

45. $y = (\sin x)^x$ Ans: $y' = (\sin x)^x[\log\sin x + x\cot x]$

46. $y = (\sin x)^{\tan x}$ Ans: $y' = (\sin x)^{\tan x}(1 + \sec^2 x\log\sin x)$

47. Prove $\frac{d}{dx}\cos v = -\sin v\frac{dv}{dx}$, using the General Rule.

48. Prove $\frac{d}{dx}\cot v = -\csc^2 v\frac{dv}{dx}$ by replacing $\cot v$ by $\frac{\cos v}{\sin v}$.

4.25 Differentiation of $\arcsin v$

Let $y = \arcsin\ v$, then $v = \sin\ y$.

Remember this function is defined only for values of v between -1 and $+1$ inclusive and that this (inverse) function is many-valued, there being infinitely many angles (in radians) whose sines will equal v. Thus, Figure 4.4 represents only a piece of the multi-valued inverse function of $\sin(x)$, represented by taking the graph of $\sin(x)$ and flipping it about the 45^o line. In the above discussion, in order to make the function single-valued, only values of y between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ inclusive are considered; that is, the smallest angle (in radians) whose sine is v.

Differentiating v with respect to y gives, by 4.15, $\frac{dv}{dy} = \cos\ y$. Therefore $\frac{dy}{dv} = \frac{1}{\cos y}$, by (4.31). But since v is a function of x, this may be substituted into $\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx}$ (see (4.29)),giving

$$\frac{dy}{dx} = \frac{1}{\cos y}\cdot\frac{dv}{dx} = \frac{1}{\sqrt{1-v^2}}\frac{dv}{dx}.$$

Here we used the fact that $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-v^2}$. The positive sign of the square root is taken since $\cos y$ is positive for all values of y between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ inclusive. Therefore,

$$\frac{d}{dx}(\arcsin\ v) = \frac{\frac{dv}{dx}}{\sqrt{1-v^2}}$$

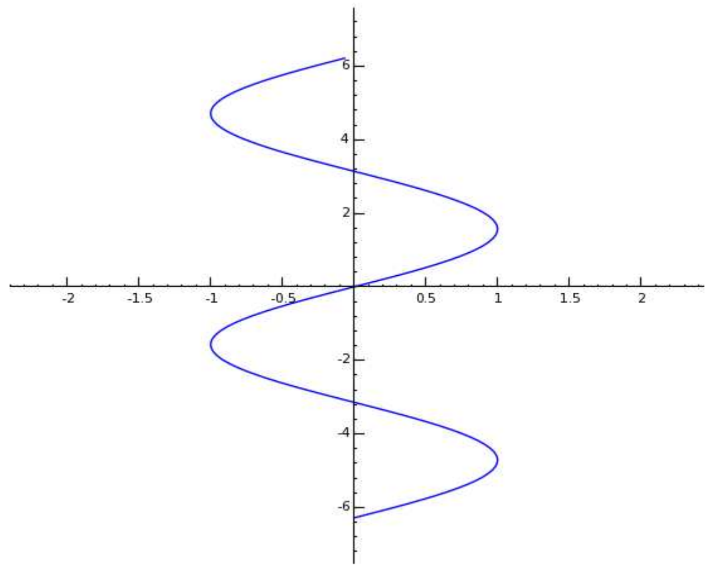

Figure 4.3: The inverse sine $\sin^{-1}\ x$ using Sage .

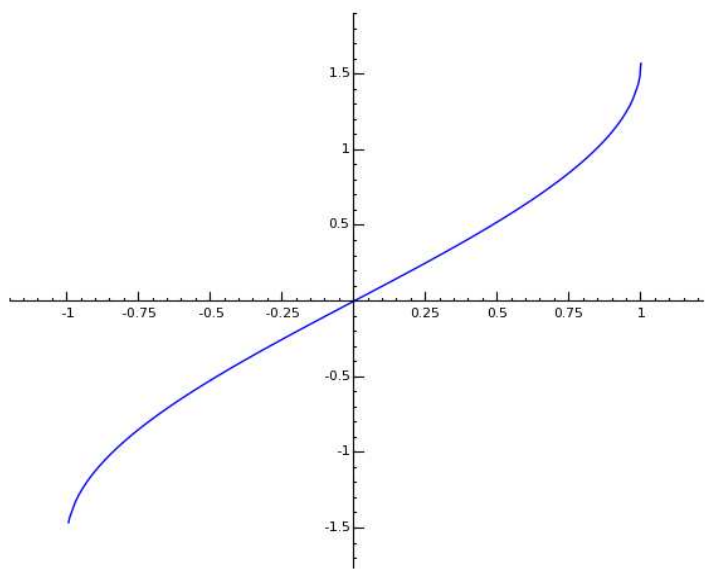

Figure 4.4: A single branch of the function $f(x) = \arcsin(x)$.

(equation (4.21) in §4.1 above).

4.26 Differentiation of $\arccos v$

Let[5] $y = \arccos\ v$; then $y = \cos\ y$.

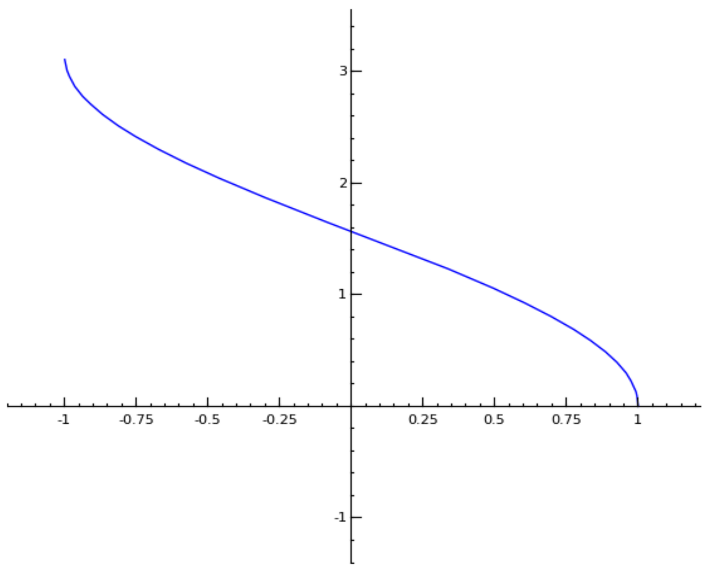

Figure 4.5: A single branch of the function $f(x) = \arccos(x)$.

Differentiating with respect to y by 4.16, $\frac{dv}{dy} = -\sin\ y$, therefore, $\frac{dy}{dv} = -\frac{1}{\sin y}$, by (4.31). But since v is a function of x, this may be substituted in the formula $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$, by (4.29), giving

$$\frac{dy}{dx} = -\frac{1}{\sin y} \cdot \frac{dv}{dx} = -\frac{1}{\sqrt{1-v^2}} \frac{dv}{dx}$$

($\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-v^2}$, the plus sign of the radical being taken, since $\sin\ y$ is positive for all values of y between 0 and π inclusive). Therefore,

$$\frac{d}{dx}(\arccos\ v) = -\frac{\frac{dv}{dx}}{\sqrt{1-v^2}}.$$

[5]This function is defined only for values of v between -1 and $+1$ inclusive, and is many-valued. In order to make the function single-valued, only values of y between 0 and π inclusive are considered; that is, y the smallest angle whose cosine is v.

(equation (4.22) in §4.1 above).

Here's how to use Sage to compute an example of this rule:

Sage

```
sage: t = var("t")
sage: x = var("x")
sage: solve(x == cos(t),t)
[t == acos(x)]
sage: f = solve(x == cos(t),t)[0].rhs()
sage: f
acos(x)
sage: diff(f,x)
-1/sqrt(1 - xˆ2)
```

This (a) computes $\arccos$ directly as the inverse function of $\cos$ (Sage can use the notation acos in addition to $\arccos$), (b) computes its derivative.

4.27 Differentiation of $\arctan\ v$

Let[6] $y = \arctan v$; then $y = \tan y$.

Differentiating with respect to y by (4.18),

$$\frac{dv}{dy} = \sec^2 y;$$

therefore $\frac{dy}{dv} = \frac{1}{\sec^2 y}$, by (4.31). But since v is a function of x, this may be substituted in the formula $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$, by (4.29),giving

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} \cdot \frac{dv}{dx} = \frac{1}{1+v^2}\frac{dv}{dx},$$

(since $\sec^2 y = 1 + \tan^2 y = 1 + v^2$). Therefore

$$\frac{d}{dx}(\arctan\ v) = \frac{\frac{dv}{dx}}{1+v^2}$$

(equation (4.23) in §4.1 above).

[6] This function is defined for all values of v, and is many-valued. In order to make it single-valued, only values of y between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are considered; that is, the smallest angle (in radians) whose tangent is v.

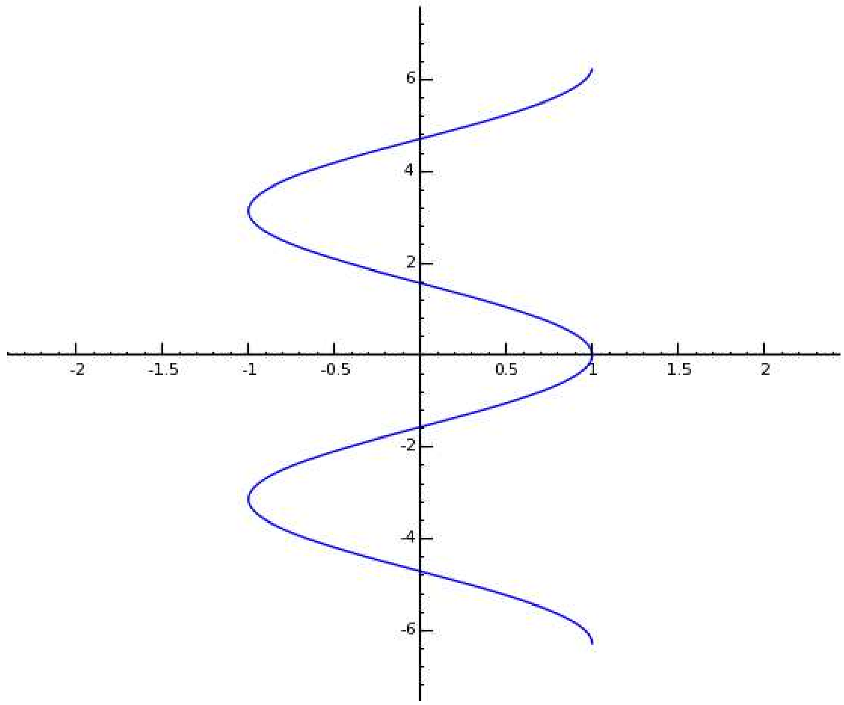

Figure 4.6: The inverse cosine $\cos^{-1}\ x$ using Sage .

4.28 Differentiation of $\operatorname{arccot} v$

Let $y = \operatorname{arccot} v$; then $y = \cot\ y$. This function is defined for all values of v, and is many-valued.In order to make it single-valued, only values of y between 0 and π are considered; that is, the smallest angle whose cotangent is v.

Following the method of the last section, we get

$$\frac{d}{dx}(\operatorname{arccot} v) = -\frac{\frac{dv}{dx}}{1+v^2}$$

(equation (4.24) in §4.1 above).

Sage

```
sage: t = var('t')
sage: f = function('f', t)
sage: acot(f(t)).diff(t)
```

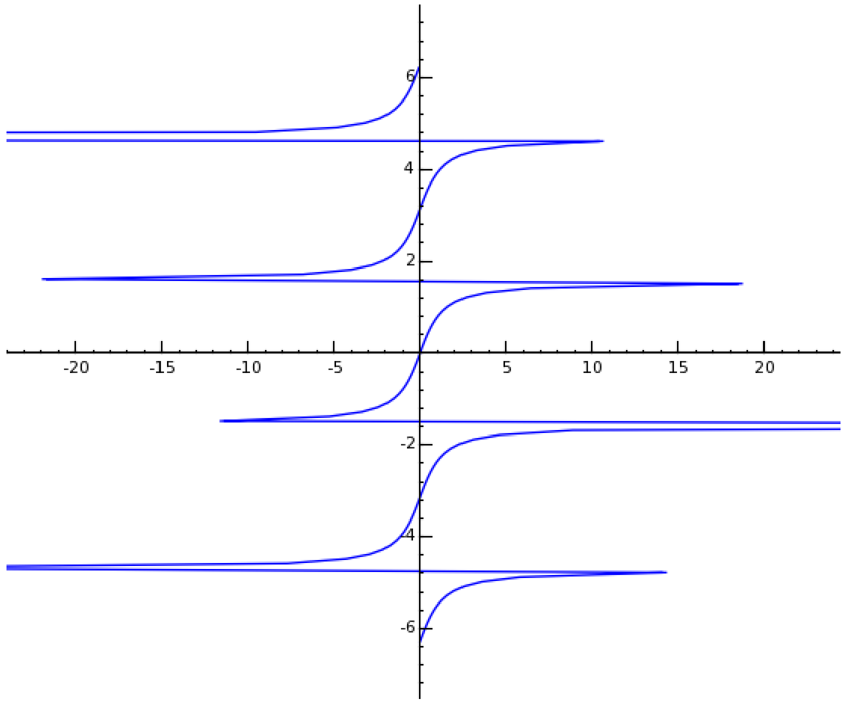

Figure 4.7: The inverse tangent $\tan^{-1} x$ using Sage .

```
-diff(f(t), t, 1)/(f(t)^2 + 1)
sage: arccot(f(t)).diff(t)
-diff(f(t), t, 1)/(f(t)^2 + 1)
sage: f = t^7
sage: arccot(f(t)).diff(t)
-7*t^6/(t^14 + 1)
```

4.29 Differentiation of $\operatorname{arcsec} v$

Let $y = \operatorname{arcsec} v$, so $v = \sec y$. This function is defined for all values of v except those lying between -1 and $+1$, and is seen to be many-valued. To make the function single-valued, y is taken as the smallest angle whose secant is v. This means that if v is positive, we confine ourselves to points on arc AB (Figure 4.9),

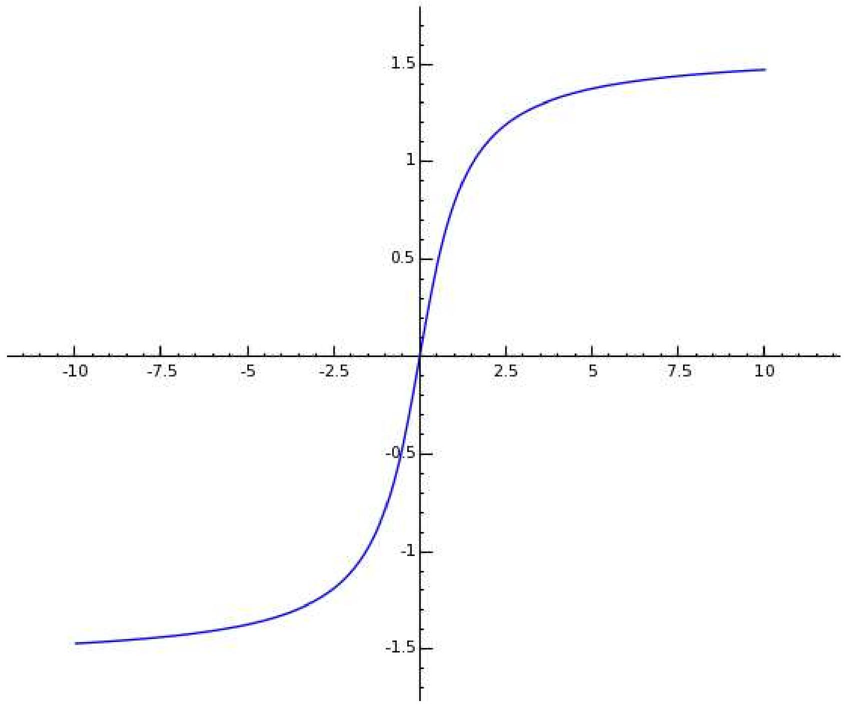

Figure 4.8: The standard branch of $\arctan\ x$ using Sage .

y taking on values between 0 and $\frac{\pi}{2}$ (0 may be included); and if v is negative, we confine ourselves to points on arc DC, y taking on values between $-\pi$ and $-\frac{\pi}{2}$ ($-\pi$ may be included).

Differentiating with respect to y by (4.4), gives $\frac{dv}{dy} = \sec y \tan y$. Therefore $\frac{dy}{dv} = \frac{1}{\sec y \tan y}$, by (4.31). But since v is a function of x, this may be substituted in the formula $\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$, by (4.29).giving

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}\frac{dv}{dx} = \frac{1}{v\sqrt{v^2-1}}\frac{dv}{dx}$$

(since $\sec y = v$, and $\tan y = \sqrt{\sec y - 1} = \sqrt{v^2-1}$, the plus sign of the radical being taken, since $\tan y$ is positive for an values of y between 0 and $\frac{\pi}{2}$ and between $-\pi$ and $-\frac{\pi}{2}$, including 0 and $-\pi$). Therefore,

$$\frac{d}{dx}(\text{arcsec} v) = \frac{\frac{dv}{dx}}{v\sqrt{v^2-1}}$$

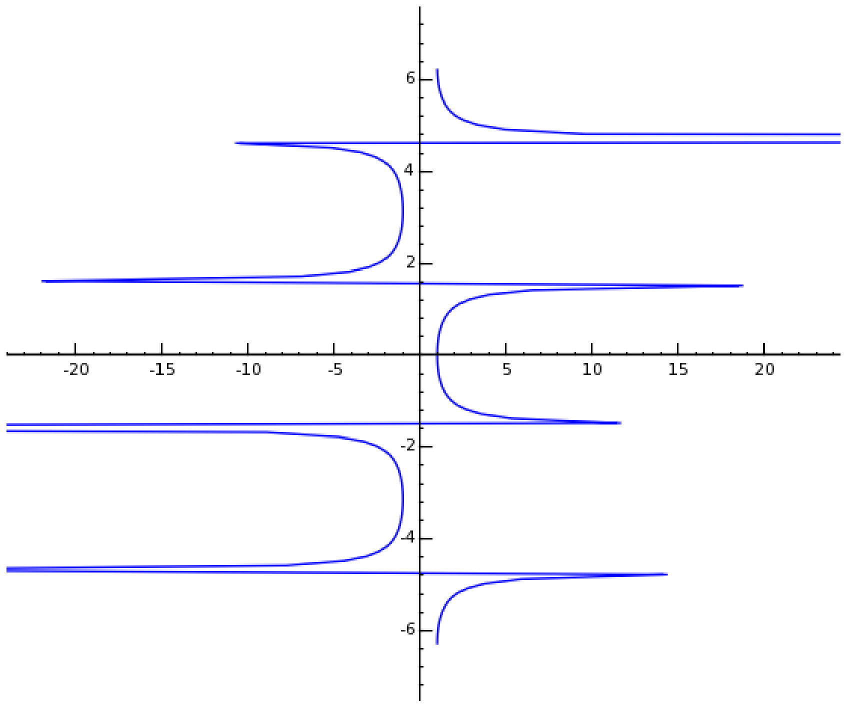

Figure 4.9: The inverse secant $\sec^{-1} x$ using Sage .

(equation (4.25) in §4.1 above).

4.30 Differentiation of $\operatorname{arccsc} v$

Let

$$y = \operatorname{arccsc} v;$$

then

$$v = \csc y.$$

This function is defined for all values of v except those lying between -1 and $+1$, and is seen to be many-valued. To make the function single-valued, y is taken as the smallest angle whose cosecant is v. This means that if v is positive, we confine ourselves to points on the arc AB (Figure 4.11), y taking on values between 0 and

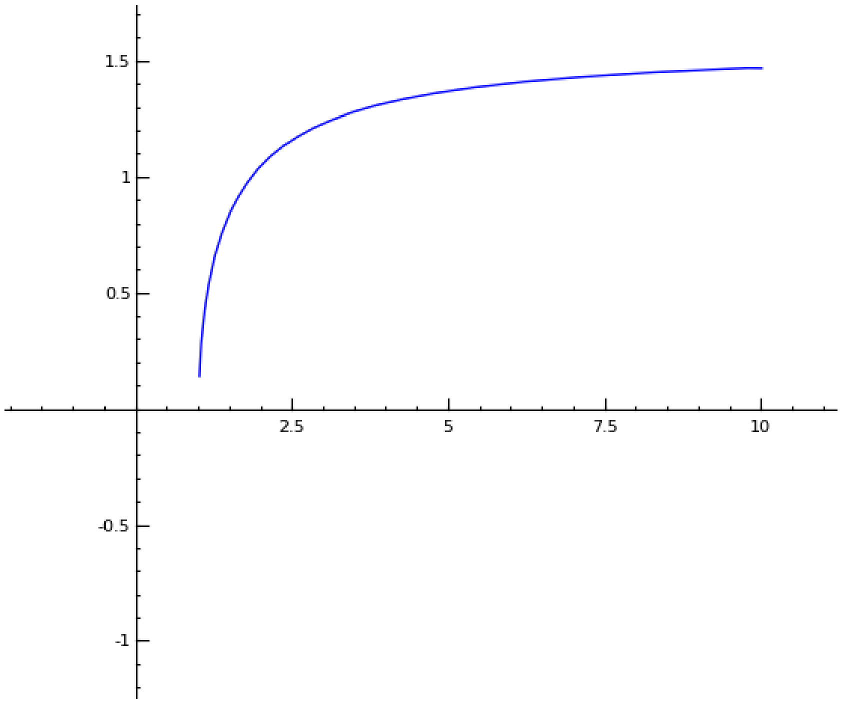

Figure 4.10: The standard branch of $\operatorname{arcsec} x$ using Sage .

$\frac{\pi}{2}$ ($\frac{\pi}{2}$ may be included); and if v is negative, we confine ourselves to points on the arc CD, y taking on values between $-\pi$ and $-\frac{\pi}{2}$ ($-\frac{\pi}{2}$ may be included).

Differentiating with respect to y by 4.20 and following the method of the last section, we get

$$\frac{d}{dx}(\operatorname{arccsc} v) = -\frac{\frac{dv}{dx}}{v\sqrt{v^2-1}}$$

(equation (4.26) in §4.1 above).

4.31 Example

Differentiate the following:

1. $y = \arctan(ax^2)$.

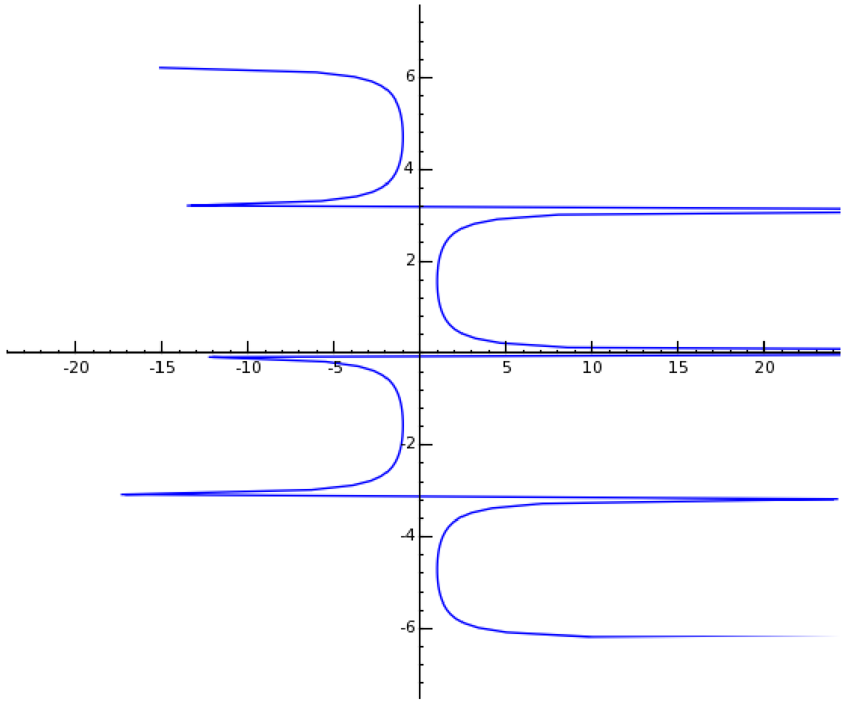

Figure 4.11: The inverse secant function $\operatorname{arccsc} x$ using Sage .

Solution. By (4.23) ($v = ax^2$)

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(ax^2)}{1 + (ax^2)^2} = \frac{2ax}{1 + a^2x^4}.$$

2. $y = \arcsin(3x - 4x^3)$.

   Solution. By 4.21 ($v = 3x - 4x^3$),

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(3x - 4x^3)}{\sqrt{1 - (3x - 4x^3)^2}} = \frac{3 - 12x^2}{\sqrt{1 - 9x^2 + 24x^4 - 16x^6}} = \frac{3}{\sqrt{1 - x^2}}.$$

3. $y = \operatorname{arcsec} \frac{x^2+1}{x^2-1}$.

   Solution. By (4.25) ($v = \frac{x^2+1}{x^2-1}$),

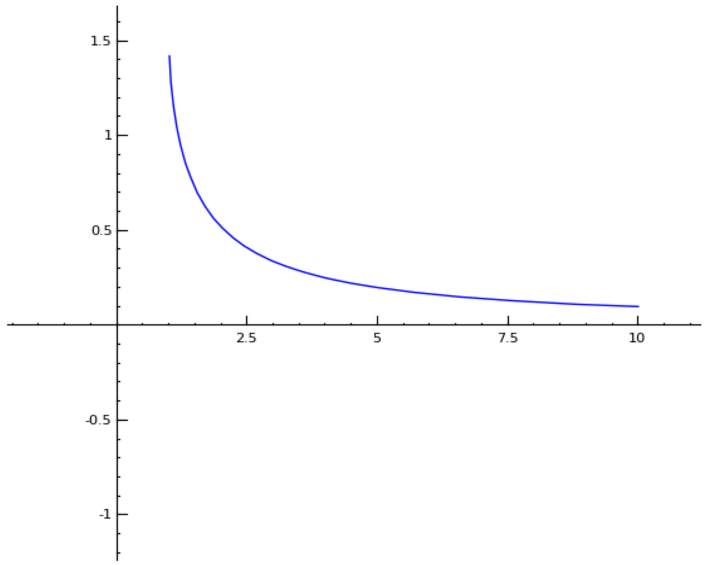

Figure 4.12: The standard branch of $\operatorname{arccsc} x$ using Sage .

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)}{\frac{x^2+1}{x^2-1}\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2 - 1}} = \frac{\frac{(x^2-1)2x-(x^2+1)2x}{(x^2-1)^2}}{\frac{x^2+1}{x^2-1} \cdot \frac{2x}{x^2-1}} = -\frac{2}{x^2+1}.$$

4. $\frac{d}{dx} \arcsin \frac{x}{a} = \frac{1}{\sqrt{a^2-x^2}}$

5. $\frac{d}{dx}\operatorname{arccot}(x^2 - 5) = \frac{-2x}{1+(x^2-5)^2}$

6. $\frac{d}{dx} \arctan \frac{2x}{1-x^2} = \frac{2}{1+x^2}$

7. $\frac{d}{dx}\operatorname{arccsc}\frac{1}{2x^2-1} = \frac{2}{\sqrt{1-x^2}}$

8. $\frac{d}{dx} \operatorname{arcvers} 2x^2 = \frac{2}{\sqrt{1-x^2}}$

9. $\frac{d}{dx} \arctan \sqrt{1 - x} = -\frac{1}{2\sqrt{1-x}(2-x)}$

10. $\frac{d}{dx}\operatorname{arccsc}\frac{3}{2x} = \frac{2}{9-4x^2}$

11. $\frac{d}{dx} \operatorname{arcvers} \frac{2x^2}{1+x^2} = \frac{2}{1+x^2}$

12. $\frac{d}{dx} \arctan \frac{x}{a} = \frac{a}{a^2+x^2}$

13. $\frac{d}{dx} \arcsin \frac{x+1}{\sqrt{2}} = \frac{1}{\sqrt{1-2x-x^2}}$

14. $f(x) = x\sqrt{a^2 - x^2} + a^2 \arcsin \frac{x}{a}$ Ans: $f'(x) = 2\sqrt{a^2 - x^2}$

15. $f(x) = \sqrt{a^2 - x^2} + a \arcsin \frac{x}{a}$ Ans: $f'(x) = \left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}$

16. $x = r \operatorname{arcvers} \frac{y}{r} - \sqrt{2ry - y^2}$ Ans: $\frac{dx}{dy} = \frac{y}{\sqrt{2ry-y^2}}$

17. $\theta = \arcsin(3r - 1)$ Ans: $\frac{d\theta}{dr} = \frac{3}{\sqrt{6r-9r^2}}$

18. $\phi = \arctan \frac{r+a}{1-ar}$ Ans: $\frac{d\phi}{dr} = \frac{1}{1+r^2}$

19. $s = \operatorname{arcsec} \frac{1}{\sqrt{1-t^2}}$ Ans: $\frac{ds}{dt} = \frac{1}{\sqrt{1-t^2}}$

20. $\frac{d}{dx}(x \arcsin x) = \arcsin x + \frac{x}{\sqrt{1-x^2}}$

21. $\frac{d}{d\theta}(\tan \theta \arctan \theta) = \sec^2 \theta \arctan \theta \frac{\tan \theta}{1+\theta^2}$

22. $\frac{d}{dt}[\log(\arccos t)] = -\frac{1}{\arccos t\sqrt{1-t^2}}$

23. $f(y) = \arccos(\log y)$ Ans: $f'(y) = -\frac{1}{y\sqrt{1-(\log y)^2}}$

24. $f(\theta) = \arcsin \sqrt{\sin \theta}$ Ans: $f'(\theta) = \frac{1}{2}\sqrt{1 + \csc \theta}$

25. $f(\phi) = \arctan \sqrt{\frac{1-\cos \phi}{1+\cos \phi}}$ Ans: $f'(\phi) = \frac{1}{2}$

26. $p = e^{\arctan q}$ Ans: $\frac{dp}{dq} = \frac{e^{\arctan q}}{1+q^2}$

27. $u = \arctan \frac{e^v - e^{-v}}{2}$ Ans: $\frac{du}{dv} = \frac{2}{e^v + e^{-v}}$

28. $s = \arccos \frac{e^t - e^{-t}}{e^t + e^{-t}}$ Ans: $\frac{ds}{dt} = -\frac{2}{e^v + e^{-v}}$

29. $y = x^{\arcsin x}$ Ans: $y' = x^{\arcsin x} \left(\frac{\arcsin x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right)$

30. $y = e^{x^x} \arctan x$ Ans: $y' = e^{x^x} \left[\frac{1}{1+x^2} + x^x \arctan x(1 + \log x)\right]$

31. $y = \arcsin(\sin x)$ Ans: $y' = 1$

32. $y = \arctan \frac{4\sin x}{3+5\cos x}$ Ans: $y' = \frac{4}{5+3\cos x}$

33. $y = \text{arccot}\frac{a}{x} + \log\sqrt{\frac{x-a}{x+a}}$ Ans: $y' = \frac{2ax^2}{x^4-a^4}$

34. $y = \log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} - \frac{1}{2}\arctan\ x$ Ans: $y' = \frac{x^2}{1-x^4}$

35. $y = \sqrt{1-x^2}\arcsin\ x - x$ Ans: $y' = -\frac{x\arcsin\ x}{\sqrt{1-x^2}}$

36. Compute the following derivatives:

$(a)\ \frac{d}{dx}\arcsin 2x^2$	$(f)\ \frac{d}{dt}t^3\arcsin\frac{t}{3}$	$(k)\ \frac{d}{dy}\arcsin\sqrt{1-y^2}$
$(b)\ \frac{d}{dx}\arctan a^2x$	$(g)\ \frac{d}{dt}e^{\arctan at}$	$(l)\ \frac{d}{dz}\arctan(\log 3az)$
$(c)\ \frac{d}{dx}\text{arcsec}\frac{x}{a}$	$(h)\ \frac{d}{d\phi}\tan\phi^2\cdot\arctan\phi^{\frac{1}{2}}$	$(m)\ \frac{d}{ds}(a^2+s^2)\text{arcsec}\frac{s}{2}$
$(d)\ \frac{d}{dx}x\arccos x$	$(i)\ \frac{d}{d\theta}\arcsin a^\theta$	$(n)\ \frac{d}{d\alpha}\text{arccot}\frac{2\alpha}{3}$
$(e)\ \frac{d}{dx}x^2\text{arccot}ax$	$(j)\ \frac{d}{d\theta}\arctan\sqrt{1+\theta^2}$	$(o)\ \frac{d}{dt}\sqrt{1-t^2}\arcsin t$

Formulas (4.29) for differentiating a function of a function, and (4.31) for differentiating inverse junctions, have been added to the list of formulas at the beginning of this chapter as (4.27) and (4.28) respectively.

In the next eight examples, first find $\frac{dy}{dv}$ and $\frac{dv}{dx}$ by differentiation and then substitute the results in $\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{dx}$ (by (4.27)) to find $\frac{dy}{dx}$. (As was pointed out in §4.11, it might be possible to eliminate v between the two given expressions so as to find y directly as a function of x, but in most cases the above method is to be preferred.)

In general our results should be expressed explicitly in terms of the independent variable; that is, $\frac{dy}{dx}$ in terms of x, $\frac{dx}{dy}$ in terms of y, $\frac{d\phi}{d\theta}$ in terms of θ, etc.

37. $y = 2v^2 - 4$, $v = 3x^2 + 1$.

$\frac{dy}{dv} = 4v$; $\frac{dv}{dx} = 6x$; substituting in (4.27), $\frac{dy}{dx} = 4v\cdot 6x = 24x(3x^2+1)$.

38. $y = \tan 2v$, $v = \arctan(2x-1)$.
$\frac{dy}{dv} = 2\sec^2 2v$; $\frac{dv}{dx} = \frac{1}{2x^2-2x+1}$; substituting in (4.27),

$$\frac{dy}{dx} = \frac{2\sec^2 2v}{2x^2-2x+1} = 2\frac{\tan^2 2v+1}{2x^2-2x+1} = \frac{2x^2-2x+1}{2(x-x^2)^2}$$

(since $v = \arctan(2x-1)$, $\tan v = 2x-1$, $\tan 2v = \frac{2x-1}{2x-2x^2}$).

39. $y = 3v^2 - 4v + 5, v = 2x^3 - 5$ Ans: $\frac{dy}{dx} = 72x^5 - 204x^2$

40. $y = \frac{2v}{3v-2}, v = \frac{x}{2x-1}$ Ans: $\frac{dy}{dx} = \frac{4}{(x-2)^2}$

41. $y = \log(a^2 - v^2)$ Ans: $\frac{dy}{dx} = -2\tan x$

42. $y = \arctan(a+v), v = e^x$ Ans: $\frac{dy}{dx} = \frac{e^x}{1+(a+e^x)^2}$

43. $r = e^{2s} + e^s, s = \log(t - t^2)$ Ans: $\frac{dr}{dt} = 4t^3 - 6t^2 + 1$

In the following examples first find $\frac{dx}{dy}$ by differentiation and then substitute in

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \text{by (4.28)}$$

to find $\frac{dy}{dx}$.

44. $x = y\sqrt{1+y}$ Ans: $\frac{dy}{dx} = \frac{2\sqrt{1+y}}{2+3y} = \frac{2x}{2y+3y^2}$

45. $x = \sqrt{1+\cos y}$ Ans: $\frac{dy}{dx} = -\frac{2\sqrt{1+\cos y}}{\sin y} = -\frac{2}{\sqrt{2-x^2}}$

46. $x = \frac{y}{1+\log y}$ Ans: $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$

47. $x = a\log\frac{a+\sqrt{a^2-y^2}}{y}$ Ans: $\frac{dy}{dx} = -\frac{y\sqrt{a^2-y^2}}{a^2}$

48. $x = r\,\text{arcvers}\,\frac{y}{r} - \sqrt{2ry - y^2}$ Ans: $\frac{dy}{dx} = \sqrt{\frac{2r-y}{y}}$

49. Show that the geometrical significance of (4.28) is that the tangent makes complementary angles with the two coordinate axes.

4.32 Implicit functions

When a relation between x and y is given by means of an equation not solved for y, then y is called an *implicit function* of x. For example, the equation

$$x^2 - 4y = 0$$

defines y as an implicit function of x. Evidently x is also defined by means of this equation as an implicit function of y. Similarly,

$$x^2 + y^2 + z^2 - a^2 = 0$$

defines anyone of the three variables as an implicit function of the other two.

It is sometimes possible to solve the equation defining an implicit function for one of the variables and thus change it into an explicit function. For instance, the above two implicit functions may be solved for y, giving $y = \frac{x^2}{4}$ and $y = \pm\sqrt{a^2 - x^2 - z^2}$; the first showing y as an explicit function of x, and the second as an explicit function of x and z. In a given case, however, such a solution may be either impossible or too complicated for convenient use.

The two implicit functions used in this section for illustration may be respectively denoted by $f(x, y) = 0$ and $F(x, y, z) = 0$.

4.33 Differentiation of implicit functions

When y is defined as an implicit function of x by means of an equation in the form

$$f(x, y) = 0, \tag{4.33}$$

it was explained in the last section how it might be inconvenient to solve for y in terms of x; that is, to find y as an explicit function of x so that the formulas we have deduced in this chapter may be applied directly. Such, for instance, would be the case for the equation

$$ax^6 + 2x^3y - y^7x - 10 = 0. \tag{4.34}$$

We then follow the rule:

Differentiate, regarding y as a function of x, and put the result equal to zero [7]. That is,

[7] Only corresponding values of x and y which satisfy the given equation may be substituted in the derivative.

$$\frac{d}{dx}f(x,y) = 0. \tag{4.35}$$

Let us apply this rule in finding $\frac{dy}{dx}$ from (4.34): by (4.35),

$$\frac{d}{dx}(ax^6 + 2x^3y - y^7x - 10) = 0,$$
$$\frac{d}{dx}(ax^6) + \frac{d}{dx}(2x^3y) - \frac{d}{dx}(y^7x) - \frac{d}{dx}(10) = 0;$$
$$6ax^5 + 2x^3\frac{dy}{dx} + 6x^2y - y^7 - 7xy^6\frac{dy}{dx} = 0;$$
$$(2x^3 - 7xy^6)\frac{dy}{dx} = y^7 - 6ax^5 - 6x^2y;$$
$$\frac{dy}{dx} = \frac{y^7 - 6ax^5 - 6x^2y}{2x^3 - 7xy^6}.$$

This is the final answer.

The student should observe that in general the result will contain both x and y.

4.34 Exercises

Differentiate the following by the above rule:

1. $y^2 = 4px$ Ans: $\frac{dy}{dx} = \frac{2p}{y}$
2. $x^2 + y^2 = r^2$ Ans: $\frac{dy}{dx} = -\frac{x}{y}$
3. $b^2x^2 + a^2y^2 = a^2b^2$ Ans: $\frac{dy}{dx} = -\frac{b^2x}{a^2y}$
4. $y^3 - 3y + 2ax = 0$ Ans: $\frac{dy}{dx} = \frac{2a}{3(1-y^2)}$
5. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ Ans: $\frac{dy}{dx} = -\sqrt{\frac{y}{x}}$
6. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ Ans: $\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}$
7. $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$ Ans: $\frac{dy}{dx} = -\frac{3b^{\frac{2}{3}}xy^{\frac{1}{3}}}{a^2}$
8. $y^2 - 2xy + b^2 = 0$ Ans: $\frac{dy}{dx} = \frac{y}{y-x}$

9. $x^3 + y^3 - 3axy = 0$ Ans: $\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$

10. $x^y = y^x$ Ans: $\frac{dy}{dx} = \frac{y^2-xy\log\, y}{x^2-xy\log\, x}$

11. $\rho^2 = a^2 \cos 2\theta$ Ans: $\frac{d\rho}{d\theta} = -\frac{a^2 \sin 2\theta}{\rho}$

12. $\rho^2 \cos\, \theta = a^2 \sin\, 3\theta$ Ans: $\frac{d\rho}{d\theta} = \frac{3a^2 \cos\, 3\theta + \rho^2 \sin\, \theta}{2\rho \cos\, \theta}$

13. $\cos(uv) = cv$ Ans: $\frac{du}{dv} = \frac{c+u\sin(uv)}{-v\sin(uv)}$

14. $\theta = \cos(\theta + \phi)$ Ans: $\frac{d\theta}{d\phi} = -\frac{\sin(\theta+\phi)}{1+\sin(\theta+\phi)}$

15. Find $\frac{dy}{dx}$ from the following equations:

$(a)\ x^2 = ay$	$(f)\ xy + y^2 + 4x = 0$	$(k)\ \tan\, x + y^3 = 0$
$(b)\ x^2 + 4y^2 = 16$	$(g)\ yx^2 - y^3 = 5$	$(l)\ \cos\, y + 3x^2 = 0$
$(c)\ b^2x^2 - a^2y^2 = a^2b^2$	$(h)\ x^2 - 2x^3 = y^3$	$(m)\ x\cot\, y + y = 0$
$(d)\ y^2 = x^3 + a$	$(i)\ x^2y^3 + 4y = 0$	$(n)\ y^2 = \log\, x$
$(e)\ x^2 - y^2 = 16$	$(j)\ y^2 = \sin\, 2x$	$(o)\ e^{x^2} + 2y^3 = 0$

16. A race track has the form of the circle $x^2 + y^2 = 12500$. The x-axis and y-axis are east and north respectively, and the unit is 1 meter. If a runner starts east at the extreme north point, in what direction will he be going

    (a) when $25\sqrt{10}$ m east of OY? Ans. Southeast or southwest.
    (b) when $25\sqrt{10}$ m north of OX? Ans. Southeast or northeast.
    (c) when 30 rods west of OY? Ans. E. 36^o 52' 12" N. or W. 36^o 52' 12" N.
    (d) when 200 m south of OX?
    (e) when 50 m east of OY?

17. An automobile course is elliptic in form, the major axis being 6 miles long and running east and west, while the minor axis is 2 miles long. If a car starts north at the extreme east point of the course, in what direction will the car be going

    (a) when 2 miles west of the starting point?

    (b) when $1/2$ mile north of the starting point?

4.35 Miscellaneous Exercises

Differentiate the following functions:

1. $\arcsin\sqrt{1-4x^2}$ — Ans: $\frac{-2}{\sqrt{1-4x^2}}$
2. xe^{x^2} — Ans: $e^{x^2}(2x^2+1)$
3. $\log\sin\frac{v}{2}$ — Ans: $\frac{1}{2}\cot\frac{v}{2}$
4. $\arccos\frac{a}{y}$ — Ans: $\frac{a}{y\sqrt{y^2-a^2}}$
5. $\frac{x}{\sqrt{a^2-x^2}}$ — Ans: $\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$
6. $\frac{x}{1+\log x}$ — Ans: $\frac{\log x}{(1+\log x)^2}$
7. $\log\sec(1-2x)$ — Ans: $-2\tan(1-2x)$
8. x^2e^{2-3x} — Ans: $xe^{2-3x}(2-3x)$
9. $\log\sqrt{\frac{1-\cos t}{1+\cos t}}$ — Ans: $\csc t$

   Here's how Sage tackles this one:

   Sage

```
sage: t = var("t")
sage: diff(log(sqrt((1-cos(t))/(1+cos(t)))),t)
(cos(t) + 1)*(sin(t)/(cos(t) + 1)
 + (1 - cos(t))*sin(t)/(cos(t) + 1)^2)/(2*(1 - cos(t)))
sage: diff(log(sqrt((1-cos(t))/(1+cos(t)))),t).simplify_trig()
-sin(t)/(cos(t)^2 - 1)
```

   Since $\cos(t)^2 - 1 = -\sin(t)^2$, the result returned by Sage agrees with the answer given.

10. $\arcsin\sqrt{\frac{1}{2}(1-\cos x)}$ — Ans: $\frac{1}{2}$, for $x > 0$; $-\frac{1}{2}$, for $x < 0$.

   Here's how Sage tackles this one:

Sage

```
sage: diff(arcsin(sqrt((1-cos(x))/2)),x)
sin(x)/(2*sqrt(2)*sqrt(1 - (1 - cos(x))/2)*sqrt(1 - cos(x)))
sage: diff(arcsin(sqrt((1-cos(x))/2)),x).simplify_trig()
sin(x)/(2*sqrt(1 - cos(x))*sqrt(cos(x) + 1))
sage: diff(arcsin(sqrt((1-cos(x))/2)),x).simplify_radical()
sin(x)/(2*sqrt(1 - cos(x))*sqrt(cos(x) + 1))
```

Here we see again that Sage does not simplify the result down to the final answer. Nonetheless, `simplify_trig` is useful. Since

$$\sqrt{1-\cos(x)}\sqrt{\cos(x)+1} = \sqrt{1-\cos(x)^2} = \sqrt{\sin(x)^2} = \pm\sin(x),$$

we see the answer given is correct (at least for the interval $-\pi < x < \pi$).

11. $\arctan \frac{2s}{\sqrt{s^2-1}}$ Ans: $\frac{2}{(1-5s^2)\sqrt{s^2-1}}$

12. $(2x-1)\sqrt[3]{\frac{2}{1+x}}$ Ans: $\frac{7+4x}{3(1+x)}\sqrt[3]{\frac{2}{1+x}}$

13. $\frac{x^3 \arcsin x}{3} + \frac{(x^2+2)\sqrt{1-x^2}}{9}$ Ans: $x^2 \arcsin x$

14. $\tan^2 \frac{\theta}{3} + \log \sec^2 \frac{\theta}{3}$

15. $\arctan \frac{1}{2}(e^{2x} + e^{-2x})$

16. $\left(\frac{3}{x}\right)^{2x}$

17. $x^{\tan x}$

18. $\frac{(x+2)^{\frac{1}{3}}(x^2-1)^{\frac{2}{5}}}{x^{\frac{3}{2}}}$

19. $e^{\sec(1-3x)}$

20. $\arctan \sqrt{1-x^2}$

21. $\frac{z^2}{\cos z}$

22. $e^{\tan x^2}$

23. $\log \sin^2 \frac{1}{2}\theta$

24. $e^{ax} \log \sin\ ax$

Here's how Sage tackles this Exercise:

Sage

```
sage: a = var("a")
sage: diff(exp(a*x)*log(sin(a*x)),x)
a*e^(a*x)*log(sin(a*x)) + a*e^(a*x)*cos(a*x)/sin(a*x)
```

25. $\sin\ 3\phi \cos\ \phi$

26. $\frac{a}{2\sqrt{(b-cx^n)^m}}$

27. $\frac{m+x}{1+m^2} \cdot \frac{e^{m \arctan x}}{\sqrt{1+x^2}}$

28. $\tan^2 x - \log \sec^2 x$

29. $\frac{3\log(2\cos x + 3\sin x) + 2x}{13}$

30. $\operatorname{arccot} \frac{a}{x} + \log \sqrt{\frac{x-a}{x+a}}$

31. $(\log \tan(3 - x^2)^3$

32. $\frac{2 - 3t^{\frac{1}{2}} + 4t^{\frac{1}{3}} + t^2}{t}$

33. $\frac{(1+x)(1-2x)(2+x)}{(3+x)(2-3x)}$

34. $\arctan(\log\ 3x)$

Here's how Sage tackles this one:

Sage

```
sage: diff(arctan(log(3*x)),x)
1/(x*(log(3*x)^2 + 1))
```

35. $\sqrt[3]{(b-ax^m)^n}$

    Here's how Sage tackles this one:

Sage

```
sage: a,b,m,n = var("a,b,m,n")
sage: diff((b-a*x^m)^(n/3),x)
-a*m*n*x^(m - 1)*(b - a*x^m)^(n/3 - 1)/3
```

36. $\log\sqrt{(a^2-bx^2)^m}$

37. $\log\sqrt{\frac{y^2+1}{y^2-1}}$

38. $e^{\operatorname{arcsec} 2\theta}$

39. $\sqrt{\frac{(2-3x)^3}{1+4x}}$

40. $\frac{\sqrt[3]{a^2-x^2}}{\cos x}$

41. $e^x \log\sin\ x$

42. $\arcsin\frac{x}{\sqrt{1+x^2}}$

43. $\arctan\ a^x$

44. $a^{\sin^2 mx}$

    Here's how Sage solves this one:

Sage

```
sage: a,m = var("a,m")
sage: diff(a^(sin(m*x)^2),x)
2*a^sin(m*x)^2*log(a)*m*cos(m*x)*sin(m*x)
```

45. $\cot^3(\log\ ax)$

46. $(1-3x^2)e^{\frac{1}{x}}$

47. $\log\frac{\sqrt{1-x^2}}{\sqrt[3]{1+x^3}}$

CHAPTER

FIVE

Simple applications of the derivative

5.1 Direction of a curve

It was shown in §3.9, that if

$$y = f(x)$$

is the equation of a curve (see Figure 5.2), then

$$\frac{dy}{dx} = \tan\tau = \text{slope of line tangent to the curve at any point } P.$$

Example 5.1.1. A group of hikers are climbing a hill whose height is described by the graph of

$$h(x) = -x^4 + 29x^3 - 290x^2 + 1200x.$$

Show that the hikers are climbing downhill when $x = 5$.

This can be verified "by hand" by computing $h'(5)$ and checking that it is negative (see also the plot in Figure 5.1), or using Sage :

Sage

```
sage: x = var("x")
sage: h = -x^4 + 29*x^3 - 290*x^2 + 1200*x
sage: Dh = h.diff(); Dh
-4*x^3 + 87*x^2 - 580*x + 1200
```

```
sage: Dh(5)
-25
sage: plot(h,0,15)
```

The output of the above plot command is in Figure 5.1.

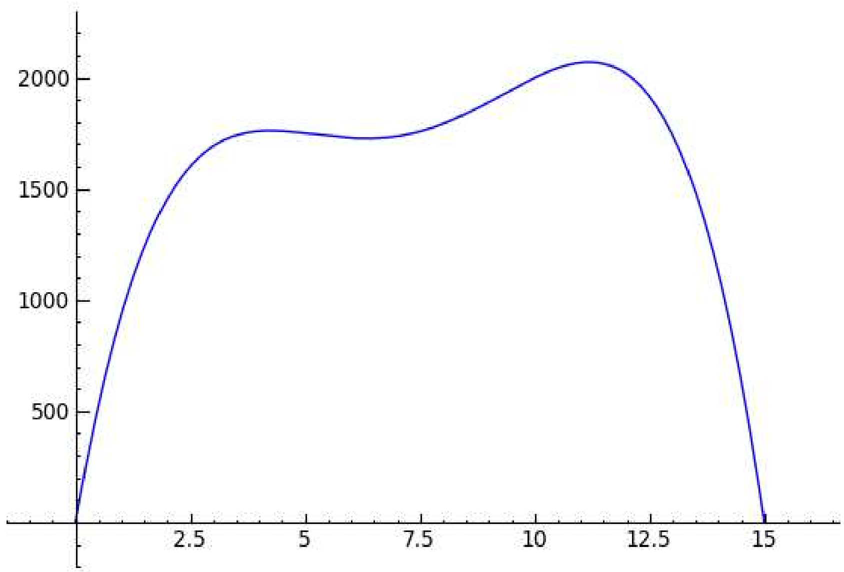

Figure 5.1: The graph of $y = -x^4 + 29x^3 - 290x^2 + 1200x$.

The *direction* of a curve at any point is defined to be the same as the direction of the line tangent to the curve at that point. From this it follows at once that

$$\frac{dy}{dx} = \tan\ \tau = \text{slope of the curve at any point } P.$$

At a particular point whose coordinates are known we write

$$\left[\frac{dy}{dx}\right]_{x=x_1,y=y_1} = \text{slope of the curve (or tangent) at point } (x_1, y_1).$$

At points such as D or F or H where the curve (or tangent) is parallel to the x-axis, $\tau = 0^o$, therefore $\frac{dy}{dx} = 0$ (see Figure 5.2 for the notation).

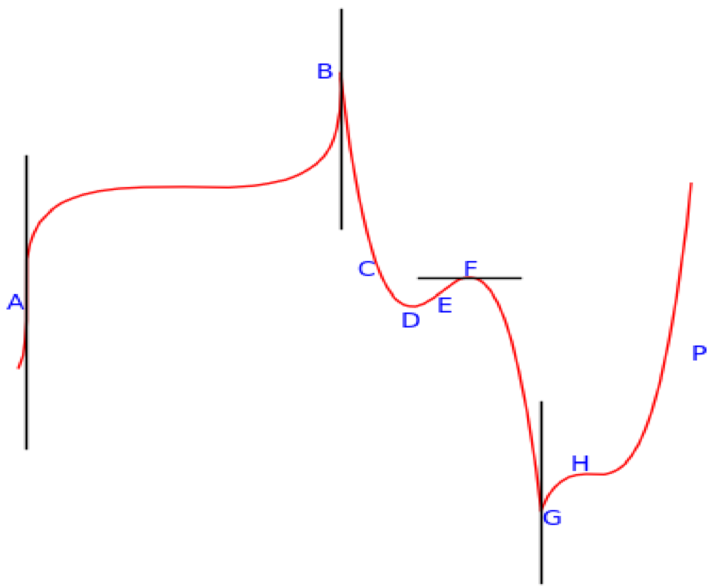

Figure 5.2: The derivative = slope of line tangent to the curve.

At points such as A, B, G, where the curve (or tangent) is perpendicular to the x-axis, $\tau = 90^o$, therefore $\frac{dy}{dx} = \infty$.
At points such as E, where the curve is rising (moving from left to right on curve),

$$\tau \;=\; \text{an acute angle; therefore } \frac{dy}{dx} \;=\; \text{a positive number.}$$

The curve (or tangent) has a positive slope

- to the left of B,
- between D and F, and
- to the right of G,

in Figure 5.2. At points such as C, where the curve is falling,

$$\tau \;=\; \text{an obtuse angle; therefore } \frac{dy}{dx} \;=\; \text{a negative number.}$$

The curve (or tangent) has a negative slope between B and D, and also between F and G.

Example 5.1.2. Given the curve $y = \frac{x^3}{3} - x^2 + 2$ (see Figure 5.3).
(a) Find τ when $x = 1$.
(b) Find τ when $x = 3$.
(c) Find the points where the curve is parallel to the x-axis.
(d) Find the points where $\tau = 45^o$.
(e) Find the points where the curve is parallel to the line $2x - 3y = 6$.

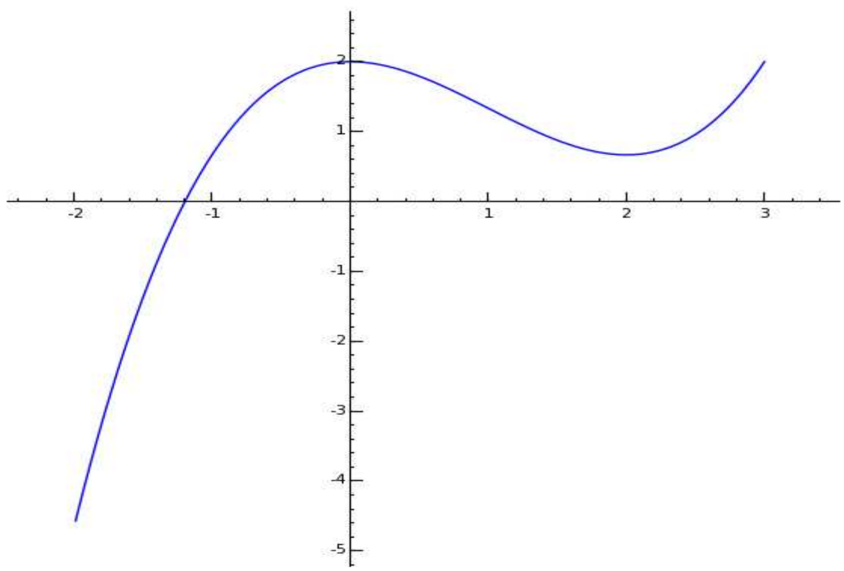

Figure 5.3: The graph of $y = \frac{x^3}{3} - x^2 + 2$.

Differentiating, $\frac{dy}{dx} = x^2 - 2x$ = slope at any point.
(a) $\tan\tau = \left[\frac{dy}{dx}\right]_{x=1} = 1 - 2 = -1$; therefore $\tau = 135^o = 3\pi/4$.
(b) $\tan\tau = \left[\frac{dy}{dx}\right]_{x=3} = 9 - 6 = 3$; therefore $\tau = \arctan 3 = 1.249....$
(c) $\tau = 0^o$, $\tan\tau = \frac{dy}{dx} = 0$; therefore $x^2 - 2x = 0$. Solving this equation, we find that $x = 0$ or 2, giving points C and D where the curve (or tangent) is parallel to the x-axis.

(d) $\tau = 45^o$, $\tan\tau = \frac{dy}{dx} = 1$; therefore $x^2 - 2x = 1$. Solving, we get $x = 1 \pm \sqrt{2}$, giving two points where the slope of the curve (or tangent) is unity.

(e) Slope of line = $\frac{2}{3}$; therefore $x^2 - 2x = \frac{2}{3}$. Solving, we get $x = 1 \pm \sqrt{\frac{5}{3}}$, giving points E and F where curve (or tangent) is parallel to $2x - 3y = 6$.

The *angle between two curves* at a common point will be the angle between their tangents at that point. This definition is analogous to the fact that the direction of curve at any point is defined to be the direction of its tangent at that point.

Example 5.1.3. Find the angle of intersection of the circles

(A) $x^2 + y^2 - 4x = 1$,

(B) $x^2 + y^2 - 2y = 9$.

Solution. Solving simultaneously, we find the points of intersection to be $(3, 2)$ and $(1, -2)$. This can be verified "by hand" or using the Sage `solve` command:

Sage

```
sage: x = var("x")
sage: y = var("y")
sage: F = x^2 + y^2 - 4*x - 1
sage: G = x^2 + y^2 - 2*y - 9
sage: solve([F == 0,G == 0],x,y)
[[x == 1, y == -2], [x == 3, y == 2]]
```

Using (A), formulas in §4.33 give $\frac{dy}{dx} = \frac{2-x}{y}$. Using (B), formulas in §4.33 give $\frac{dy}{dx} = \frac{x}{1-y}$. Therefore,

$$\left[\frac{2-x}{y}\right]_{x=3,y=2} = -\frac{1}{2} = \text{ slope of tangent to (A) at } (3, 2).$$

$$\left[\frac{x}{1-y}\right]_{x=3,y=2} = -3 = \text{ slope of tangent to (B) at } (3, 2).$$

We can check this using the commands

Sage

```
sage: x = var("x")
sage: y = function("y",x)
```

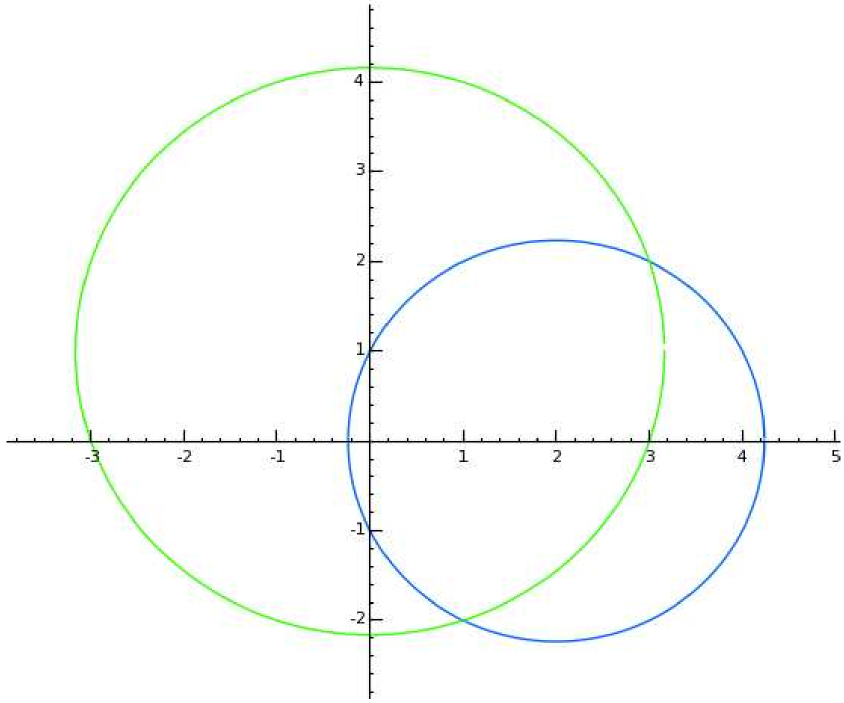

Figure 5.4: The graphs of $x^2 + y^2 - 4x = 1$, $x^2 + y^2 - 2y = 9$.

```
sage: F = x^2 + y^2 - 4*x - 1
sage: F.diff(x)
2*y(x)*diff(y(x), x, 1) + 2*x - 4
sage: solve(F.diff(x) == 0, diff(y(x), x, 1))
[diff(y(x), x, 1) == (2 - x)/y(x)]
sage: G = x^2 + y^2 - 2*y - 9
sage: G.diff(x)
2*y(x)*diff(y(x), x, 1) - 2*diff(y(x), x, 1) + 2*x
sage: solve(G.diff(x) == 0, diff(y(x), x, 1))
[diff(y(x), x, 1) == -x/(y(x) - 1)]
```

The formula for finding the angle between two lines whose slopes are m_1 and m_2 is

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2},$$

by item 55, §12.1. Substituting, $\tan\theta = \frac{-\frac{1}{2}+3}{1+\frac{3}{2}} = 1$; therefore $\theta = \pi/4 = 45^o$. This is also the angle of intersection at the point $(1, -2)$.

5.2 Exercises

The corresponding figure should be drawn in each of the following examples:

1. Find the slope of $y = \frac{x}{1+x^2}$ at the origin.

   Ans. $1 = \tan\,\tau$.

2. What angle does the tangent to the curve $x^2y^2 = a^3(x + y)$ at the origin make with the x-axis?

   Ans. $\tau = 135^o = 3\pi/4$.

3. What is the direction in which the point generating the graph of $y = 3x^2 - x$ tends to move at the instant when $x = 1$?

   Ans. Parallel to a line whose slope is 5.

4. Show that $\frac{dy}{dx}$ (or slope) is constant for a straight line.

5. Find the points where the curve $y = x^3 - 3x^2 - 9x + 5$ is parallel to the x-axis.

   Ans. $x = 3$, $x = -1$.

6. At what point on $y^2 = 2x^3$ is the slope equal to 3?

   Ans. $(2, 4)$.

7. At what points on the circle $x^2 + y^2 = r^2$ is the slope of the tangent line equal to $-\frac{3}{4}$?

   Ans. $\left(\pm\frac{3r}{5}, \pm\frac{4r}{5}\right)$

8. Where will a point moving on the parabola $y = x^2 - 7x + 3$ be moving parallel to the line $y = 5x + 2$?

   Ans. $(6, -3)$.

9. Find the points where a particle moving on the circle $x^2 + y^2 = 169$ moves perpendicular to the line $5x + 12y = 60$.

   Ans. $(\pm 12, \mp 5)$.

10. Show that all the curves of the system $y = \log\, kx$ have the same slope; i.e. the slope is independent of k.

11. The path of the projectile from a mortar cannon lies on the parabola $y = 2x - x^2$; the unit is 1 mile, the x-axis being horizontal and the y-axis vertical, and the origin being the point of projection. Find the direction of motion of the projectile

   (a) at instant of projection;

   (b) when it strikes a vertical cliff $\frac{3}{2}$ miles distant.

   (c) Where will the path make an inclination of $45^o = \pi/4$ with the horizontal?

   (d) Where will the projectile travel horizontally?

   Ans. (a) $\arctan\, 2$; (b) $135^o = 3\pi/4$; (c) $(\frac{1}{2}, \frac{3}{4})$; (d) $(1, 1)$.

12. If the cannon in the preceding example was situated on a hillside of inclination $45^o = \pi/4$, at what angle would a shot fired up strike the hillside?

   Ans. $45^o = \pi/4$.

13. At what angles does a road following the line $3y - 2x - 8 = 0$ intersect a railway track following the parabola $y^2 = 8x$?

   Ans. $\arctan \frac{1}{5}$, and $\arctan \frac{1}{8}$.

14. Find the angle of intersection between the parabola $y^2 = 6x$ and the circle $x^2 + y^2 = 16$.

   Ans. $\arctan \frac{5}{3}\sqrt{3}$.

15. Show that the hyperbola $x^2 - y^2 = 5$ and the ellipse $\frac{x^2}{18} + \frac{y^2}{8} = 1$ intersect at right angles.

16. Show that the circle $x^2 + y^2 = 8ax$ and the cissoid $y^2 = \frac{x^3}{2a-x}$

   (a) are perpendicular at the origin;

   (b) intersect at an angle of $45^o = \pi/4$ at two other points.

17. Find the angle of intersection of the parabola $x^2 = 4ay$ and the Witch of Agnesi, $y = \frac{8a^3}{x^2+4a^2}$.

    Ans. $\arctan 3 = 71^o 33' = 1.249....$

    For the interesting history of this curve, see for example `http://en.wikipedia.org/wiki/Witch_of_Agnesi`.

18. Show that the tangents to the Folium of Descartes, $x^3 + y^3 = 3axy$ at the points where it meets the parabola $y^2 = ax$ are parallel to the y-axis.

    For some history of this curve, see for example `http://en.wikipedia.org/wiki/Folium_of_Descartes`.

19. At how many points will a particle moving on the curve $y = x^3 - 2x^2 + x - 4$ be moving parallel to the x-axis? What are the points?

    Ans. Two; at $(1, -4)$ and $(\frac{1}{3}, -\frac{104}{27})$.

20. Find the angle at which the parabolas $y = 3x^2 - 1$ and $y = 2x^2 + 3$ intersect.

    Ans. $\arctan \frac{4}{97}$.

21. Find the relation between the coefficients of the conics $a_1x^2 + b_1y^2 = 1$ and $a_2x^2 + b_2y^2 = 1$ when they intersect at right angles.

    Ans. $\frac{1}{a_1} - \frac{1}{b_1} = \frac{1}{b_2} - \frac{1}{b_2}$.

5.3 Equations of tangent and normal lines

This section will discuss equations of tangent and normal lines, lengths of subtangent and subnormal, and rectangular coordinates.

The equation of a straight line passing through the point (x_1, y_1) and having the slope m is

$$y - y_1 = m(x - x_1)$$

(this is item 54, §12.1).

If this line is tangent to the curve $y = f(x)$ at the point $P = (x_1, y_1)$ (see Figure 5.5 to visualize how these can be situated in relationship to the graph of the curve), then from §5.1,

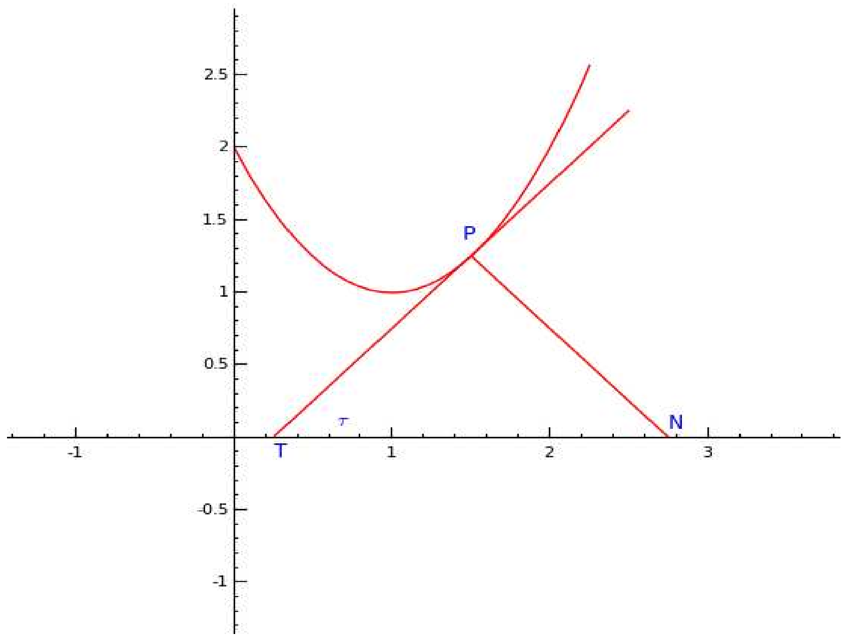

Figure 5.5: The tangent and normal line to a curve.

$$m = \tan\tau = \left[\frac{dy}{dx}\right]_{x=x_1, y=y_1}.$$

Therefore at a point $P = (x_1, y_1)$ on the curve, the equation of the *tangent line* (containing the segment TP) is

$$y - y_1 = (\left[\frac{dy}{dx}\right]_{x=x_1, y=y_1})(x - x_1). \tag{5.1}$$

The normal being perpendicular to tangent, its slope is

$$-\frac{1}{m} = -\left[\frac{dx}{dy}\right]_{x=x_1, y=y_1} = -(\left[\frac{dy}{dx}\right]_{x=x_1, y=y_1})^{-1}$$

(item 55 in §12.1). And since it also passes through the point $P = (x_1, y_1)$, we have for the equation of the *normal line* (containing the segment PN)

$$y - y_1 = -(\left[\frac{dx}{dy}\right]_{x=x_1, y=y_1})(x - x_1). \tag{5.2}$$

The length of the segment on the tangent line which is between $P = (x_1, y_1)$ and the point of contact with the x-axis is called the *length of the tangent* (= TP), and the projection of this segment on the x-axis is called the *length of the subtangent*[1] (= TM). Similarly, we have the *length of the normal* (= PN) and the *length of the subnormal* (= MN).

In the triangle TPM, $\tan\,\tau = \frac{MP}{TM}$; therefore[2]

$$TM = \frac{MP}{\tan\tau} = y_1 \left[\frac{dx}{dy}\right]_{x=x_1, y=y_1} = \text{ length of subtangent.} \tag{5.3}$$

In the triangle MPN, $\tan\,\tau = \frac{MN}{MP}$; therefore[3]

$$MN = MP\tan\tau = y_1 \left[\frac{dy}{dx}\right]_{x=x_1, y=y_1} = \text{ length of subnormal.} \tag{5.4}$$

The length of tangent ($=$ TP) and the length of normal ($=$ PN) may then be found directly from Figure 5.5, each being the hypotenuse of a right triangle having the two legs known. Thus

$$\begin{aligned} TP &= \sqrt{T\bar{M}^2 + \bar{M}P^2} \\ &= \sqrt{\left(y_1 \left[\frac{dx}{dy}\right]_{x=x_1,y=y_1}\right)^2 + (y_1)^2} \\ &= y_1\sqrt{\left(\left[\frac{dx}{dy}\right]_{x=x_1,y=y_1}\right)^2 + 1} \\ &= \text{ length of tangent.} \end{aligned} \tag{5.5}$$

Likewise,

$$\begin{aligned} PN &= \sqrt{\bar{M}N^2 + \bar{M}P^2} \\ &= \sqrt{\left(\left[\frac{dy}{dx}\right]_{x=x_1,y=y_1}\right)^2 + (y_1)^2} \\ &= y_1\sqrt{\left(\left[\frac{dy}{dx}\right]_{x=x_1,y=y_1}\right)^2 + 1} \\ &= \text{ length of normal.} \end{aligned} \tag{5.6}$$

[1]The *subtangent* is the segment obtained by projecting the portion TP of the tangent line onto the x-axis).

[2]If subtangent extends to the right of T, we consider it positive; if to the left, negative.

[3] If subnormal extends to the right of M, we consider it positive; if to the left, negative.

The student is advised to get the lengths of the tangent and of the normal directly from the figure rather than by using these equations.

When the length of subtangent or subnormal at a point on a curve is determined, the tangent and normal may be easily constructed.

5.4 Exercises

1. Find the equations of tangent and normal, lengths of subtangent, subnormal tangent, and normal at the point (a, a) on the cissoid $y^2 = \frac{x^3}{2a-x}$.

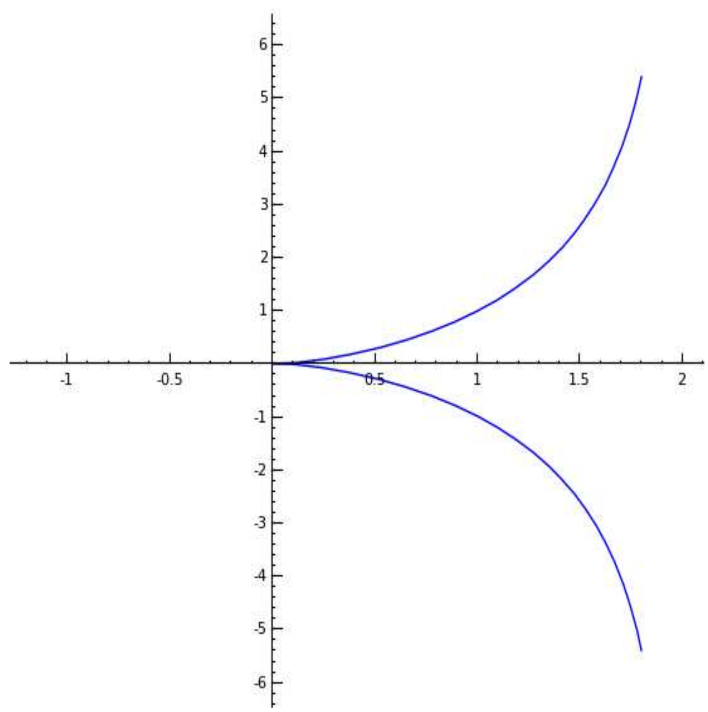

Figure 5.6: Graph of cissoid $y^2 = \frac{x^3}{2a-x}$ with $a = 1$.

Solution. $\frac{dy}{dx} = \frac{3ax^2-x^3}{y(2a-x)^2}$. Hence

$$\frac{dy_1}{dx_1} = \left[\frac{dy}{dx}\right]_{x=a,y=a} = \frac{3a^3 - a^3}{a(2a-a)^2} = 2$$

is the slope of tangent. Substituting in (5.1) gives

$$y = 2x - a,$$

the equation of the tangent line. Substituting in (5.2) gives

$$2y + x = 3a,$$

the equation of the normal line. Substituting in (5.3) gives

$$TM = \frac{a}{2},$$

the length of subtangent. Substituting in (5.4) gives

$$MN = 2a,$$

the length of subnormal. Also

$$PT = \sqrt{(TM)^2 + (MP)^2} = \sqrt{\frac{a^2}{4} + a^2} = \frac{a}{2}\sqrt{5},$$

which is the length of tangent, and

$$PN = \sqrt{(MN)^2 + (MP)^2} = \sqrt{4a^2 + a^2} = a\sqrt{5},$$

the length of normal.

2. Find equations of tangent and normal to the ellipse $x^2 + 2y^2 - 2xy - x = 0$ at the points where $x = 1$.

   Ans. At $(1, 0)$, $2y = x - 1$, $y + 2x = 2$. At $(1, 1)$, $2y = x + 1$, $y + 2x = 3$.

3. Find equations of tangent and normal, lengths of subtangent and subnormal at the point (x_1, y_1) on the circle[4] $x^2 + y^2 = r^2$.

   Ans. $x_l x + y_1 y = r^2$, $x_1 y - y_1 x = 0$, $-x_1, -\frac{{y_1}^2}{x_1}$.

4. Show that the subtangent to the parabola $y^2 = 4px$ is bisected at the vertex, and that the subnormal is constant and equal to $2p$.

[4]In Exs. 3 and 5 the student should notice that if we drop the subscripts in equations of tangents, they reduce to the equations of the curves themselves.

5. Find the equation of the tangent at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

   Ans. $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$.

   Here's how to find the length of tangent, normal, subtangent and subnormal of this in Sage using the values $a = 1$, $b = 2$ (so $x^2 + \frac{y^2}{4} = 1$) and $x_1 = 4/5$, $y_1 = 6/5$.

   Sage

```
sage: x = var("x")
sage: y = var("y")
sage: F = x^2 + y^2/4 - 1
sage: Dx = -diff(F,y)/diff(F,x); Dx; Dx(4/5,6/5)
-y/(4*x)
-3/8
sage: Dy = -diff(F,x)/diff(F,y); Dy; Dy(4/5,6/5)
-4*x/y
-8/3
```

   (For this Sage calculation, we have used the fact that $F(x, y) = 0$ implies $F_x(x, y) + \frac{dy}{dx} F_y(x, y) = 0$, where y is regarded as a function of x.) Therefore, we have (using (5.3))

$$\text{length of subtangent} = y_1 \left[\frac{dx}{dy}\right]_{x=x_1, y=y_1} = (6/5)(-3/8) = -9/20,$$

   (using (5.4))

$$\text{length of subnormal} = y_1 \left[\frac{dy}{dx}\right]_{x=x_1, y=y_1} = (6/5)(-8/3) = -16/5,$$

   (using (5.5))

$$\begin{aligned}\text{length of tangent} = y_1 \sqrt{\left(\left[\frac{dx}{dy}\right]_{x=x_1, y=y_1}\right)^2 + 1} &= (6/5)\sqrt{1 + \tfrac{9}{64}} \\ &= 3\sqrt{73}/20 = 1.2816\ldots ,\end{aligned}$$

and (using (5.6))

$$\text{length of normal} = y_1\sqrt{\left(\left[\frac{dy}{dx}\right]_{x=x_1,y=y_1}\right)^2+1} \;= (6/5)\sqrt{1+\tfrac{64}{9}} = 2\sqrt{73}/5 = 3.4176\ldots\,.$$

6. Find equations of tangent and normal to the Witch of Agnesi $y = \frac{8a^3}{4a^2+x^2}$ as at the point where $x = 2a$.

   Ans. $x + 2y = 4a$, $y = 2x - 3a$.

7. Prove that at any point on the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$ the lengths of subnormal and normal are $\frac{a}{4}(e^{\frac{2x}{a}} - e^{-\frac{2x}{a}})$ and $\frac{y^2}{a}$ respectively.

8. Find equations of tangent and normal, lengths of subtangent and subnormal, to each of the following curves at the points indicated:

(a) $y = x^3$ at $(\frac{1}{2}, \frac{1}{8})$	(e) $y = 9 - x^2$ at $(-3, 0)$
(b) $y^2 = 4x$ at $(9, -6)$	(f) $x^2 = 6y$ where $x = -6$
(c) $x^2 + 5y^2 = 14$ where $y = 1$	(g) $x^2 - xy + 2x - 9 = 0$ at $(3, 2)$
(d) $x^2 + y^2 = 25at(-3, -4)$	(h) $2x^2 - y^2 = 14$ at $(3, -2)$

9. Prove that the length of subtangent to $y = a^x$ is constant and equal to $\frac{1}{\log a}$.

10. Get the equation of tangent to the parabola $y^2 = 20x$ which makes an angle of $45^o = \pi/4$ with the x-axis.

    Ans. $y = x + 5$. (Hint: First find point of contact by method of Example 5.1.2.)

11. Find equations of tangents to the circle $x^2 + y^2 = 52$ which are parallel to the line $2x + 3y = 6$.

    Ans. $2x + 3y \pm 26 = 0$

12. Find equations of tangents to the hyperbola $4x^2 - 9y^2 + 36 = 0$ which are perpendicular to the line $2y + 5x = 10$.

    Ans. $2x - 5y \pm 8 = 0$.

13. Show that in the equilateral hyperbola $2xy = a^2$ the area of the triangle formed by a tangent and the coordinate axes is constant and equal to a^2.

14. Find equations of tangents and normals to the curve $y^2 = 2x^2 - x^3$ at the points where $x = 1$.

    Ans. At $(1,1)$, $2y = x + 1$, $y + 2x = 3$. At $(1,-1)$, $2y = -x - 1$, $y - 2x = -3$.

15. Show that the sum of the intercepts of the tangent to the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

16. Find the equation of tangent to the curve $x^2(x+y) = a^2(x-y)$ at the origin.

17. Show that for the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ that portion of the tangent included between the coordinate axes is constant and equal to a.

    (This curve is parameterized by $x = a\cos(t)^3$, $y = a\sin(t)^3$, $0 \le t \le 2\pi$. Parametric equations shall be discussed in the next section.)

18. Show that the curve $y = ae^{\frac{x}{c}}$ has a constant subtangent.

5.5 Parametric equations of a curve

Let the equation of a curve be

$$F(x, y) = 0. \tag{5.7}$$

If x is given as a function of a third variable, t say, called a *parameter*, then by virtue of (5.7) y is also a function of t, and the same functional relation (5.7) between x and y may generally be expressed by means of equations in the form

$$\begin{cases} x = f(t), \\ y = g(t) \end{cases} \tag{5.8}$$

each value of t giving a value of x and a value of y. Equations (5.8) are called *parametric equations of the curve*. If we eliminate t between equations (5.8), it is evident that the relation (5.7) must result.

Example 5.5.1. For example, take equation of circle

$$x^2 + y^2 = r^2 \text{ or } y = \sqrt{r^2 - x^2}.$$

We have

$$\begin{cases} x = r\cos\, t \\ y = r\sin\, t \end{cases} \tag{5.9}$$

as parametric equations of the circle, t being the parameter[5].

If we eliminate t between equations (5.9) by squaring and adding the results, we have

$$x^2 + y^2 = r^2(\cos^2 t + \sin^2 t) = r^2,$$

the rectangular equation of the circle. It is evident that if t varies from 0 to 2π, the point $P = (x, y)$ will describe a complete circumference.

In §5.13 we shall discuss the motion of a point P, which motion is defined by equations such as

$$\begin{cases} x = f(t), \\ y = g(t) \end{cases}$$

We call these the *parametric equations of the path*, the time t being the parameter.

Example 5.5.2. Newtonian physics tells us that

$$\begin{cases} x = v_0 \cos\alpha \cdot t, \\ y = -\frac{1}{2}gt^2 + v_0 \sin\alpha \cdot t \end{cases}$$

are really the parametric equations of the trajectory of a projectile[6], the time t being the parameter. The elimination of t gives the rectangular equation of the trajectory

$$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}.$$

[5]Parameterizations are not unique. Another set of parametric equations of the first quadrant of the circle is given by $x = \frac{\sqrt{2t}}{\sqrt{1+t^2}}$, $y = \frac{1-t}{\sqrt{1+t^2}}$, for example.

[6]Subject to (downward) gravitational force but no wind resistance or other external forces.

Since from (5.8) y is given as a function of t, and t as a function of x, we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \text{by (4.27)} \\ &= \frac{dy}{dt} \cdot \frac{1}{\frac{dx}{dt}} \quad \text{by (4.28)} \end{aligned}$$

that is,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}. \tag{5.10}$$

Hence, if the parametric equations of a curve are given, we can find equations of tangent and normal, lengths of subtangent and subnormal at a given point on the curve, by first finding the value of $\frac{dy}{dx}$ at that point from (5.10) and then substituting in formulas (5.1), (5.2), (5.3), (5.4) of the last section.

Example 5.5.3. Find equations of tangent and normal, lengths of subtangent and subnormal to the ellipse

$$\begin{cases} x = a\cos\phi, \\ y = b\sin\phi, \end{cases} \tag{5.11}$$

at the point where $\phi = \frac{\pi}{4}$.

As in Figure 5.7 draw the major and minor auxiliary circles of the ellipse. Through two points B and C on the same radius draw lines parallel to the axes of coordinates. These lines will intersect in a point $P = (x, y)$ on the ellipse, because $x = OA = OB\cos\phi = a\cos\phi$ and $y = AP = OD = OC\sin\phi = b\sin\phi$, or, $\frac{x}{a} = \cos\phi$ and $\frac{y}{b} = \sin\phi$. Now squaring and adding, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\phi + \sin^2\phi = 1,$$

the rectangular equation of the ellipse. ϕ is sometimes called the *eccentric angle* of the ellipse at the point P.

Solution. The parameter being ϕ, $\frac{dx}{d\phi} = -a\sin\phi$, $\frac{dy}{d\phi} = b\cos\phi$.

Substituting $\phi = \frac{\pi}{4}$ in the given equations (5.11), we get $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ as the point of contact. Hence $\left[\frac{dy}{dx}\right]_{x=x_1, y=y_1} = -\frac{b}{a}$. Substituting in (5.1),

$$y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right),$$

or, $bx + ay = \sqrt{2}ab$, the equation of tangent. Substituting in (5.2),

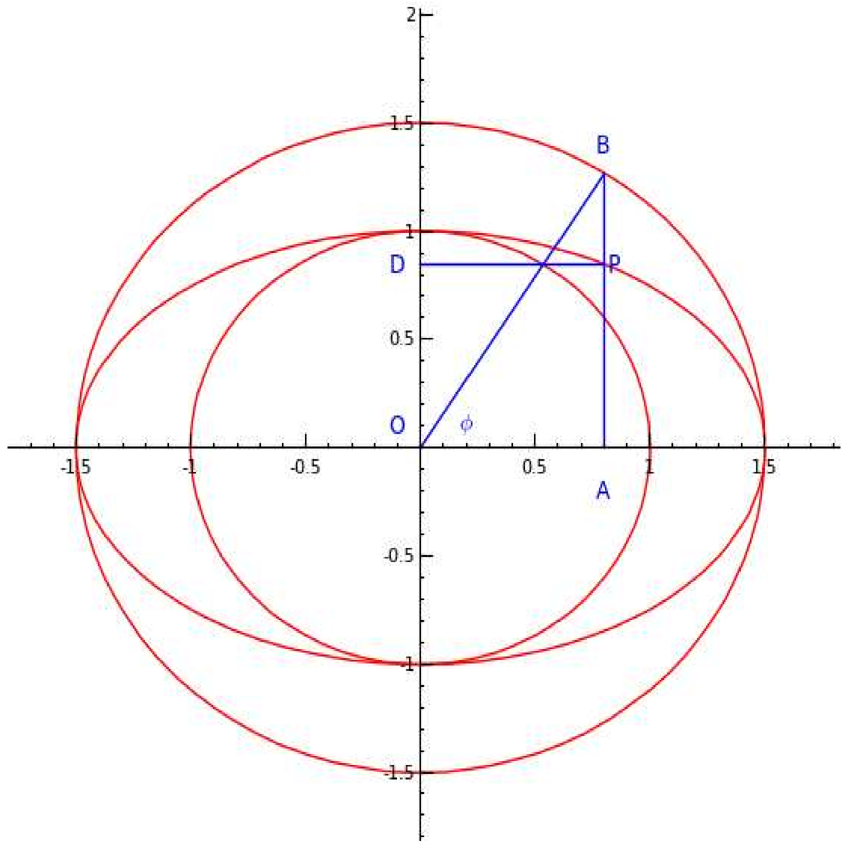

Figure 5.7: Auxiliary circles of an ellipse.

$$y - \frac{b}{\sqrt{2}} = \frac{a}{b}\left(x - \frac{a}{\sqrt{2}}\right),$$

or, $\sqrt{2}(ax - by) = a^2 - b^2$, the equation of normal. Substituting in (5.3) and (5.4), we find

$$\frac{b}{\sqrt{2}}\left(-\frac{b}{a}\right) = -\frac{b^2}{a\sqrt{2}},$$

the length of subnormal, and

$$\frac{b}{\sqrt{2}}\left(-\frac{a}{b}\right) = -\frac{a}{\sqrt{2}},$$

the length of subtangent.

Example 5.5.4. Given equation of the cycloid in parametric form

$$\begin{cases} x = a(\theta - \sin\theta), \\ y = a(1 - \cos\theta), \end{cases}$$

θ being the variable parameter; find lengths of subtangent, subnormal, tangent, and normal at the point where $\theta = \frac{\pi}{2}$.

The path described by a point on the circumference of a circle which rolls without sliding on a fixed straight line is called the *cycloid*. Let the radius of the rolling circle be a, P the generating point, and M the point of contact with the fixed line OX, which is called *the base*. If arc PM equals OM in length, then P will touch at O if the circle is rolled to the left. We have, denoting angle POM by θ,

$$\begin{aligned} x &= OM - NM = a\theta - a\sin\theta = a(\theta - \sin\theta), \\ y &= PN = MC - AC = a - a\cos\theta = a(1 - \cos\theta), \end{aligned}$$

the parametric equations of the cycloid, the angle θ through which the rolling circle turns being the parameter. $OD = 2\pi a$ is called the *base of one arch* of the cycloid, and the point V is called the *vertex*. Eliminating θ, we get the rectangular equation

$$x = a \arccos\left(\frac{a - y}{a}\right) - \sqrt{2ay - y^2}.$$

The Sage commands for creating this plot are as follows:

Sage

```
sage: t = var("t")
sage: f1 = lambda t: [t-sin(t),1-cos(t)]
sage: p1 = parametric_plot(f1(t), 0.0, 2*pi, rgbcolor=(1,0,0))
sage: f2 = lambda t: [t+RR(pi)/2-1,t+1]
sage: p2 = parametric_plot(f2(t), -1, 1, rgbcolor=(1,0,0))
sage: f3 = lambda t: [-t+RR(pi)/2,t]
sage: p3 = parametric_plot(f3(t), -1, 1, rgbcolor=(1,0,0))
sage: t1 = text("P", (RR(pi)/2-1+0.1,1-0.1))
sage: t2 = text("T", (-0.4,0.1))
```

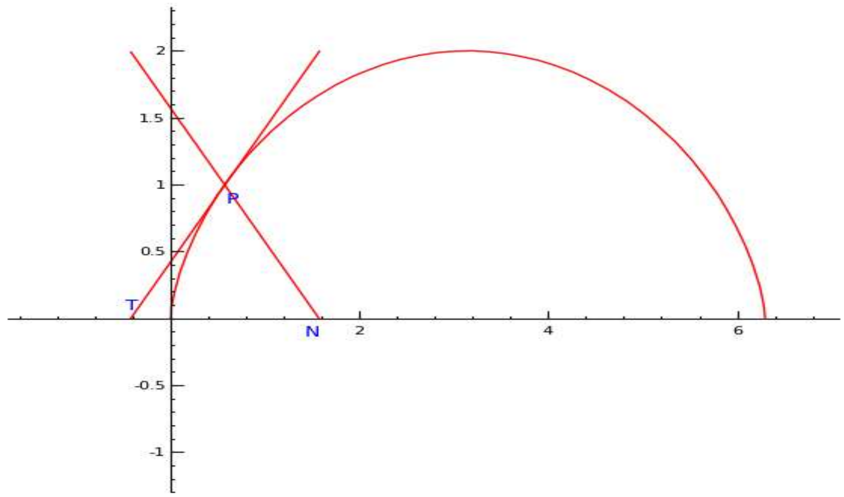

Figure 5.8: Tangent line of a cycloid.

```
sage: t3 = text("N", (RR(pi)/2,0))
sage: show(p1+p2+p3+t1+t2+t3)
```

Solution:

$$\frac{dx}{d\theta} = a(1-\cos\theta), \quad \frac{dy}{d\theta} = a\sin\theta.$$

Substituting in (5.10),

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta},$$

the slope at any point. Since $\theta = \frac{\pi}{2}$, the point of contact is $\left(\frac{\pi a}{2} - a, a\right)$, and $\left[\frac{dy}{dx}\right]_{x=x_1,y=y_1} = 1$.

Substituting in (5.3), (5.4), (5.5), (5.6) of the last section, we get
length of subtangent = a,
length of subnormal = a,
length of tangent = $a\sqrt{2}$,
length of normal = $a\sqrt{2}$.

5.6 Exercises

Find equations of tangent and normal, lengths of subtangent and subnormal to each of the following curves at the point indicated:

1. Curve: $x = t^2$, $2y = t$;

   Point: $t = 1$.

   Tangent line: $x - 4y + 1 = 0$;

   Normal line: $8x + 2y - 9 = 0$;

   Subtangent: 2;

   Subnormal: $\frac{1}{8}$.

2. Curve: $x = t$, $y = t^3$;

   Point: $t = 2$.

   Tangent line: $12x - y - 16 = 0$;

   Normal line: $x + 12y - 98 = 0$;

   Subtangent: $\frac{2}{3}$;

   Subnormal: 96.

3. Curve: $x = t^2$, $y = t^3$;

   Point: $t = 1$.

   Tangent line: $3x - 2y - 1 = 0$;

   Normal line: $2x + 3y - 5 = 0$;

   Subtangent: $\frac{2}{3}$;

   Subnormal: $\frac{3}{2}$.

4. Curve: $x = 2e^t$, $y = e^{-t}$;

   Point: $t = 0$.

   Tangent line: $x + 2y - 4 = 0$;

   Normal line: $2x - y - 3 = 0$;

   Subtangent: -2;

   Subnormal: $-\frac{1}{2}$.

5. Curve: $x = \sin\, t$, $y = \cos\, 2t$;

   Point: $t = \frac{\pi}{6}$.

   Tangent line: $2y + 4x - 3 = 0$;

   Normal line: $4y - 2x - 1 = 0$;

   Subtangent: $-\frac{1}{4}$;

   Subnormal: -1.

   Sage can help with the computations here:

   Sage

```
sage: t = var("t")
sage: x = sin(t)
sage: y = cos(2*t)
sage: t0 = pi/6
sage: y_x = diff(y,t)/diff(x,t)
sage: y_x
-2*sin(2*t)/cos(t)
sage: y_x(t0)
-2
sage: m = y_x(t0); x0 = x(t0); y0 = y(t0)
sage: X,Y = var("X,Y")
sage: Y - y0 == m*(X - x0)
Y - 1/2 == -2*(X - 1/2)
```

   The last line is the point-slope form of the tangent line of the parametric curve at that point $t_0 = \pi/6$ (so, $(x_0, y_0) = (\sin(t_0), \cos(2t_0)) = (1/2, 1/2)$). We use X and Y in place of x and y so as to not over-ride the entries that Sage has stored for them. Continuing the above Sage computations:

   Sage

```
sage: x_y = diff(x,t)/diff(y,t)
sage: len_subtan = y(t0)*x_y(t0); len_subtan
-1/4
sage:
```

```
sage: len_subnor = y(t0)*y_x(t0); len_subnor
-1
sage: len_tan = y(t0)*sqrt(x_y(t0)^2+1); len_tan
sqrt(5)/4
sage: len_nor = y(t0)*sqrt(y_x(t0)^2+1); len_nor
sqrt(5)/2
```

These tell us the length of the subtangent is $-\frac{1}{4}$ (as expected), as well as the lengths of the subnormal, tangent and normal, using formulas (5.10), (5.3), (5.4), (5.5), (5.6) of the last section.

6. Curve: $x = 1 - t$, $y = t^2$;

   Point: $t = 3$.

7. Curve: $x = 3t$; $y = 6t - t^2$;

   Point: $t = 0$.

8. Curve: $x = t^3$; $y = t$;

   Point: $t = 2$.

9. Curve: $x = t^3$, $y = t^2$;

   Point: $t = -1$.

10. Curve: $x = 2 - t$; $y = 3t^2$;

    Point: $t = 1$.

11. Curve: $x = \cos\ t$, $y = \sin\ 2t$;

    Point: $t = \frac{\pi}{3}$.

12. Curve: $x = 3e^{-t}$, $y = 2e^t$;

    Point: $t = 0$.

13. Curve: $x = \sin\ t$, $y = 2\cos\ t$;

    Point: $t = \frac{\pi}{4}$.

14. Curve: $x = 4\cos\ t$, $y = 3\sin\ t$;

    Point: $t = \frac{\pi}{2}$.

15. Curve:

    Point:

In the following curves find lengths of (a) subtangent, (b) subnormal, (c) tangent, (d) normal, at any point:

16. The curve

$$\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t). \end{cases}$$

    Ans. (a) $y\cot\ t$, (b) $y\tan\ t$, (c) $\frac{y}{\sin\ t}$, (d) $\frac{y}{\cos\ t}$.

17. The hypocycloid (astroid)

$$\begin{cases} x = 4a\cos^3 t, \\ y = 4a\sin^3 t. \end{cases}$$

    Ans. (a) $-y\cot\ t$, (b) $-y\tan\ t$, (c) $\frac{y}{\sin\ t}$, (d) $\frac{y}{\cos\ t}$.

18. The circle

$$\begin{cases} x = r\cos\ t, \\ y = r\sin\ t. \end{cases}$$

19. The cardioid

$$\begin{cases} x = a(2\cos\ t - \cos\ 2t), \\ y = a(2\sin\ t - \sin\ 2t). \end{cases}$$

20. The folium

$$\begin{cases} x = \frac{3t}{1+t^3} \\ y = \frac{3t^2}{1+t^3}. \end{cases}$$

21. The hyperbolic spiral

$$\begin{cases} x = \frac{a}{t}\cos\, t \\ y = \frac{a}{t}\sin\, t \end{cases}$$

5.7 Angle between radius vector and tangent

Angle between the radius vector drawn to a point on a curve and the tangent to the curve at that point. Let the equation of the curve in polar coordinates be $\rho = f(\theta)$.

Let P be any fixed point (ρ, θ) on the curve. If θ, which we assume as the independent variable, takes on an increment $\Delta\theta$, then ρ will take on a corresponding increment $\Delta\rho$.

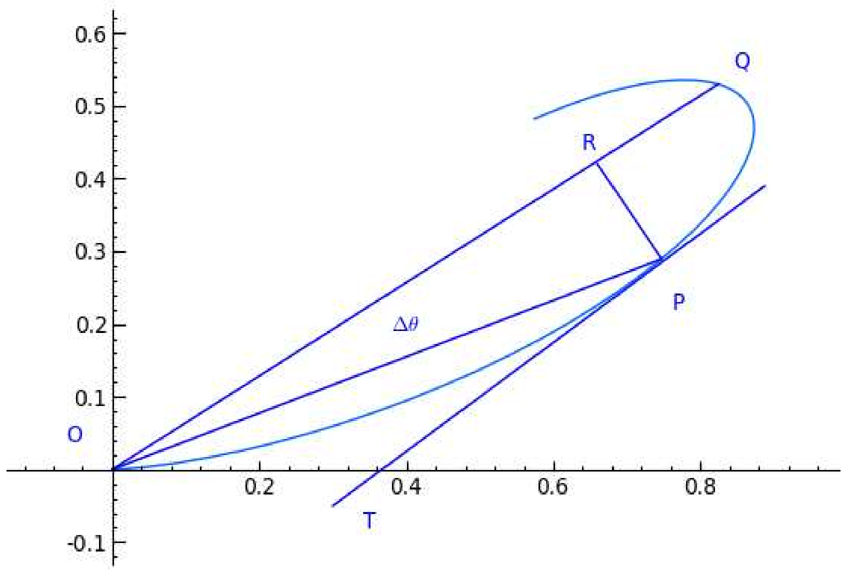

Figure 5.9: Angle between the radius vector drawn to a point on a curve and the tangent to the curve at that point.

Denote by Q the point $(\rho+\Delta\rho, \theta+\Delta\theta)$, as in Figure 5.9, Draw PR perpendicular to OQ where R is a point at a distance of $\rho\cos\Delta\theta$ from the origin. Then $OQ = \rho + \Delta\rho$, $PR = \rho\sin\Delta\theta$, and $OR = \rho\cos\Delta\theta$. Also,

$$\tan PQR = \frac{PR}{RQ} = \frac{PR}{OQ - OR} = \frac{\rho \sin \Delta\theta}{\rho + \Delta\rho - \rho \cos \Delta\theta}.$$

Denote by ψ the angle between the radius vector OP and the tangent PT. If we now let $\Delta\theta$ approach the limit zero, then

(a) the point Q will approach indefinitely near P;

(b) the secant PQ will approach the tangent PT as a limiting position; and

(c) the angle PQR will approach ψ as a limit.

Hence

$$\tan\ \psi = \lim_{\Delta\theta \to 0} \frac{\rho\Delta\theta}{\rho + \Delta\rho - \rho\cos\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{\rho\Delta\theta}{2\rho\sin^2\frac{\Delta\theta}{2} + \Delta\rho}$$

(since, from 39, §12.1, $\rho - \rho\cos\Delta\theta = \rho(1 - \cos\Delta\theta) = 2\rho\sin^2\frac{\Delta\theta}{2}$). Dividing both numerator and denominator by $\Delta\theta$, this is

$$= \lim_{\Delta\theta \to 0} \frac{\frac{\rho\sin\Delta\theta}{\Delta\theta}}{\frac{2\rho\sin^2\frac{\Delta\theta}{2}}{\Delta\theta} + \frac{\Delta\rho}{\Delta\theta}} = \lim_{\Delta \to 0} \frac{\rho \cdot \frac{\sin\Delta\theta}{\Delta\theta}}{\rho\sin\frac{\Delta\theta}{2} \cdot \frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} + \frac{\Delta\rho}{\Delta\theta}}.$$

Since

$$\lim_{\Delta\theta \to 0}\left(\frac{\Delta\rho}{\Delta\theta}\right) = \frac{d\rho}{d\theta} \text{ and } \lim_{\Delta\theta \to 0}\left(\sin\frac{\Delta\theta}{2}\right) = 0,$$

also

$$\lim_{\Delta\theta \to 0}\left(\frac{\sin\Delta\theta}{\Delta\theta}\right) = 1$$

and

$$\lim_{\Delta\theta \to 0}\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} = 1$$

by §2.10, we have

$$\tan\ \psi = \frac{\rho}{\frac{d\rho}{d\theta}} \tag{5.12}$$

From the triangle OPT we get

$$\tau = \theta + \psi. \tag{5.13}$$

Having found τ, we may then find $\tan\ \tau$, the slope of the tangent to the curve at P. Or since, from (5.13),

$$\tan\tau = \tan(\theta+\psi) = \frac{\tan\theta + \tan\psi}{1 - \tan\theta\tan\psi}$$

we may calculate $\tan\ \psi$ from (5.12) and substitute in the formula

$$\text{slope of tangent } = \tan\tau = \frac{\tan\theta + \tan\psi}{1 - \tan\theta\tan\psi}. \tag{5.14}$$

Example 5.7.1. Find ψ and τ in the cardioid $\psi = a(1 - \cos\ \theta)$. Also find the slope at $\theta = \frac{\pi}{6}$.

Solution. $\frac{d\psi}{d\theta} = a\sin\theta$. Substituting in (5.12) gives

$$\tan\psi = \frac{\rho}{\frac{d\rho}{d\theta}} = \frac{a(1-\cos\theta)}{a\sin\theta} = \frac{2a\sin^2\frac{\theta}{2}}{2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2},$$

by items 39 and 37, §12.1. Since $\tan\psi = \tan\frac{\theta}{2}$, we have $\psi = \frac{\theta}{2}$.

Substituting in (5.13), $\tau = \theta + \frac{\theta}{2} = \frac{3\theta}{2}$. so

$$\tan\ \tau = \tan\frac{\pi}{4} = 1.$$

To find the angle of intersection ϕ of two curves C and C' whose equations are given in polar coordinates, we may proceed as follows:

angle TPT′ = angle OPT′ - angle OPT,

or, $\phi = \psi' - \psi$. Hence

$$\tan\ \phi = \frac{\tan\ \psi' - \tan\ \psi}{1 + \tan\ \psi' \tan\ \psi}, \tag{5.15}$$

where $\tan\ \psi'$ and $\tan\ \psi$ are calculated by (5.12) from the two curves and evaluated for the point of intersection.

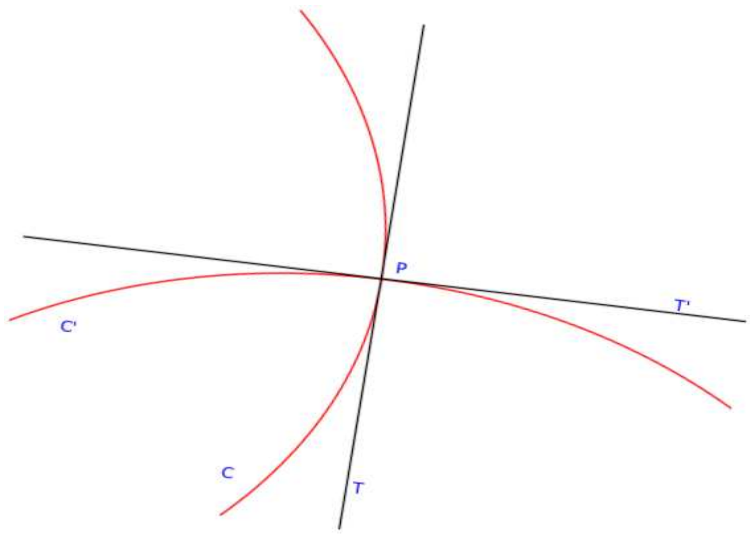

Figure 5.10: The angle between two curves.

Example 5.7.2. Find the angle of of intersection of the curves $\rho = a\sin 2\theta$, $\rho = a\cos 2\theta$.

Solution. Solving the two equations simultaneously, we get at the point of intersection

$$\tan\, 2\theta = 1, \quad 2\theta = 45^o = \pi/4, \quad \theta = \frac{45^o}{2} = \pi/8.$$

From the first curve, using (5.12),

$$\tan\psi' = \frac{1}{2}\tan 2\theta = \frac{1}{2},$$

for $\theta = \frac{45}{2}^o = \pi/8$. From the second curve,

$$\tan\psi = -\frac{1}{2}\cot 2\theta = -\frac{1}{2},$$

for $\theta = \frac{45}{2}^o = \pi/8$.

Substituting in ((5.15),

$$\tan\psi = \frac{\frac{1}{2}+\frac{1}{2}}{1-\frac{1}{4}} = \frac{4}{3}.$$

therefore $\psi = \arctan\frac{4}{3}$.

5.8 Lengths of polar subtangent and polar subnormal

Draw a line NT through the origin perpendicular to the radius vector of the point P on the curve. If PT is the tangent and PN the normal to the curve at P, then[7]

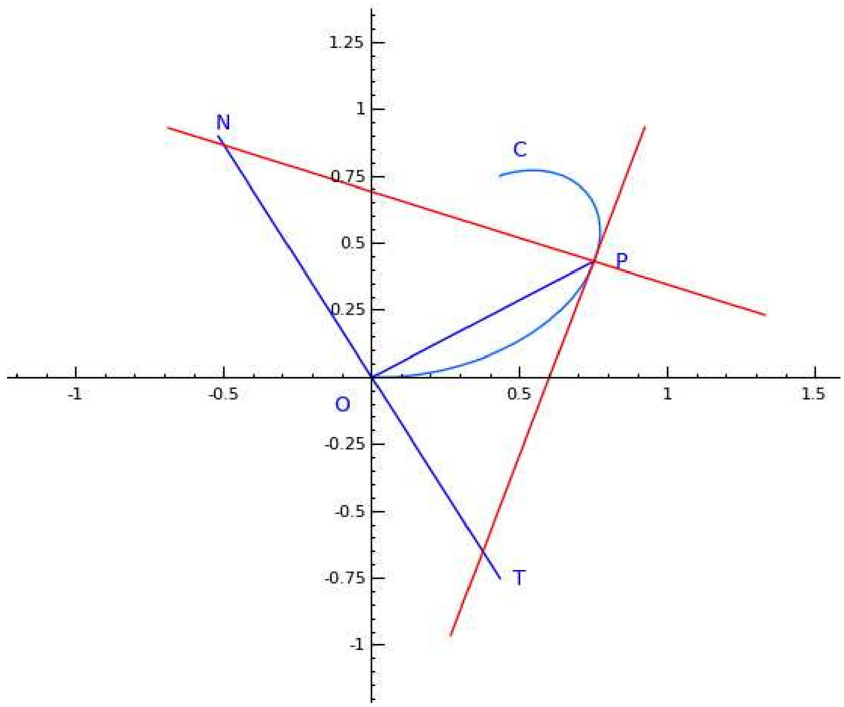

Figure 5.11: The polar subtangent and polar subnormal.

$$OT = \text{length of polar subtangent},$$

and

$$ON = \text{length of polar subnormal}$$

[7]When θ increases with ρ, $\frac{d\theta}{d\rho}$ is positive and ρ is an acute angle, as in Figure 5.11. Then the subtangent OT is positive and is measured to the right of an observer placed at O and looking along OP. When $\frac{d\theta}{d\rho}$ is negative, the subtangent is negative and is measured to the left of the observer.

of the curve at P.

In the triangle OPT, $\tan\,\psi = \frac{OT}{\rho}$. Therefore

$$OT = \rho \tan\psi = \rho^2 \frac{d\theta}{d\rho} = \text{ length of polar subtangent.} \tag{5.16}$$

In the triangle OPN, $\tan\psi = \frac{\rho}{ON}$. Therefore

$$ON = \frac{\rho}{\tan\psi} = \frac{d\rho}{d\theta} = \text{ length of polar subnormal.} \tag{5.17}$$

The length of the polar tangent (= PT) and the length of the polar normal (= PN) may be found from the figure, each being the hypotenuse of a right triangle.

Example 5.8.1. Find lengths of polar subtangent and subnormal to the lemniscate $\rho^2 = a^2 \cos\, 2\theta$.

Solution. Differentiating the equation of the curve as an implicit function with respect to θ, or, $2\rho\frac{d\rho}{d\theta} = -2a^2 \sin 2\theta$, $\frac{d\rho}{d\theta} = -\frac{a^2 \sin 2\theta}{\rho}$.

Substituting in (5.16) and (5.17), we get

length of polar subtangent = $-\frac{\rho^3}{a^2 \sin 2\theta}$,
length of polar subnormal = $-\frac{a^2 \sin 2\theta}{\rho}$.

If we wish to express the results in terms of θ, find ρ in terms of θ from the given equation and substitute. Thus, in the above, $\rho = \pm a\sqrt{\cos 2\theta}$; therefore

length of polar subtangent = $\pm a \cot 2\theta \sqrt{\cos 2\theta}$.

5.9 Examples

1. In the circle $\rho = r \sin\, \theta$, find ψ and τ in terms of θ.

   Solution: $\psi = \theta$, $\tau = 2\theta$.

2. In the parabola $\rho = a \sec^{\frac{\theta}{2}}$, show that $\tau + \psi = \pi$.

3. In the curve $\rho^2 = a^2 \cos\, 2\theta$, show that $2\psi = \pi + 4\theta$.

4. Show that ψ is constant in the logarithmic spiral $\rho = e^{a\theta}$. Since the tangent makes a constant angle with the radius vector, this curve is also called the equiangular spiral.

5. Given the curve $\rho = a\sin^3\frac{\theta}{3}$, prove that $\tau = 4\psi$.

   Sage can help with this problem. Using (5.12) but with t in place of θ for typographical simplicity, we have

   Sage

   ```
   sage: a,t = var("a,t")
   sage: r = a*sin(t/3)^3
   sage: tanpsi = r/diff(r,t); tanpsi
   sin(t/3)/cos(t/3)
   ```

   Therefore, $\tan(\psi) = \tan(\theta/3)$, so $\theta = 3\psi$. Therefore, according to (5.13), we have $\tau = \theta + \psi = 3\psi + \psi = 4\psi$, as expected.

6. Show that $\tan\ \psi = \theta$ in the spiral of Archimedes $\rho = a\theta$. Find values of ψ when $\theta = 2\pi$ and 4π.

   Solution: $\psi = 80^o57' = 1.4128...$ and $85^o27' = 1.4913....$

7. Find the angle between the straight line $\rho\cos\ \theta = 2a$ and the circle $\rho = 5a\sin\ \theta$.

   Solution: $\arctan\frac{3}{4}$.

8. Show that the parabolas $\rho = a\sec^2\frac{\theta}{2}$ and $\rho = b\csc^2\frac{\theta}{2}$ intersect at right angles.

9. Find the angle of intersection of $\rho = a\sin\ \theta$ and $\rho = a\sin\ 2\theta$.

   Solution: At origin 0^o; at two other points $\arctan 3\sqrt{3}$.

10. Find the slopes of the following curves at the points designated:

curve	point	solution (if given)
(a) $\rho = a(l - \cos, \theta)$	$\theta = \frac{\pi}{2}$	-1
(b) $\rho = a \sec^2 \theta$	$\rho = 2a$	3
(c) $\rho = a \sin\, 4\theta$	origin	$0, 1, \infty, -1$
(d) $\rho^2 = a^2 \sin\, 4\theta$	origin	$0, 1, \infty, -1$
(e) $\rho = a \sin\, 3\theta$	origin	$0, \sqrt{3}, -\sqrt{3}$
(f) $\rho = a \cos\, 3\theta$	origin	
(g) $\rho = a \cos\, 2\theta$	origin	
(h) $\rho = a \sin\, 2\theta$	$\theta = \frac{\pi}{4}$	
(i) $\rho = a \sin\, 3\theta$	$\theta = \frac{pi}{6}$	
(j) $\rho = a\theta$	$\theta = \frac{\pi}{2}$	
(k) $\rho\theta = a$	$\theta = \frac{\pi}{2}$	
(l) $\rho = e^\theta$	$\theta = 0$	

11. Prove that the spiral of Archimedes $\rho = a\theta$, and the reciprocal spiral $\rho = \frac{a}{\theta}$, intersect at right angles.

12. Find the angle between the parabola $\rho = asec^2 \frac{\theta}{2}$ and the straight line $\rho \sin\, \theta = 2a$.

    Solution: $45^o = \pi/4$.

13. Show that the two cardioids $\rho = a(1 + \cos\, \theta)$ and $\rho = a(1 - \cos\, \theta)$ cut each other perpendicularly.

14. Find lengths of subtangent, subnormal, tangent, and normal of the spiral of Archimedes $\rho = a\theta$.

    Solution: subt. = $\frac{\rho^2}{a}$, tan. = $\frac{\rho}{a}\sqrt{a^2 + \rho^2}$, subn. = a, nor. = $\sqrt{a^2 + \rho^2}$. The student should note the fact that the subnormal is constant.

15. Get lengths of subtangent, subnormal, tangent, and normal in the logarithmic spiral $\rho = a^\theta$.

    Solution: subt. = $\frac{\rho}{\log a}$, tan. = $\rho\sqrt{1 + \frac{1}{\log^2 a}}$, subn. = $\rho \log\, a$, nor. = $\rho\sqrt{1 + \log^2 a}$.

    When $a = e$, we notice that subt. = subn., and tan. = nor.

16. Find the angles between the curves $\rho = a(1 + \cos\, \theta)$ and $\rho = b(1 - \cos\, \theta)$.

    Solution: 0 and $\frac{\pi}{2}$.

17. Show that the reciprocal spiral $\rho = \frac{a}{\theta}$ has a constant subtangent.

18. Show that the equilateral hyperbolas $\rho^2 \sin 2\theta = a^2$, $\rho^2 \cos 2\theta = b^2$ intersect at right angles.

5.10 Solution of equations having multiple roots

Any root which occurs more than once in an equation is called a *multiple root*. Thus 3, 3, 3, -2 are the roots of

$$x^4 - 7x^3 + 9x^2 + 27x - 54 = 0;$$

hence 3 is a multiple root occurring three times. Evidently this equation may also be written in the form

$$(x-3)^3(x+2) = 0.$$

Let $f(x)$ denote an integral rational function of x having a multiple root a, and suppose it occurs m times. Then we may write

$$f(x) = (x-a)^m \phi(x), \tag{5.18}$$

where $\phi(x)$ is the product of the factors corresponding to all the roots of $f(x)$ differing from a. Differentiating (5.18),

$$f'(x) = (x-a)^m \phi'(x) + m\phi(x)(x-a)^{m-1},$$

or,

$$f'(x) = (x-a)^{m-1}[(x-a)\phi'(x) + m\phi(x)]. \tag{5.19}$$

Therefore $f'(x)$ contains the factor $(x-a)$ repeated $m-1$ times and no more; that is, the *greatest common divisor* (G.C.D.) of $f(x)$ and $f'(x)$ has $m-1$ roots equal to a.

In case $f(x)$ has a second multiple root β occurring r times, it is evident that the G.C.D. would also contain the factor $(x-\beta)^{r-1}$ and so on for any number of different multiple roots, each occurring once more in $f(x)$ than in the G.C.D.

We may then state a *rule for finding the multiple roots* of an equation $f(x) = 0$ as follows:

- FIRST STEP. Find $f'(x)$.

- SECOND STEP. Find the G.C.D.of $f(x)$ and $f'(x)$.

- THIRD STEP. Find the roots of the G.C.D.Each different root of the G.C.D.will occur once more in $f(x)$ than it does in the G.C.D.

If it turns out that the G.C.D.does not involve x, then $f(x)$ has no multiple roots and the above process is of no assistance in the solution of the equation, but it may be of interest to know that the equation has no equal, i.e. multiple, roots.

Example 5.10.1. Solve the equation $x^3 - 8x^2 + 13x - 6 = 0$.
Solution. Place $f(x) = x^3 - 8x^2 + 13x - 6$.
First step. $f'(x) = 3x^2 - 16x + 13$.
Second step. G.C.D. = $x - 1$.
Third step. $x - 1 = 0$, therefore $x = 1$.
Since 1 occurs once as a root in the G.C.D.,it will occur twice in the given equation; that is, $(x-1)^2$ will occur there as a factor. Dividing $x^3-8x^2+13x-6$ by $(x-1)^2$ gives the only remaining factor $(x-6)$, yielding the root 6. The roots of our equation are then 1, 1, 6. Drawing the graph of the function, we see that at the double root $x = 1$ the graph touches the x-axis but does not cross it.
Note: Since the first derivative vanishes for every multiple root, it follows that the x-axis is tangent to the graph at all points corresponding to multiple roots. If a multiple root occurs an even number of times, the graph will not cross the x-axis at such a point (see Figure 5.12); if it occurs an odd number of times, the graph will cross.

5.11 Examples

1. $x^3 - 7x^2 + 16x - 12 = 0$.

   Ans. 2, 2, 3.

2. $x^4 - 6x^2 - 8x - 3 = 0$.

3. $x^4 - 7x^3 + 9x^2 + 27x - 64 = 0$.

   Ans. 3, 3, 3, -2.

4. $x^4 - 5x^3 - 9x^2 + 81x - 108 = 0$.

   Ans. 3, 3, 3, -4.

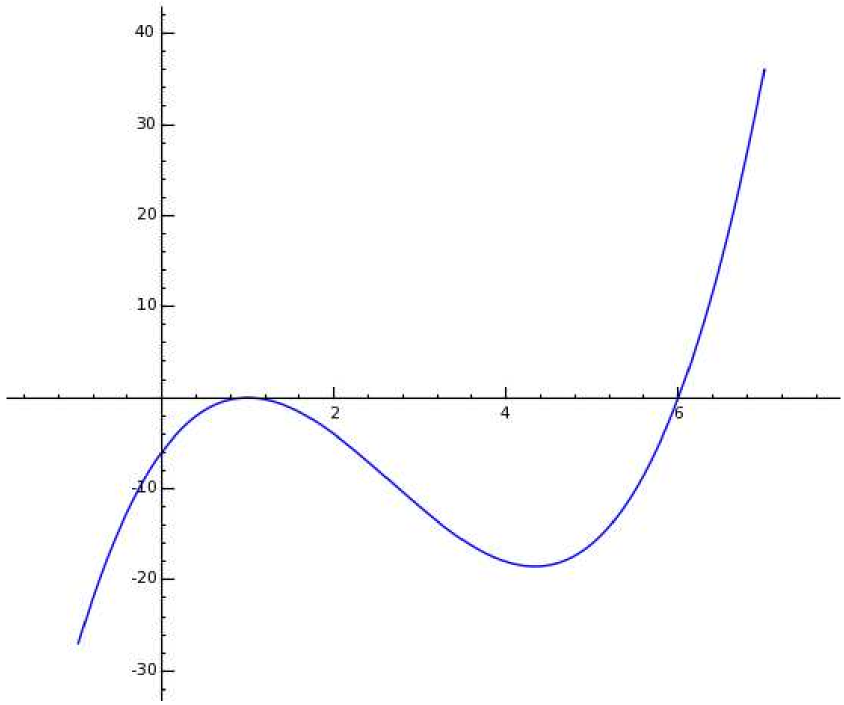

Figure 5.12: plot of $f(x) = (x-1)^2(x-6)$ illustrating a multiple root.

5. $x^4 + 6x^3 + x^2 - 24x + 16 = 0.$

   Ans. 1, 1, -4, -4.

6. $x^4 - 9x^3 + 23x^2 - 3x - 36 = 0.$

   Ans. 3, 3, -1, 4.

7. $x^4 - 6x^3 + 10x^2 - 8 = 0.$

   Ans. 2, 2, $1 \pm \sqrt{3}$.

Sage can help with this problem.

Sage

```
sage: x = var("x")
sage: solve(x^4 - 6*x^3 + 10*x^2 - 8 == 0,x)
[x == 1 - sqrt(3), x == sqrt(3) + 1, x == 2]
```

```
sage: factor(x^4 - 6*x^3 + 10*x^2 - 8)
(x - 2)^2*(x^2 - 2*x - 2)
```

This tells use that the root 2 occurs with multiplicity 2.

8. $x^5 - x^4 - 5x^3 + x^2 + 8x + 4 = 0.$

   Sage can help with this problem.

   Sage

```
sage: x = var("x")
sage: solve(x^5 - 15*x^3 + 10*x^2 + 60*x - 72 == 0,x)
[x == -3, x == 2]
sage: factor(x^5 - 15*x^3 + 10*x^2 + 60*x - 72)
(x - 2)^3*(x + 3)^2
```

   This tells use that the root 2 occurs with multiplicity 3 amd the root -3 occurs with multiplicity 2, as expected.

9. $x^5 - 15x^3 + 10x^2 + 60x - 72 = 0.$

   Ans. $2, 2, 2, -3, -3$.

10. $x^5 - 3x^4 - 5x^3 + 13x^2 + 24x + l0 = 0.$

Show that the following four equations have no multiple (equal) roots:

11. $x^3 + 9x^2 + 2x - 48 = 0.$

12. $x^4 - 15x^2 - 10x + 24 = 0.$

13. $x^4 - 3x^3 - 6x^2 + 14x + 12 = 0.$

14. $x^n - a^n = 0.$

15. Show that the condition that the equation

$$x^3 + 3qx + r = 0$$

shall have a double root is $4q^3 + r^2 = 0$.

16. Show that the condition that the equation

$$x^3 + 3px^2 + r = 0$$

shall have a double root is $r(4p^3 + r) = 0$.

5.12 Applications of the derivative in mechanics

Included also are applications to velocity and rectilinear motion.

Consider the motion of a point P on the straight line AB.

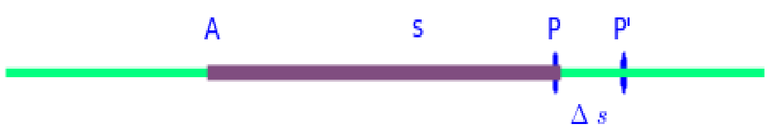

Figure 5.13: Illustration of rectilinear motion.

Let s be the distance measured from some fixed point as A to any position of P, and let t be the corresponding elapsed time. To each value of t corresponds a position of P and therefore a distance (or space) s. Hence s will be a function of t, and we may write

$$s = f(t)$$

Now let t take on an increment Δt; then s takes on an increment[8] Δs, and

[8] s being the space or distance passed over in the time Δt.

$$\frac{\Delta s}{\Delta t} = \text{the average velocity} \tag{5.20}$$

of P during the time interval Δt. If P moves with uniform motion, the above ratio will have the same value for every interval of time and is the velocity at any instant.

For the general case of any kind of motion, uniform or not, we define the *velocity* (or, time rate of change of s) at any instant as the limit of the ratio $\frac{\Delta s}{\Delta t}$ as Δt approaches the limit zero; that is,

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t},$$

or

$$v = \frac{ds}{dt} \tag{5.21}$$

The velocity is the derivative of the distance (= space) with respect to the time.

To show that this agrees with the conception we already have of velocity, let us find the velocity of a falling body at the end of two seconds.

By experiment it has been found that a body falling freely from rest in a vacuum near the earth's surface follows approximately the law

$$s = 16.1t^2 \tag{5.22}$$

where s = space fallen in feet, t = time in seconds. Apply the General Rule, §3.7, to (5.22).

FIRST STEP. $s + \Delta s = 16.1(t + \Delta t)^2 = 16.1t^2 + 32.2t \cdot \Delta t + 16.1(\Delta t)^2$.

SECOND STEP. $\Delta s = 32.2t \cdot \Delta t + 16.1(\Delta t)^2$.

THIRD STEP. $\frac{\Delta s}{\Delta t} = 32.2t + 16.1\Delta t =$ average velocity throughout the time interval Δt.

Placing $t = 2$,

$$\frac{\Delta s}{\Delta t} = 64.4 + 16.1\Delta t \tag{5.23}$$

which equals the average velocity throughout the time interval Δt after two seconds of falling. Our notion of velocity tells us at once that (5.23) does not give us the actual velocity at the end of two seconds; for even if we take Δt very small, say $\frac{1}{100}$ or $\frac{1}{1000}$ of a second, (5.23) still gives only the average velocity during the corresponding small interval of time. But what we do mean by the velocity at the end of two seconds is the limit of the average velocity when Δt diminishes towards zero; that is, the velocity at the end of two seconds is from (5.23), 64.4 ft. per second.

Thus even the everyday notion of velocity which we get from experience involves the idea of a limit, or in our notation

$$v = \lim_{\Delta t \to 0} \left(\frac{\Delta s}{\Delta t} \right) = 64.4\ ft./sec.$$

The above example illustrates well the notion of a limiting value. The student should be impressed with the idea that a limiting value is a definite, fixed value, not something that is only approximated. Observe that it does not make any difference how small $16.1\Delta t$ may be taken; it is only the limiting value of $64.4 + 16.1\Delta t$, when Δt diminishes towards zero, that is of importance, and that value is exactly 64.4.

5.13 Component velocities. Curvilinear motion

The coordinates x and y of a point P moving in the xy-plane are also functions of time, and the motion may be defined by means of two equations[9], $x = f(t)$, $y = g(t)$. These are the *parametric equations* of the path (see §5.5).

The horizontal component[10] v_x of v is the velocity along the x-axis of the projection M of P, and is therefore the time rate of change of x. Hence, from (5.21), when s is replaced by x, we get

$$v_x = \frac{dx}{dt}. \tag{5.24}$$

In the same way we get the vertical component, or time rate of change of y,

$$v_y = \frac{dy}{dt}. \tag{5.25}$$

Representing the velocity and its components by vectors, we have at once from the figure

$$v^2 = {v_x}^2 + {v_y}^2,$$

[9]The equation of the path in rectangular coordinates may often be found by eliminating t between their equations.

[10]The direction of v is along the tangent to the path.

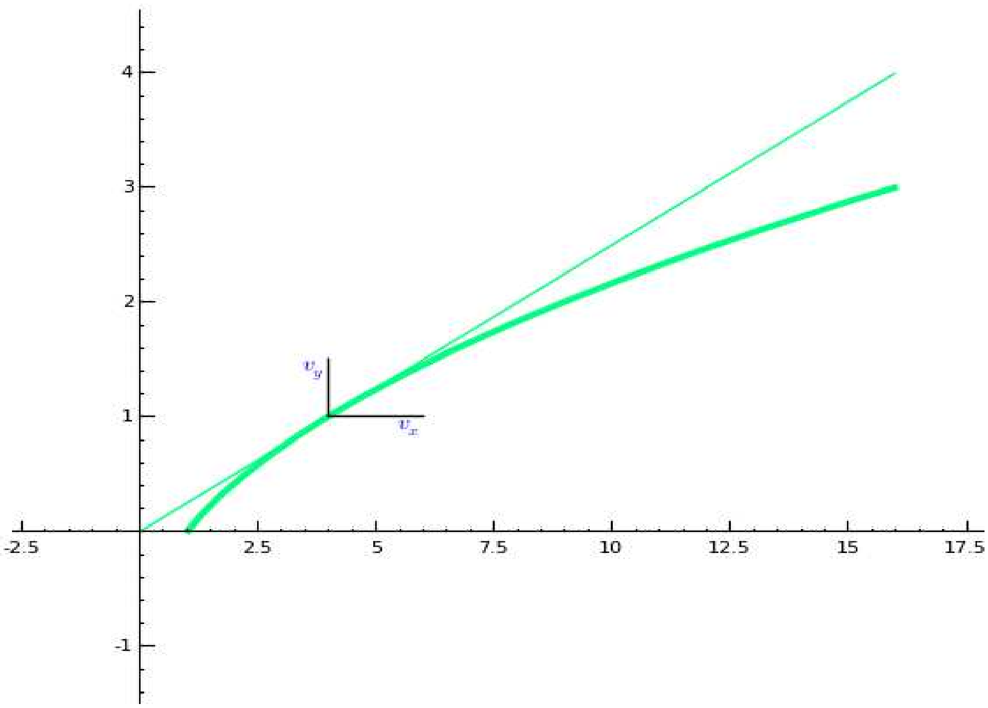

Figure 5.14: The components of velocity.

or,

$$v = \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}, \tag{5.26}$$

giving the magnitude of the velocity at any instant.

If τ be the angle which the direction of the velocity makes with the x-axis; we have from the figure, using (5.21), (5.24), (5.25),

$$\sin\tau = \frac{v_y}{v} = \frac{\frac{dy}{dt}}{\frac{ds}{dt}};\ \ \cos\tau = \frac{v_x}{v} = \frac{\frac{dx}{dt}}{\frac{ds}{ds}};\ \ \tan\tau = \frac{v_y}{v_x} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{5.27}$$

5.14 Acceleration. Rectilinear motion

In general, v will be a function of t. Now let t take on an increment Δt, then v takes on an increment Δv, and $\frac{\Delta v}{\Delta t}$ is the average acceleration of P during the time interval Δt. We define the *acceleration* a at any instant as the limit of the ratio $\frac{\Delta v}{\Delta t}$ as Δt approaches the limit zero; that is,

$$a = \lim_{\Delta t \to 0} \left(\frac{\Delta v}{\Delta t} \right),$$

or,

$$a = \frac{dv}{dt} \tag{5.28}$$

The acceleration is the derivative of the velocity with respect to time.

5.15 Component accelerations. Curvilinear motion

In treatises on Mechanics it is shown that in curvilinear motion the acceleration is not, like the velocity, directed along the tangent, but toward the concave side, of the path of motion. It may be resolved into a tangential component, a_t, and a normal component, a_n where

$$a_t = \frac{dv}{dt}; \quad a_n = \frac{v^2}{R}. \tag{5.29}$$

(R is the radius of curvature. See §11.5.)

The acceleration may also be resolved into components parallel to the axes of the path of motion. Following the same plan used in §5.13 for finding component velocities, we define the component accelerations parallel to the x-axis and y-axis,

$$a_x = \frac{dv_x}{dt}; \quad a_y = \frac{dv_y}{dt}. \tag{5.30}$$

Also,

$$a = \sqrt{\left(\frac{dv_x}{dt} \right)^2 + \left(\frac{dv_y}{dt} \right)^2}, \tag{5.31}$$

which gives the magnitude of the acceleration at any instant.

5.16 Examples

1. By experiment it has been found that a body falling freely from rest in a vacuum near the earth's surface follows approximately the law $s = 16.1t^2$, where s = space (height) in feet, t = time in seconds. Find the velocity and acceleration

   (a) at any instant;

   (b) at end of the first second;

   (c) at end of the fifth second.

   Solution. We have $s = 16.1t^2$.

   (a) Differentiating, $\frac{ds}{dt} = 32.2t$, or, from (5.21), $v = 32.2t$ ft./sec. Differentiating again, $\frac{dv}{dt} = 32.2$, or, from (5.28), $a = 32.2 \text{ ft.}/(\text{sec.})^2$, which tells us that the acceleration of a falling body is constant; in other words, the velocity increases 32.2 ft./sec. every second it keeps on falling.

   (b) To find v and a at the end of the first second, substitute $t = 1$ to get $v = 32.2$ ft./sec., $a = 32.2 \text{ ft.}/(\text{sec.})^2$.

   (c) To find v and a at the end of the fifth second, substitute $t = 5$ to get $v = 161$ ft./sec., $a = 32.2 \text{ ft.}/(\text{sec.})^2$.

2. Neglecting the resistance of the air, the equations of motion for a projectile are

$$x = v_0 \cos\phi \cdot t, \quad y = v_0 \sin\phi \cdot t - 16.1t^2;$$

   where v_0 = initial velocity, ϕ = angle of projection with horizon, t = time of flight in seconds, x and y being measured in feet. Find the velocity, acceleration, component velocities, and component accelerations

   (a) at any instant;

   (b) at the end of the first second, having given $v_0 = 100$ ft. per sec., $\phi = 30^0 = \pi/6$;

   (c) find direction of motion at the end of the first second.

Solution. From (5.24) and (5.25), (a) $v_x = v_0 \cos\phi$; $v_y = v_0 \sin\phi - 32.2t$. Also, from (5.26), $v = \sqrt{v_0{}^2 - 64.4tv_0 \sin\phi + 1036.8t^2}$. From (5.30) and (5.31), $a_x = 0$; $a_y = 32.2$; $a = 32.2$.

(b) Substituting $t = 1$, $v_0 = 100$, $\phi = 30^0 = \pi/6$ in these results, we get $v_x = 86.6$ ft./sec., $a_x = 0$; $v_y = 17.8$ ft./sec., $a_y = -32.2 \text{ ft.}/(\text{sec.})^2$; $v = 88.4$ ft./sec., $a = 32.2 \text{ ft.}/(\text{sec.})^2$.

(c) $\tau = \arctan \frac{v_y}{v_x} = \arctan \frac{17.8}{86.6} = 0.2027... \approx 11^o$, which is the angle of direction of motion with the horizontal.

3. Given the following equations of rectilinear motion. Find the distance, velocity, and acceleration at the instant indicated:

(a) $s = t^3 + 2t^2$; $t = 2$.

Ans. $s = 16$, $v = 20$, $a = 16$.

(b) $s = t^2 + 2t$; $t = 3$.

Ans. $s = 15$, $v = 8$, $a = 2$.

(c) $s = 3 - 4t$; $t = 4$.

Ans. $s = -13$, $v = -4$, $a = 0$.

(d) $x = 2t - t^2$; $t = 1$.

Ans. $x = 1$, $v = 0$, $a = -2$.

(e) $y = 2t - t^3$; $t = 0$.

Ans. $y = 0$, $v = 2$, $a = 0$.

(f) $h = 20t + 16t^2$; $t = 10$.

Ans. $h = 1800$, $v = 340$, $a = 32$.

(g) $s = 2\sin\, t$; $t = \frac{\pi}{4}$.

Ans. $s = \sqrt{2}$, $v = \sqrt{2}$, $a = -\sqrt{2}$.

(h) $y = a\cos\frac{\pi t}{3}$; $t = 1$.

Ans. $y = \frac{a}{2}$, $v = -\frac{\pi a\sqrt{3}}{6}$, $a = -\frac{\pi^2 a}{18}$.

(i) $s = 2e^{3t}$; $t = 0$.

Ans. $s = 2$, $v = 6$, $a = 18$.

(j) $s = 2t^2 - 3t$; $t = 2$.

(k) $x = 4 + t^3$; $t = 3$.

(l) $y = 5\cos 2t; t = \frac{\pi}{6}$.

(m) $s = b\sin\frac{\pi t}{4}; t = 2$.

(n) $x = ae^{-2t}; t = 1$.

(o) $s = \frac{a}{t} + bt^2; t = t_0$.

(p) $s = 10\log\frac{4}{4+t}; t = 1$.

4. If a projectile be given an initial velocity of 200 ft. per sec. in a direction inclined $45^o = \pi/4$ with the horizontal, find

   (a) the velocity and direction of motion at the end of the third and sixth seconds;

   (b) the component velocities at the same instants.

   Conditions are the same as for Exercise 2.

   Ans.
   (a) When $t = 3$, $v = 148.3$ ft. per sec., $\tau = 0.3068... = 17^o35'$; when $t = 6$, $v = 150.5$ ft. per sec., $\tau = 2.79049... = 159^o53'$;

   (b) When $t = 3$, $v_x = 141.4$ ft. per sec., $v_y = 44.8$ ft. per sec.; when $t = 6$, $v_x = 141.4$ ft. per sec., $v_y = -51.8$ ft. per sec.

5. The height (= s) in feet reached in t seconds by a body projected vertically upwards with a velocity of v_0 ft. per sec. is given by the formula $s = v_0t - 16.1t^2$. Find

   (a) velocity and acceleration at any instant; and, if $v_0 = 300$ ft. per sec., find velocity and acceleration

   (b) at end of 2 seconds;

   (c) at end of 15 seconds. Resistance of air is neglected.

   Ans. (a) $v = v_0 - 32.2t$, $a = -32.2$; (b) $v = 235.6$ ft. per sec. Upwards, $a = 32.2$ ft. per (sec.)^2 downwards; (c) $v = 183$ ft. per sec. Downwards, $a = 32.2$ ft. per (sec.)^2 downwards.

6. A cannon ball is fired vertically upwards with a muzzle velocity of 644 ft. per sec. Find (a) its velocity at the end of 10 seconds; (b) for how long it will continue to rise. Conditions same as for Exercise 5.

   Ans. (a) 322 ft. per sec. Upwards; (b) 20 seconds.

7. A train left a station and in t hours was at a distance (space) of

$$s = t^3 + 2t^2 + 3t$$

miles from the starting point. Find its acceleration (a) at the end of t hours; (b) at the end of 2 hours.

Ans. (a) $a = 6t + 4$; (b) $a = 16$ miles/(hour)^2.

8. In t hours a train had reached a point at the distance of $\frac{1}{4}t^4 - 4t^3 + 16t^2$ miles from the starting point.

   (a) Find its velocity and acceleration.

   (b) When will the train stop to change the direction of its motion?

   (c) Describe the motion during the first 10 hours.

   Ans. (a) $v = t^3 - 12t^2 + 32t$, $a = 3t^2 - 24t + 32$;

   (b) at end of fourth and eighth hours;

   (c) forward first 4 hours, backward the next 4 hours, forward again after 8 hours.

9. The space in feet described in t seconds by a point is expressed by the formula

$$s = 48t - 16t^2.$$

Find the velocity and acceleration at the end of $\frac{3}{2}$ seconds.

Ans. $v = 0$,$a = -32 \text{ ft.}/(\text{sec.})^2$.

10. Find the acceleration, having given

   (a) $v = t^2 + 2t$; $t = 3$.
   Ans. $a = 8$.

   (b) $v = 3t - t^3$; $t = 2$.
   Ans. $a = -9$.

   (c) $v = 4 \sin \frac{t}{2}$; $t = \frac{\pi}{3}$.
   Ans. $a = \sqrt{3}$.

(d) $v = r\cos 3t$; $t = \frac{\pi}{6}$.
Ans. $a = -3r$.

(e) $v = 5e^{2t}$; $t = 1$.
Ans. $a = 10e^2$.

11. At the end of t seconds a body has a velocity of $3t^2 + 2t$ ft. per sec.; find its acceleration (a) in general; (b) at the end of 4 seconds.

    Ans. (a) $a = 6t + 2 \text{ ft.}/(\text{sec.})^2$; (b) $a = 26 \text{ ft.}/(\text{sec.})^2$

12. The vertical component of velocity of a point at the end of t seconds is

$$v_y = 3t^2 - 2t + 6$$

    in ft. per sec. Find the vertical component of acceleration (a) at any instant; (b) at the end of 2 seconds.

    Ans. (a) $a_y = 6t - 2$; (b) $10 \text{ ft.}/(\text{sec.})^2$.

13. If a point moves in a fixed path so that

$$s = \sqrt{t},$$

    show that the acceleration is negative and proportional to the cube of the velocity.

14. If the distance travelled at time t is given by

$$s = c_1e^t + c_2e^{-t},$$

    for some constants c_1 and c_2, show that the acceleration is always equal in magnitude to the space passed over.

15. If a point referred to rectangular coordinates moves so that

$$x = a_1 + a_2 \cos\, t, \quad y = b_1 + b_2 \sin\, t,$$

    for some constants a_i and b_i,show that its velocity has a constant magnitude.

16. If the path of a moving point is the sine curve

$$\begin{cases} x = at, \\ y = b \sin\, at \end{cases}$$

show (a) that the x-component of the velocity is constant; (b) that the acceleration of the point at any instant is proportional to its distance from the x-axis.

17. Given the following equations of curvilinear motion, find at the given instant

- v_x, v_y, v;
- a_x, a_y, a;
- position of point (coordinates);
- direction of motion.
- the equation of the path in rectangular coordinates.

(a) $x = t^2$, $y = t$; $t = 2$.

(g) $x = 2\sin\, t$, $y = 3\cos\, t$; $t = \pi$.

(b) $x = t$, $y = t^3$; $t = 1$.

(h) $x = \sin\, t$, $y = \cos\, 2t$; $t = \frac{\pi}{4}$.

(c) $x = t^2$, $y = t^3$; $t = 3$.

(i) $x = 2t$, $y = 3e^t$; $t = 0$.

(d) $x = 2t$, $y = t^2 + 3$; $t = 0$.

(e) $x = 1 - t^2$, $y = 2t$; $t = 2$.

(j) $x = 3t$, $y = \log\, t$; $t = 1$.

(f) $x = r\sin\, t$, $y = r\cos\, t$; $t = \frac{3\pi}{4}$.

(k) $x = t$, $y = 12/t$; $t = 3$.

5.17 Application: Newton's method

[11]

Newton's method (also known as the Newton-Raphson method) is an efficient algorithm for finding approximations to the zeros (or roots) of a real-valued function. As such, it is an example of a root-finding algorithm. It produces iteratively a sequence of approximations to the root. It can also be used to find a minimum or maximum of such a function, by finding a zero in the function's first derivative.

5.17.1 Description of the method

The idea of the method is as follows: one starts with an initial guess which is reasonably close to the true root, then the function is approximated by its tangent line (which can be computed using the tools of calculus), and one computes the x-intercept of this tangent line (which is easily done with elementary algebra). This x-intercept will typically be a better approximation to the function's root than the original guess, and the method can be iterated.

Suppose $f : [a, b] \to \mathbb{R}$ is a differentiable function defined on the interval $[a, b]$ with values in the real numbers $\mathbb{R}$. The formula for converging on the root can be easily derived. Suppose we have some current approximation x_n. Then we can derive the formula for a better approximation, x_{n+1} by referring to the diagram on the right. We know from the definition of the derivative at a given point that it is the slope of a tangent at that point.

That is

$$f'(x_n) = \frac{\text{rise}}{\text{run}} = \frac{\Delta \text{y}}{\Delta \text{x}} = \frac{f(x_n) - 0}{x_n - x_{n+1}} = \frac{0 - f(x_n)}{(x_{n+1} - x_n)}.$$

Here, f' denotes the derivative of the function f. Then by simple algebra we can derive

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

We start the process off with some arbitrary initial value x_0. (The closer to the zero, the better. But, in the absence of any intuition about where the zero might lie, a "guess and check" method might narrow the possibilities to a reasonably small interval by appealing to the intermediate value theorem.) The method will

[11] This section uses material modified from Wikipedia [N].

usually converge, provided this initial guess is close enough to the unknown zero, and that $f'(x_0) \neq 0$. Furthermore, for a zero of multiplicity 1, the convergence is at least quadratic (see rate of convergence) in a neighbourhood of the zero, which intuitively means that the number of correct digits roughly at least doubles in every step. More details can be found in the analysis section below.

Example 5.17.1. Consider the problem of finding the positive number x with $\cos(x) = x^3$. We can rephrase that as finding the zero of $f(x) = \cos(x) - x^3$. We have $f'(x) = -\sin(x) - 3x^2$. Since $\cos(x) \leq 1$ for all x and $x^3 > 1$ for $x > 1$, we know that our zero lies between 0 and 1. We try a starting value of $x_0 = 0.5$.

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} &&= 0.5 - \frac{\cos(0.5) - 0.5^3}{-\sin(0.5) - 3 \times 0.5^2} = 1.112141637097 \\
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} &&= \underline{0.9}09672693736 \\
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} &&= \underline{0.86}7263818209 \\
x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} &&= \underline{0.86547}7135298 \\
x_5 &= x_4 - \frac{f(x_4)}{f'(x_4)} &&= \underline{0.8654740331}11 \\
x_6 &= x_5 - \frac{f(x_5)}{f'(x_5)} &&= \underline{0.865474033102}
\end{aligned}$$

The correct digits are underlined in the above example. In particular, x_6 is correct to the number of decimal places given. We see that the number of correct digits after the decimal point increases from 2 (for x_3) to 5 and 10, illustrating the quadratic convergence.

5.17.2 Analysis

Suppose that the function f has a zero at a, i.e., $f(a) = 0$.

If f is continuously differentiable and its derivative does not vanish at a, then there exists a neighborhood of a such that for all starting values x_0 in that neighborhood, the sequence $\{x_n\}$ will converge to a.

In practice this result is "local" and the neighborhood of convergence is not known a priori, but there are also some results on "global convergence." For instance, given a right neighborhood U of a, if f is twice differentiable in U and if $f' \neq 0$, $f \cdot f'' > 0$ in U, then, for each $x_0 \in U$ the sequence x_k is monotonically decreasing to a.

5.17.3 Fractals

For complex functions $f : \mathbb{C} \to \mathbb{C}$, however, Newton's method can be directly applied to find their zeros. For many complex functions, the boundary of the set (also known as the basin of attraction) of all starting values that cause the method to converge to a particular zero is a fractal[12]

For example, the function $f(x) = x^5 - 1$, $x \in \mathbb{C}$, has five roots, equally spaced around the unit circle in the complex plane. If x_0 is a starting point which converges to the root at $x = 1$, color x_0 yellow. Repeat this using four other colors (blue, red, green, purple) for the other four roots of f. The resulting image is in Figure 5.15.

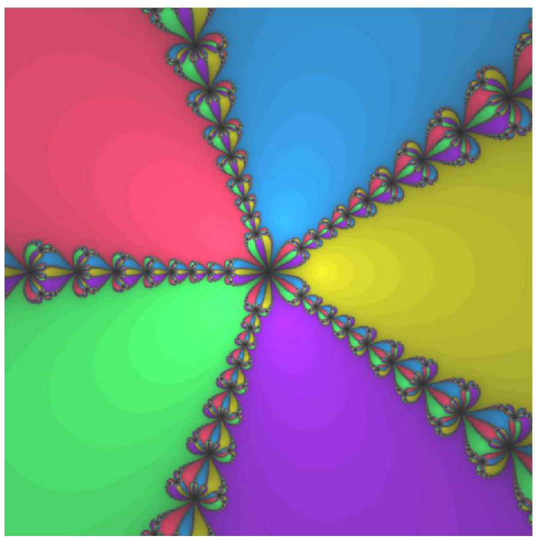

Figure 5.15: Basins of attraction for $x^5 - 1 = 0$; darker means more iterations to converge.

[12]The definition of a fractal would take us too far afield. Roughly speaking, it is a geometrical object with certain self-similarity properties [F].

CHAPTER

SIX

Successive differentiation

6.1 Definition of successive derivatives

We have seen that the derivative, if it exists, of a function of x is also a function of x. This new function may itself be differentiable, in which case the derivative of the first derivative is called the second derivative of the original function. Similarly, the derivative of the second derivative is called the third derivative; and so on. Thus, if

$$\begin{aligned} y &= 3x^4, \\ \frac{dy}{dx} &= 12x^3, \\ \frac{d}{dx}\left(\frac{dy}{dx}\right) &= 36x^2, \\ \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] &= 72x, \end{aligned}$$

and so on.

6.2 Notation

The symbols for the successive derivatives are usually abbreviated as follows:

$$\begin{aligned} \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d^2y}{dx^2}, \\ \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] &= \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}, \\ \ldots & \quad \ldots \\ \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) &= \frac{d^ny}{dx^n}. \end{aligned}$$

If $y = f(x)$, the successive derivatives are also denoted by

$$f'(x),\ f''(x),\ f'''(x),\ f^{(4)}(x),\ ...,\ f^{(n)}(x);$$

or

$$y',\ y'',\ y''',\ y^{(4)},\ ...,\ y^{(n)};$$

or,

$$\frac{d}{dx}f(x),\ \frac{d^2}{dx^2}f(x),\ \frac{d^3}{dx^3}f(x),\ \frac{d^4}{dx^4}f(x),\ ...,\ \frac{d^n}{dx^n}f(x).$$

6.3 The n-th derivative

For certain functions a general formula involving n may be found in the expression for the n-th derivative. To discsover this formula, the usual plan is to find a number of successive derivatives, as many as may be necessary to discover by induction the formula. This formula can then (hopefully) be proven by the method of mathematical induction[1].

Example 6.3.1. Given $y = e^{ax}$, find $\frac{d^n y}{dx^n}$.
Solution. $\frac{dy}{dx} = ae^{ax}$, $\frac{d^2y}{dx^2} = a^2e^{ax}, \ldots, \frac{d^ny}{dx^n} = a^ne^{ax}$.

Example 6.3.2. Given $y = \log\ x$, find $\frac{d^n y}{dx^n}$.
Solution. $\frac{dy}{dx} = \frac{1}{x}$, $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$, $\frac{d^3y}{dx^3} = \frac{1\cdot 2}{x^3}$, $\frac{d^4y}{dx^4} = \frac{1\cdot 2\cdot 3}{x^4}, \ldots \frac{d^ny}{dx^n} = (-1)^{n-1}\frac{(n-1)!}{x^n}$.

Example 6.3.3. Given $y = \sin\ x$, find $\frac{d^n y}{dx^n}$.
Solution. $\frac{dy}{dx} = \cos\ x = \sin\left(x + \frac{\pi}{2}\right)$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + \frac{2\pi}{2}\right),$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\sin\left(x + \frac{2\pi}{2}\right) = \cos\left(x + \frac{2\pi}{2}\right) = \sin\left(x + \frac{3\pi}{2}\right)$$

$$\cdots$$

$$\frac{d^ny}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right).$$

[1]The mathod of inductin is usually taught in a course after calculus. For the curious reader, we recommend the discussion and references in Wikipedia `http://en.wikipedia.org/wiki/Mathematical_induction` as a good start.

6.4 Leibnitz's Formula for the n-th derivative of a product

This formula expresses the n-th derivative of the product of two variables in terms of the variables themselves and their successive derivatives.

If u and v are functions of x, we have, from equation (4.5) in §4.1 above,

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}.$$

Differentiating again with respect to x,

$$\frac{d^2}{dx^2}(uv) = \frac{d^2u}{dx^2}v + \frac{du}{dx}\frac{dv}{dx} + \frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2} = \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}.$$

Similarly,

$$\begin{aligned}\frac{d^3}{dx^3}(uv) &= \frac{d^3u}{dx^3} + \frac{d^2u}{dx^2}\frac{dv}{dx} + 2\frac{d^2u}{dx^2}\frac{dv}{dx} + 2\frac{du}{dx}\frac{d^2v}{dx^2} + \frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3} \\ &= \frac{d^3u}{dx^3}v + 3\frac{d^2u}{dx^2}\frac{dv}{dx} + 3\frac{du}{dx}\frac{d^2v}{dx^2} + u\frac{d^3v}{dx^3}.\end{aligned}$$

However far this process may be continued, it will be seen that the numerical coefficients follow the same law as those of the Binomial Theorem, and the indices of the derivatives correspond[2] to the exponents of the Binomial Theorem. Reasoning then by mathematical induction from the m-th to the $(m+1)$-st derivative of the product, we can prove *Leibnitz's Formula*

$$\frac{d^n}{dx^n}(uv) = \frac{d^nu}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \frac{n(n-1)}{2!}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2} + \cdots + n\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + u\frac{d^nv}{dx^n}, \tag{6.1}$$

for all $n > 0$.

Example 6.4.1. Given $y = e^x \log x$, find $\frac{d^3y}{dx^3}$ by Leibnitz's Formula.

Solution. Let $u = e^x$, and $v = \log x$; then $\frac{du}{dx} = e^x$, $\frac{dv}{dx} = \frac{1}{x}$, $\frac{d^2u}{dx^2} = e^x$, $\frac{d^2v}{dx^2} = -\frac{1}{x^2}$, $\frac{d^3u}{dx^3} = e^x$, $\frac{d^3v}{dx^3} = \frac{2}{x^3}$.

Substituting in (6.1), we get

$$\frac{d^3y}{dx^3} = e^x \log x + \frac{3e^x}{x} - \frac{3e^x}{x^2} = e^x\left(\log x + \frac{3}{x} - \frac{3}{x^2} + \frac{2}{x^3}\right).$$

This can be verified using the Sage commands:

[2] To make this correspondence complete, u and v are considered as $\frac{d^0u}{dx^0}$ and $\frac{d^0v}{dx^0}$.

Sage

```
sage: x = var("x")
sage: f = exp(x)*log(x)
sage: diff(f,x,1); diff(f,x,2); diff(f,x,3)
e^x*log(x) + e^x/x
e^x*log(x) + 2*e^x/x - e^x/x^2
e^x*log(x) + 3*e^x/x - 3*e^x/x^2 + 2*e^x/x^3
sage: diff(f*g,x,1)
f(x)*diff(g(x), x, 1) + g(x)*diff(f(x), x, 1)
sage: diff(f*g,x,2)
f(x)*diff(g(x), x, 2)+2*diff(f(x), x, 1)*diff(g(x), x, 1)\
 + g(x)*diff(f(x), x, 2)
```

Example 6.4.2. Given $y = x^2 e^{ax}$, find $\frac{d^n y}{dx^n}$ by Leibnitz's Formula.

Solution. Let $u = x^2$, and $v = e^{ax}$; then $\frac{du}{dx} = 2x$, $\frac{dv}{dx} = ae^{ax}$, $\frac{d^2u}{dx^2} = 2x$, $\frac{d^2v}{dx^2} = a^2e^{ax}$, $\frac{d^3u}{dx^3} = 0$, $\frac{d^3v}{dx^3} = a^3e^{ax}$, $\dots$, $\frac{d^nu}{dx^n} = 0$, $\frac{d^nv}{dx^n} = a^ne^{ax}$. Substituting in (6.1), we get

$$\frac{d^n y}{dx^n} = x^2a^ne^{ax}+2na^{n-1}xe^{ax}+n(n-1)a^{n-2}e^{ax} = a^{n-2}e^{ax}[x^2a^2+2nax+n(n-1)].$$

6.5 Successive differentiation of implicit functions

To illustrate the process we shall find $\frac{d^2y}{dx^2}$ from the equation of the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2.$$

Differentiating with respect to x, as in §4.33,

$$2b^2x - 2a^2y\frac{dy}{dx} = 0,$$

or,

$$\frac{dy}{dx} = \frac{b^2x}{a^2y}. \tag{6.2}$$

Differentiating again, remembering that y is a function of x,

$$\frac{d^2y}{dx^2} = \frac{a^2yb^2 - b^2xa^2\frac{dy}{dx}}{a^4y^2}.$$

Substituting for $\frac{dy}{dx}$ its value from (6.2),

$$\frac{d^2y}{dx^2} = \frac{a^2b^2y - a^2b^2x\left(\frac{b^2y}{a^2y}\right)}{a^4y^2} = -\frac{b^2(b^2x^2 - a^2y^2)}{a^4y^3}.$$

The given equation, $b^2x^2 - a^2y^2 = a^2b^2$, therefore gives,

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}.$$

Sage can be made to do a lot of this work for you (though the notation doesn't get any prettier):

Sage

```
sage: x = var("x")
sage: y = function("y",x)
sage: a = var("a")
sage: b = var("b")
sage: F = x^2/a^2 - y^2/b^2 - 1
sage: F.diff(x)
2*x/a^2 - 2*y(x)*diff(y(x), x, 1)/b^2
sage: F.diff(x,2)
-2*y(x)*diff(y(x), x, 2)/b^2 - 2*diff(y(x), x, 1)^2/b^2 + 2/a^2
sage: solve(F.diff(x) == 0, diff(y(x), x, 1))
[diff(y(x), x, 1) == b^2*x/(a^2*y(x))]
sage: solve(F.diff(x,2) == 0, diff(y(x), x, 2))
[diff(y(x), x, 2) == (b^2 - a^2*diff(y(x), x, 1)^2)/(a^2*y(x))]
```

This basically says

$$y' = \frac{dy}{dx} = \frac{b^2x}{a^2y},$$

and

$$y'' = \frac{d^2y}{dx^2} = -\frac{b^2 - a^2(y')^2}{a^2y}.$$

Now simply plug the first equation into the second, obtaining $y'' = -b^2 \frac{1-a^{-2}b^2x^2/y^2}{a^2y}$. Next, use the given equation in the form $a^{-2}b^2x^2/y^2 - 1 = b^2/y^2$ to get the result above.

6.6 Exercises

Verify the following derivatives:

1. $y = 4x^3 - 6x^2 + 4x + 7$.

   Ans. $\frac{d^2y}{dx^2} = 12(2x-1)$.

2. $f(x) = \frac{x^3}{1-x}$.

   Ans. $f^{(4)}(x) = \frac{4!}{(1-x)^5}$.

3. $f(y) = y^6$.

   Ans. $f^{(6)}(y) = 6!$.

4. $y = x^3 \log x$.

   Ans. $\frac{d^4y}{dx^4} = \frac{6}{x}$.

5. $y = \frac{c}{x^n}$. $y'' = \frac{n(n+1)c}{x^{n+2}}$.

6. $y = (x-3)e^{2x} + 4xe^x + x$.

   Ans. $y'' = 4e^x[(x-2)e^x + x + 2]$.

7. $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.

   Ans. $y'' = \frac{1}{2a}(e^{\frac{x}{a}} + e^{-\frac{x}{a}}) = \frac{y}{a^2}$.

8. $f(x) = ax^2 + bx + c$.

   Ans. $f'''(x) = 0$.

9. $f(x) = \log(x+1)$.

   Ans. $f^{(4)}(x) = -\frac{6}{(x+1)^4}$.

10. $f(x) = log(e^x + e^{-x})$.

    Ans. $f'''(x) = -\frac{8(e^x - e^{-x})}{(e^x - e^{-x})^3}$.

11. $r = \sin\ a\theta$.

    Ans. $\frac{d^4r}{d\theta^4} = a^4 \sin a\theta = a^4 r$.

12. $r = \tan\ \phi$.

    Ans. $\frac{d^3r}{d\phi^3} = 6\sec^6\phi - 4\sec^2\phi$.

13. $r = \log\sin\ \phi$.

    Ans. $r''' = 2\cot\ \phi\csc^2\phi$.

14. $f(t) = e^{-t}\cos\ t$.

    Ans. $f^{(4)}(t) = -4e^{-t}\cos\ t = -4f(t)$.

15. $f(\theta) = \sqrt{\sec 2\theta}$.

    Ans. $f''(\theta) = 3[f(\theta)]5 - f(\theta)$.

16. $p = (q^2 + a^2)\arctan\frac{q}{a}$.

    Ans. $\frac{d^3p}{dq^3} = \frac{4a^3}{(a^2+q^2)^2}$.

17. $y = a^x$.

    Ans. $\frac{d^ny}{dx^n} = (\log a)^n a^x$.

Sage

```
sage: a,x = var("a,x")
sage: y = a^x
sage: diff(y,x); diff(y,x,2); diff(y,x,3); diff(y,x,4)
a^x*log(a)
a^x*log(a)^2
a^x*log(a)^3
a^x*log(a)^4
```

18. $y = \log(1 + x)$.

    Ans. $\frac{d^ny}{dx^n} = (-1)^{n-1}\frac{(n-1)!}{(1+x)^n}$.

19. $y = \cos\ ax$.

    Ans. $\frac{d^ny}{dx^n} = a^n\cos\left(ax + \frac{n\pi}{2}\right)$.

20. $y = x^{n-1} \log\, x$.

 Ans. $\frac{d^n y}{dx^n} = \frac{(n-1)!}{x}$.

21. $y = \frac{1-x}{1+x}$.

 Ans. $\frac{d^n y}{dx^n} = 2(-1)^n \frac{n!}{(1+x)^{n+1}}$.

 Hint: Reduce fraction to form $-1 + \frac{2}{1+x}$ before differentiating.

22. If $y = e^x \sin\, x$, prove that $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

23. If $y = a\cos(\log\, x) + b\sin(\log\, x)$, prove that $x^2\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 0$.

Use Leibnitz's Formula in the next four examples:

24. $y = x^2 a^x$.

 Ans. $\frac{d^n y}{dx^n} = a^x(\log a)^{n-2}[(x\log a + n)^2 - n]$.

25. $y = xe^x$.

 Ans. $\frac{d^n y}{dx^n} = (x+n)e^x$.

26. $f(x) = e^x \sin\, x$.

 Ans. $f^{(n)}(x) = (\sqrt{2})^n e^x \sin\left(x + \frac{n\pi}{4}\right)$.

27. $f(\theta) = \cos\, a\theta \cos\, b\theta$.

 Ans. $f^n(\theta) = \frac{(a+b)^n}{2}\cos\left[(a+b)\theta + \frac{n\pi}{2}\right] + \frac{(a-b)^n}{2}\cos\left[(a-b)\theta + \frac{n\pi}{2}\right]$.

28. Show that the formulas for acceleration, (5.28), (5.30), may be written $a = \frac{d^2 s}{dt^2}$, $a_x = \frac{d^2 x}{dt^2}$, $a_y = \frac{d^2 y}{dt^2}$.

29. $y^2 = 4ax$.

 Ans. $\frac{d^2 y}{dx^2} = -\frac{4a^2}{y^3}$.

30. $b^2x^2 + a^2y^2 = a^2b^2$.

 Ans. $\frac{d^2 y}{dx^2} = -\frac{b^4}{a^2y^3}$; $\frac{d^3 y}{dx^2} = -\frac{3b^6 x}{a^4y^5}$.

31. $x^2 + y^2 = r^2$. $\frac{d^2 y}{dx^2} = -\frac{r^2}{y^3}$.

32. $y^2 + y = x^2$.

    Ans. $\frac{d^3y}{dx^3} = -\frac{24x}{(1+2y)^5}$.

33. $ax^2 + 2hxy + by^2 = 1$.

    Ans. $\frac{d^2y}{dx^2} = \frac{h^2-ab}{(hx+by)^3}$.

34. $y^2 - 2xy = a^2$.

    Ans. $\frac{d^2y}{dx^2} = \frac{a^2}{(y-x)^3}$; $\frac{d^3y}{dx^3} = -\frac{3a^2x}{(y-x)^5}$.

35. $\sec\phi \cos\theta = c$.

    Ans. $\frac{d^2\theta}{d\phi^2} = \frac{\tan^2\theta - \tan^2\phi}{\tan^3\theta}$.

36. $\theta = \tan(\phi + \theta)$.

    Ans. $\frac{d^3\theta}{d\phi^3} = -\frac{2(5+8\theta^2+3\theta^4)}{\theta^8}$.

37. Find the second derivative in the following:

$$
\begin{array}{ll}
(a)\ \log(u+v) = u - v. & (e)\ y^3 + x^3 - 3axy = 0. \\
(b)\ e^u + u = e^v + v. & (f)\ y^2 - 2mxy + x^2 - a = 0. \\
(c)\ s = 1 + te^s. & (g)\ y = \sin(x+y). \\
(d)\ e^s + st - e = 0. & (h)\ e^{x+y} = xy.
\end{array}
$$

CHAPTER
SEVEN

Maxima, minima and inflection points

7.1 Introduction

Many practical problems occur where we have to deal with functions that have a maximum value (or a minimum value) and it is important to know where the extreme values of the function occur.

Example 7.1.1. A wooden box is to be built to contain 108 ft^3. It is to have an open top and a square base. What must be its dimensions in order that the amount of material required shall be a minimum; that is, what dimensions will make the cost the least?
Let x denote the length of side of square base in feet, and y denote the height of box. Since the volume of the box is given, y may be found in terms of x. Thus $\text{volume} = x^2y = 108$, so $y = \frac{108}{x^2}$. Let M denote the number of square feet of lumber required as a function of x. We compute M explicitly as follows:

area of base = x^2 sq. ft.,
area of four sides = $4xy = \frac{432}{x}$ sq. ft.

Hence

$$M = M(x) = x^2 + \frac{432}{x}$$

is a formula giving the number of square feet required in any such box having a capacity of 108 ft^3. Draw a graph of $M(x)$.

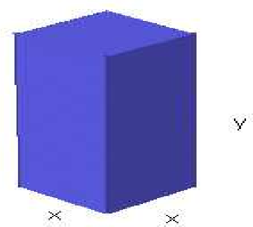

Figure 7.1: A box with square $x \times x$ base, height $y = 108/x^2$, and fixed volume.

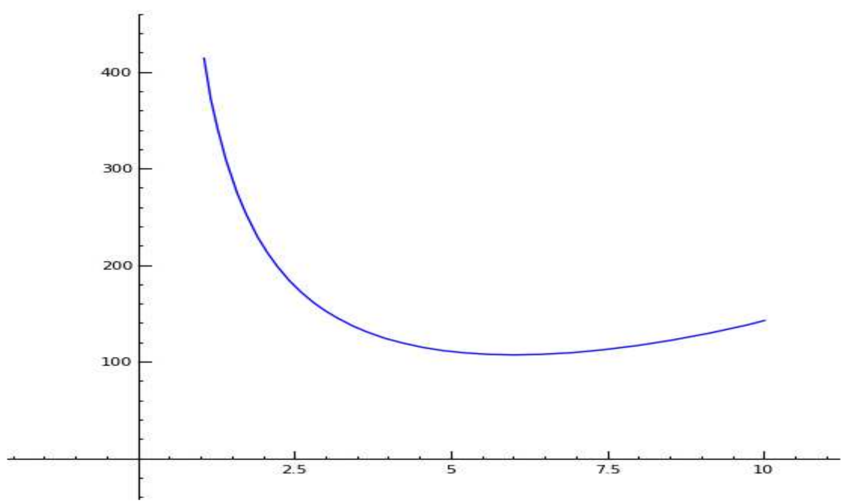

Figure 7.2: Sage plot of $y = x^2 + \frac{432}{x}$, $1 < x < 10$.

What do we learn from the graph?

(a) If the box is carefully drawn, we may measure the ordinate corresponding to any length ($= x$) of the side of the square base and so determine the number of square feet of lumber required.

(b) There is one horizontal tangent (RS). The ordinate from its point of contact T is less than any other ordinate. Hence this discovery: One of the boxes evidently takes less lumber than any of the others. In other words, we may infer that the function defined by $M = M(x)$ has a minimum value. Let us find this point on the graph exactly, using our Calculus. Differentiating $M(x)$ to get the slope at any point, we have

$$\frac{dM}{dx} = 2x - \frac{432}{x^2}.$$

At the lowest point T the slope will be zero. Hence

$$2x - \frac{432}{x^2} = 0;$$

that is, when $x = 6$ the least amount of lumber will be needed.

Substituting in $M(x)$, we see that this is $M = 108$ sq. ft.

In addition to the graph, the fact that a least value of M *exists* can be intuitively deduced by the following argument. Let the base increase from a very small square to a very large one. In the former case the height must be very great and therefore the amount of lumber required will be large. In the latter case, while the height is small, the base will take a great deal of lumber. Our intuition tells that M therefore varies from a large value, decreases for a while, then increases again to another large value. It follows, then, that the graph of $y = M(x)$ must have a "lowest" point corresponding to the dimensions which require the least amount of lumber, and therefore would involve the least cost.

Here is how to compute the critical points of M in Sage :

Sage

```
sage: x = var("x")
sage: M = x^2 + 432/x
sage: solve(M.diff(x)==0,x)
[x == 3*sqrt(3)*I - 3, x == -3*sqrt(3)*I - 3, x == 6]
```

This says that $(x^2 + 432/x)' = 0$ has three roots, but only one real root - the one reported above at $x = 6$.

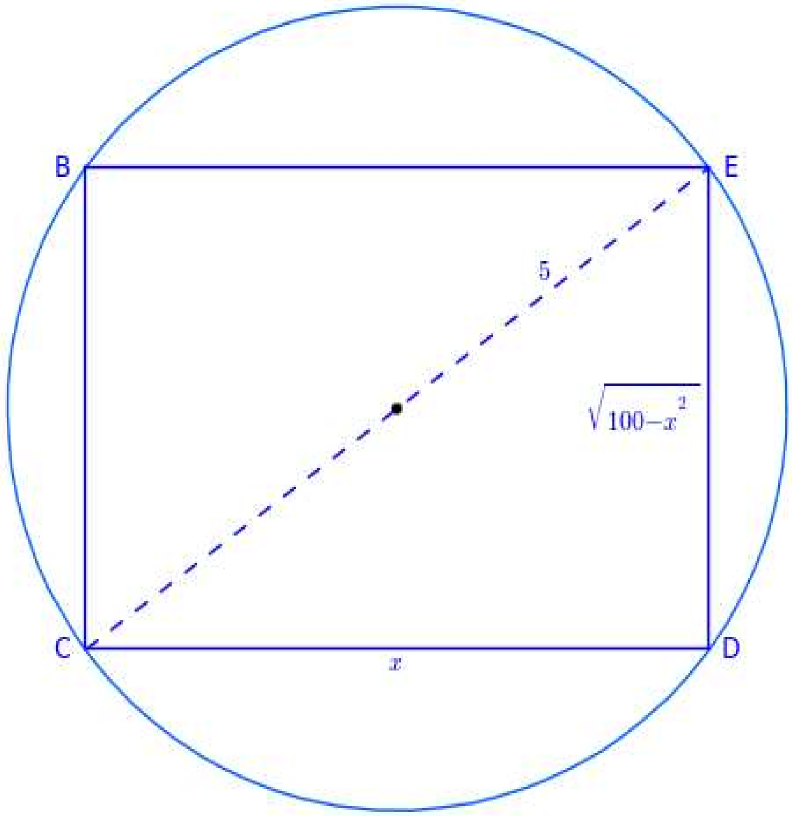

Figure 7.3: A rectangle with circumscribed circle.

Example 7.1.2. For instance, suppose that it is required to find the dimensions of the rectangle of greatest area that can be inscribed in a circle of radius 5 inches. Consider the circle in Figure 7.3:
Inscribe any rectangle, as BCDE, where CD is the base and DE is the height. Let $CD = x$, so $DE = \sqrt{100 - x^2}$, and the area of the rectangle is evidently

$$A = A(x) = x\sqrt{100 - x^2}.$$

That a rectangle of maximum area must *exist* may be seen as follows: Let the base CD ($= x$) increase to 10 inches (the diameter); then the altitude DE ($= \sqrt{100 - x^2}$) will decrease to zero and the area will become zero. Now let the base decrease to zero; then the altitude will increase to 10 inches and the area will again become zero. It is therefore intuitively evident that there exists in-between these extremes a rectangle of greatest area. By a careful study of the figure we might suspect that when the rectangle becomes a square its area would be the greatest, but this would be mere guesswork[1]. A better way would be to plot the graph of the function $y = A(x)$ and note its behavior. To aid us in drawing the graph of $A(x)$, we observe that

(a) from the nature of the problem it is clear that x and $A(x)$ must both be positive; and

(b) the values of x range from zero to 10 inclusive.

Now draw the graph (we have used Sage in Figure 7.4).

What do we learn from the graph?

(a) If the rectangle is carefully drawn, we may approximate the area of the rectangle corresponding to any value x by measuring the length of the corresponding ordinate. For example, when $x = 3$ inches, then the area is about $A(x) \approx 28.6$ inches^2; and when $x\frac{9}{2}$ inches, then the area is about $A(x) \approx 39.8$ inches^2.

(b) There is one horizontal tangent to the curve $y = A(x)$.

The y-coordinate at the point T there this tangent contacts the curve is greater than any other y-coordinate on the curve. We deduce from this that one of the inscribed rectangles has a greater area than any of the others. In other words, we may infer from this that the function defined by $y = A(x)$ has a maximum value. We can find this value very easily to using calculus. We observed that at T the tangent was horizontal, hence the slope will be zero at that point (Example 5.1.2). To find the x-coordinate of T we find the first derivative of $A(x)$, set it equal to zero, and solve for x:

[1]Reasoning that "by symmetry we must have base = height" happens to work in this particular example (as we will see) but, surprisingly enough, does not hold in general.

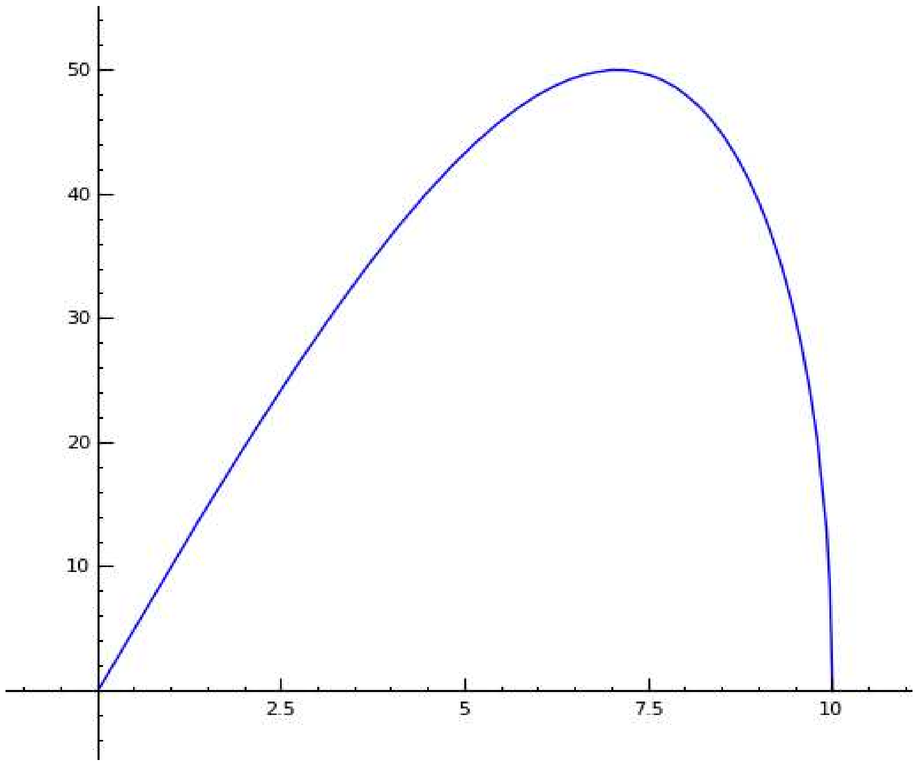

Figure 7.4: The area of a rectangle with fixed circumscribed circle.

$$\begin{aligned} A &= x\sqrt{100 - x^2}, \\ \tfrac{dA}{dx} &= \tfrac{100-2x^2}{\sqrt{100-x^2}}, \\ \tfrac{100-2x^2}{\sqrt{100-x^2}} &= 0. \end{aligned}$$

Solving this gives $x = 5\sqrt{2}$. Substituting back, we get $DE = \sqrt{100 - x^2} = 5\sqrt{2}$. Hence the rectangle of maximum area inscribed in the circle is a square of area $A = CD \times DE = 5\sqrt{2} \times 5\sqrt{2} = 50$ square inches. The length of HT is therefore 50.

We will now proceed to the treatment in detail of the subject of maxima and minima.

7.2 Increasing and decreasing functions

A function is said to be *increasing* when it increases as the variable increases and decreases as the variable decreases. A function is said to be *decreasing* when it decreases as the variable increases and increases as the variable decreases.

The graph of a function indicates plainly whether it is increasing or decreasing.

Example 7.2.1. (1) Consider the function $y = a^x$, $a > 1$, whose graph is shown in Figure 7.5.

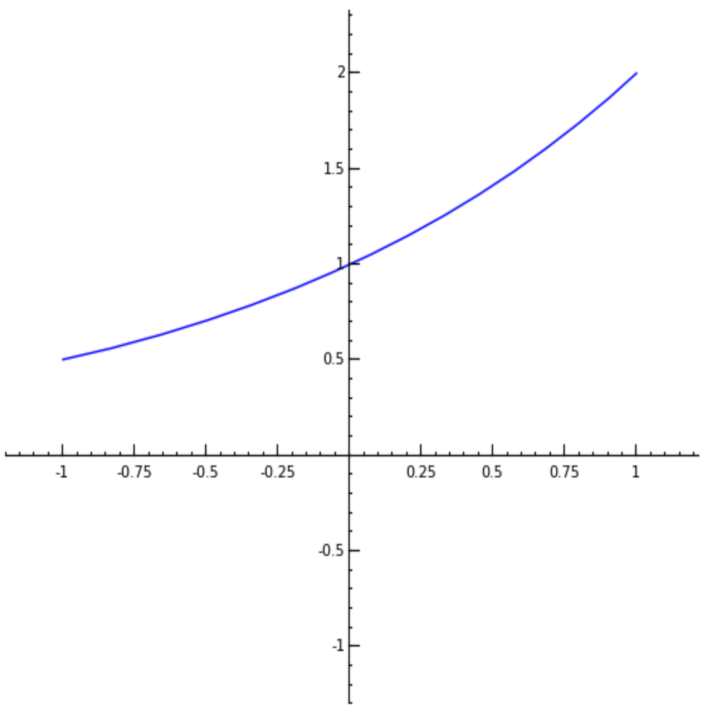

Figure 7.5: Sage plot of $y = 2^x$, $-1 < x < 1$.

As we move along the curve from left to right the curve is rising; that is, as x increases the function $y = a^x$ always increases. Therefore a^x $(a > 1)$ is an increasing function for all values of x.

(2) On the other hand, consider the function $(a - x)^3$ whose graph (Figure 7.6) is the locus of the equation $y = (a - x)^3$.

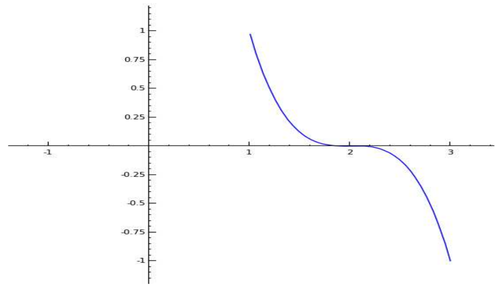

Figure 7.6: Sage plot of $y = (2 - x)^3$, $1 < x < 3$.

Now as we move along the curve from left to right the curve is falling; that is, as x increases, the function $y = (a - x)^3$ always decreases. Hence $(a - x)^3$ is a decreasing function for all values of x.

(3) That a function may be sometimes increasing and sometimes decreasing is shown by the graph (Figure 7.7) of

$$y = 2x^3 - 9x^2 + 12x - 3.$$

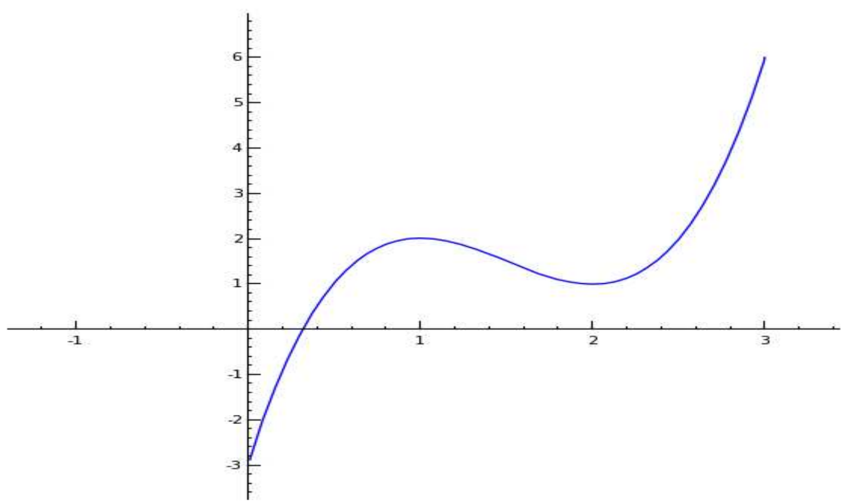

Figure 7.7: Sage plot of $y = 2x^3 - 9x^2 + 12x - 3$, $0 < x < 3$.

As we move along the curve from left to right the curve rises until we reach the point when $x = 1$, then it falls from that point to the point when $x = 2$, and to the right of $x = 2$ it is always increasing. Hence

(a) from $x = -\infty$ to $x = 1$ the function is increasing;

(b) from $x = 1$ to $x = 2$ the function is decreasing;

(c) from $x = 2$ to $x = +\infty$ the function is increasing.

The student should study the curve carefully in order to note the behavior of the function when $x = 1$ and $x = 2$. At $x = 1$ the function ceases to increase and commences to decrease; at $x = 2$, the reverse is true. At $x = 1$ and at $x = 2$ the tangent to the curve is parallel to the x-axis, and therefore the slope is zero.

7.3 Tests for determining when a function is increasing or decreasing

It is evident from Figure 7.7 that at a point where a function

$$y = f(x)$$

is increasing, the tangent in general makes an acute angle with the x-axis, so

$$\text{slope} = \tan\tau = \tfrac{dy}{dx} = f'(x) = \text{a positive number.}$$

Similarly, at a point where a function is decreasing, the tangent in general makes an obtuse angle with the x-axis; therefore[2]

$$\text{slope} = \tan\tau = \tfrac{dy}{dx} = f'(x) = \text{a negative number.}$$

It follows from this that in order for a differentiable function to change from an increasing to a decreasing function, or vice versa, it is a necessary and sufficient condition that the first derivative changes sign. But this can only happen for a continuous derivative by passing through the value zero. Thus in Figure 7.7 as

[2]Conversely, for any given value of x, if $f'(x) > 0$, then $f(x)$ is increasing; if $f'(x) < 0$, then $f(x)$ is decreasing. When $f'(x) = 0$, we cannot decide without further investigation whether $f(x)$ is increasing or decreasing.

we pass along the curve the derivative (= slope) changes sign at the points where $x = 1$ and $x = 2$. In general, then, we have at these "turning points,"

$$\frac{dy}{dx} = f'(x) = 0.$$

A value of $y = f(x)$ satisfying this condition is called a *critical point* of the function $f(x)$.

Remark 7.3.1. The derivative is continuous in nearly all our important applications, but it is interesting to note the case when the derivative (= slope) of y changes sign by "passing through ∞" (that is, its reciprocal $1/y$ passes through the value zero). This would evidently happen at the points on a curve where the tangent is perpendicular to the x-axis. At such "turning points" we have

$$\frac{dy}{dx} = f'(x) = \inf;$$

or, what amounts to the same thing,

$$\frac{1}{f'(x)} = 0.$$

For example, the function $y = 1/x^2$ has a "turning point" at $x = 0$, where the slope is infinite but the function changes from being increasing (for $x < 0$) to decreasing (for $x > 0$).

7.4 Maximum and minimum values of a function

A *maximum value* of a function is one that is greater than any values immediately preceding or following. A *minimum value* of a function is one that is less than any values immediately preceding or following.

For example, in Figure 7.7, it is clear that the function has a maximum value ($y = 2$) when $x = 1$, and a minimum value ($y = l$) when $x = 2$.

The student should observe that a maximum value is not necessarily the greatest possible value of a function nor a minimum value the least. For in Figure 7.7 it is seen that the function ($= y$) has values to the right of $x = 1$ that are greater than the maximum 2, and values to the left of $x = 1$ that are less than the minimum 1.

A function may have several maximum and minimum values. Suppose that Figure 7.8 represents the graph of a function $f(x)$.

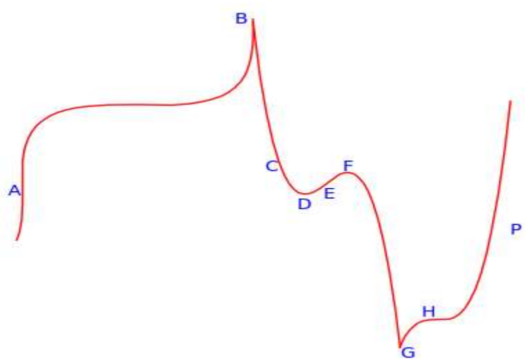

Figure 7.8: A continuous function.

At B, F the function is at a local maximum, and at D, G a minimum. That some particular minimum value of a function may be greater than some particular maximum value is shown in the figure, the minimum value at D being greater than the maximum value at G.

At the ordinary critical points D, F, H the tangent (or curve) is parallel to the x-axis; therefore

$$\text{slope} = \frac{dy}{dx} = f'(x) = 0.$$

At the exceptional critical points A, B, G the tangent (or curve) is perpendicular to the x-axis, giving

$$\text{slope} = \frac{dy}{dx} = f'(x) = \infty.$$

One of these two conditions is then necessary in order that the function shall have a maximum or a minimum value. But such a condition is not sufficient; for at H the slope is zero and at A it is infinite, and yet the function has neither a maximum nor a minimum value at either point. It is necessary for us to know, in addition, how the function behaves in the neighborhood of each point. Thus at the points of maximum value, B, F, the function changes from an increasing to a decreasing function, and at the points of minimum value, D, G, the function changes from a decreasing to an increasing function. It therefore follows from §7.3 that at maximum points

$$\text{slope} = \tfrac{dy}{dx} = f'(x) \text{ must change from + to -,}$$

and at minimum points

$$\text{slope} = \tfrac{dy}{dx} = f'(x) \text{ must change from - to +}$$

when we move along the curve from left to right.

At such points as A and H where the slope is zero or infinite, but which are neither maximum nor minimum points,

$$\text{slope} = \tfrac{dy}{dx} = f'(x) \text{ does not change sign.}$$

We may then state the conditions in general for maximum and minimum values of $f(x)$ for certain values of the variable as follows:

$$f(x) \text{ is a maximum if } f'(x) = 0, \text{ and } f'(x) \text{ changes from } + \text{ to } -. \qquad (7.1)$$

$$f(x) \text{ is a minimum if } f'(x) = 0, \text{ and } f'(x) \text{ changes from } - \text{ to } +. \qquad (7.2)$$

The values of the variable at the turning points of a function are called *critical values*; thus $x = 1$ and $x = 2$ are the critical values of the variable for the function whose graph is shown in Figure 7.7. The critical values at turning points where the tangent is parallel to the x-axis are evidently found by placing the first derivative equal to zero and solving for real values of x, just as under §5.1. (Similarly, if we wish to examine a function at exceptional turning points where the tangent is perpendicular to the x-axis, we set the reciprocal of the first derivative equal to zero and solve to find critical values.)

To determine the sign of the first derivative at points near a particular turning point, substitute in it, first, a value of the variable just a little less than the corresponding critical value, and then one a little greater[3]. If the first gives $+$ (as at L, Figure 7.8) and the second - (as at M), then the function ($= y$) has a maximum

[3]In this connection the term "little less," or "trifle less," means any value between the next smaller root (critical value) and the one under consideration; and the term "little greater," or "trifle greater," means any value between the root under consideration and the next larger one.

value in that interval (as at I). If the first gives $-$ (as at P) and the second $+$ (as at N), then the function ($= y$) has a minimum value in that interval (as at C).

If the sign is the same in both cases (as at Q and R), then the function ($= y$) has neither a maximum nor a minimum value in that interval (as at F)[4].

We shall now summarize our results into a compact working rule.

7.5 Examining a function for extremal values: first method

Working rule, sometimes referred to as the *sign test of the first derivative*.

- FIRST STEP. Find the first derivative of the function.
- SECOND STEP. Set the first derivative equal to zero[5] and solve the resulting equation for real roots in order to find the critical values of the variable.
- THIRD STEP. Write the derivative in factored form if possible.
- FOURTH STEP. Considering one critical value at a time, test the first derivative, first for a value a trifle less and then for a value a trifle greater than the critical value. If the sign of the derivative is first $+$ and then $-$, the function has a maximum value for that particular critical value of the variable; but if the reverse is true, then it has a minimum value. If the sign does not change, the function has neither.

Remark 7.5.1. It can be helpful to draw a *sign graph* for the values of the derivative. This is a sketch of the real axis, with tick marks at the critical points, labeling an interval in-between successive critical points with a "$+$" if the derivative is positive there, and labeling such an interval with a "$-$" otherwise.

Example 7.5.1. In the problem worked out in Example 7.1.2, we showed by means of the graph of the function

$$A = x\sqrt{100 - x^2}$$

[4] A similar discussion will evidently hold for the exceptional turning points B, E, and A respectively.

[5] When the first derivative becomes infinite for a certain value of the independent variable, then the function should be examined for such a critical value of the variable, for it may give maximum or minimum values, as at B, E, or A (Figure 7.8). See footnote in §7.3.

that the rectangle of maximum area inscribed in a circle of radius 5 inches contained 50 square inches. This may now be proved analytically as follows by applying the above rule.

Solution. Let $f(x) = x\sqrt{100 - x^2}$.

First step. Compute $f'(x) = \frac{100-2x^2}{\sqrt{100-x^2}}$.

Second step. $\frac{100-2x^2}{\sqrt{100-x^2}} = 0$ implies $x = 5\sqrt{2}$, which is the critical value. Only the positive sign of the radical is taken, since, from the nature of the problem, the negative sign has no meaning.

Third step. $f'(x) = \frac{2(5\sqrt{2}-x)(5\sqrt{2}+x)}{\sqrt{(10-x)(10+x)}}$.

Fourth step. When $x < 5\sqrt{2}$, $f'(x) = \frac{2(+)(+)}{\sqrt{(+)(+)}} = +$. When $x > 5\sqrt{2}$, $f'(x) = \frac{2(+)(+)}{\sqrt{(-)(+)}} = -$.

Since the sign of the first derivative changes from $+$ to $-$ at $x = 5\sqrt{2}$, the function has a maximum value

$$f(5\sqrt{2}) = 5\sqrt{2} \cdot 5\sqrt{2} = 50.$$

In Sage :

Sage

```
sage: x = var("x")
sage: f(x) = x*sqrt(100 - x^2)
sage: f1(x) = diff(f(x),x); f1(x)
sqrt(100 - x^2) - x^2/sqrt(100 - x^2)
sage: crit_pts = solve(f1(x) == 0,x); crit_pts
[x == -5*sqrt(2), x == 5*sqrt(2)]
sage: x0 = crit_pts[1].rhs(); x0
5*sqrt(2)
sage: f(x0)
50
sage: RR(f1(x0-0.1))>0
True
sage: RR(f1(x0+0.1))<0
True
```

This tells us that $x_0 = 5\sqrt{2}$ is a critical point, at which the area is 50 square inches and at which the area changes from increasing to decreasing. This implies that the area is a maximum at this point.

7.6 Examining a function for extremal values: second method

From (7.1), it is clear that in the vicinity of a maximum value of $f(x)$, in passing along the graph from left to right, $f'(x)$ changes from $+$ to 0 to $-$. Hence $f'(x)$ is a decreasing function, and by §7.3 we know that its derivative, i.e. the second derivative ($= f''(x)$) of the function itself, is negative or zero.

Similarly, we have, from (7.2), that in the vicinity of a minimum value of $f(x)$ $f'(x)$ changes from $-$ to 0 to $+$. Hence $f'(x)$ is an increasing function and by §7.3 it follows that $f''(x)$ is positive or zero.

The student should observe that $f''(x)$ is positive not only at minimum values but also at "nearby" points, P say, to the right of such a critical point. For, as a point passes through P in moving from left to right, slope $= \tan\tau = \frac{dy}{dx} = f'(x)$ is an increasing function. At such a point the curve is said to be *concave upwards*. Similarly, $f''(x)$ is negative not only at maximum points but also at "nearby "points, Q say, to the left of such a critical point. For, as a point passes through Q, slope $= \tan\tau = \frac{dy}{dx} = f'(x)$ is a decreasing function. At such a point the curve is said to be *concave downwards*.

At a point where the curve is concave upwards we sometimes say that the curve has a "positive bending,]] and where it is concave downwards a "negative bending."

We may then state the sufficient conditions for maximum and minimum values of $f(x)$ for certain values of the variable as follows:

$$f(x) \text{ is a maximum if } f'(x) = 0 \text{ and } f''(x) = \text{ a negative number.} \tag{7.3}$$

$$f(x) \text{ is a minimum if } f'(x) = 0 \text{ and } f''(x) = \text{ a positive number.} \tag{7.4}$$

Following is the corresponding working rule, sometimes referred to as the *second derivative test*.

- FIRST STEP. Find the first derivative of the function.

- SECOND STEP. Set the first derivative equal to zero and solve the resulting equation for real roots in order to find the critical values of the variable.
- THIRD STEP. Find the second derivative.
- FOURTH STEP. Substitute each critical value for the variable in the second derivative. If the result is negative, then the function is a maximum for that critical value; if the result is positive, the function is a minimum.

When $f''(x) = 0$, or does not exist, the above process fails, although there may even then be a maximum or a minimum; in that case the first method given in the last section still holds, being fundamental. Usually this second method does apply, and when the process of finding the second derivative is not too long or tedious, it is generally the shortest method.

Example 7.6.1. Let us now apply the above rule to test analytically the function

$$M = x^2 + \frac{432}{x}$$

found in Example 7.1.1.

Solution. Let $f(x) = x^2 + \frac{432}{x}$.

First step. Compute $f'(x) = 2x - \frac{432}{x^2}$.

Second step. Solve $2x - \frac{432}{x^2} = 0$. (In Example 7.1.1 we got $x = 6$.)

Third step. Compute $f''(x) = 2 + \frac{864}{x^3}$.

Fourth step. Use the second derivative test. $f''(6) = +$. Hence $f(6) = 108$, minimum value.

In Sage :

Sage

```
sage: x = var("x")
sage: f(x) = x^2 + 432/x
sage: f1(x) = diff(f(x),x); f1(x)
2*x - 432/x^2
sage: f2(x) = diff(f(x),x,2); f2(x)
864/x^3 + 2
sage: crit_pts = solve(f1(x) == 0,x); crit_pts
[x == 3*sqrt(3)*I - 3, x == -3*sqrt(3)*I - 3, x == 6]
sage: x0 = crit_pts[2].rhs(); x0
6
```

```
sage: f2(x0)
6
sage: f(x0)
108
```

This tells us that $x_0 = 6$ is a critical point and that $f''(x_0) > 0$, so it is a minimum.

The work of finding maximum and minimum values may frequently be simplified by the aid of the following principles, which follow at once from our discussion of the subject.

(a) The (local) maximum and minimum values of a continuous function must occur alternately. (In particular, you cannot have two local maximums without having a minimum in-between them.)

(b) If c is a positive constant, $c \cdot f(x)$ is a maximum or a minimum for a given value of x if and only if $f(x)$ is a maximum or a minimum at x.

Consequently, in determining the critical values and testing for maxima and minima, any constant factor may be omitted.

When c is negative, $c \cdot f(x)$ is a maximum if and only if $f(x)$ is a minimum, and conversely.

(c) If c is a constant, $f(x)$ and $c + f(x)$ have maximum and minimum values for the same values of x.

Hence a constant term may be omitted when finding critical values of x and testing.

In general we must first construct, from the conditions given in the problem, the function whose maximum and minimum values are required, as was done in the two examples worked out in §7.1. This is sometimes a problem of considerable difficulty. No rule applicable in all cases can be given for constructing the function, but in a large number of problems we may be guided by the following general directions.

(a) Express the function whose maximum or minimum is involved in the problem.

(b) If the resulting expression contains more than only variable, the conditions of the problem will furnish enough relations between the variables so that all may be expressed in terms of a single one.

(c) To the resulting function of a single variable apply one of our two rules for finding maximum and minimum values.

(d) In practical problems it is usually easy to tell which critical value will give a maximum and which a minimum value, so it is not always necessary to apply the fourth step of our rules.

(e) Draw the graph of the function in order to check the work.

7.7 Problems

1. It is desired to make an open-top box of greatest possible volume from a square piece of tin whose side is a, by cutting equal squares out of the corners and then folding up the tin to form the sides. What should be the length of a side of the squares cut out?

   Solution. Let x = side of small square = depth of box; then $a - 2x$ = side of square forming bottom of box, and volume is $V = (a - 2x)^2 x$, which is the function to be made a maximum by varying x. Applying rule:

   First step. $\frac{dV}{dx} = (a - 2x)^2 - 4x(a - 2x) = a^2 - 8ax + 12x^2$.

   Second step. Solving $a^2 - 8ax + 12x^2 = 0$ gives critical values $x = \frac{a}{2}$ and $\frac{a}{6}$.

   It is evident that $x = \frac{a}{2}$ must give a minimum, for then all the tin would be cut away, leaving no material out of which to make a box. By the usual test, $x = \frac{a}{6}$ is found to give a maximum volume $\frac{2a^3}{27}$. Hence the side of the square to be cut out is one sixth of the side of the given square.

   The drawing of the graph of the function in this and the following problems is left to the student.

2. Assuming that the strength of a beam with rectangular cross section varies directly as the breadth and as the square of the depth, what are the dimensions of the strongest beam that can be sawed out of a round log whose diameter is d?

Solution. If x = breadth and y = depth, then the beam will have maximum strength when the function xy^2 is a maximum. From the construction and the Pythagorean theorem, $y^2 = d^2 - x^2$; hence we should test the function

$$f(x) = x(d^2 - x^2).$$

First step. $f'(x) = -2x^2 + d^2 - x^2 = d^2 - 3x^2$.

Second step. $d^2 - 3x^2 = 0$. Therefore, $x = \frac{d}{\sqrt{3}}$ = critical value which gives a maximum.

Therefore, if the beam is cut so that depth = $\sqrt{\frac{2}{3}}$ of diameter of log, and breadth = $\sqrt{\frac{1}{3}}$ of diameter of log, the beam will have maximum strength.

3. What is the width of the rectangle of maximum area that can be inscribed in a given segment OAA' of a parabola?

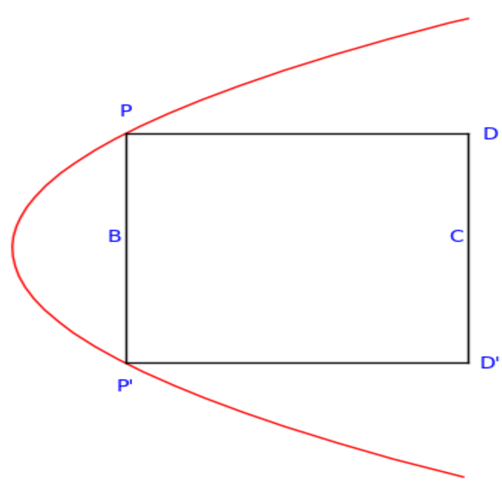

Figure 7.9: An inscribed rectangle in a parabola, $P = (x, y)$.

HINT. If $OC = h$, $BC = h - x$ and $PP' = 2y$; therefore the area of rectangle $PDD'P'$ is $2(h - x)y$.

But since P lies on the parabola $y^2 = 2px$, the function to be tested is $2(h - x)\sqrt{2px}$

Ans. Width = $\frac{2}{3}h$.

4. Find the altitude of the cone of maximum volume that can be inscribed in a sphere of radius r (see Figure 7.10).

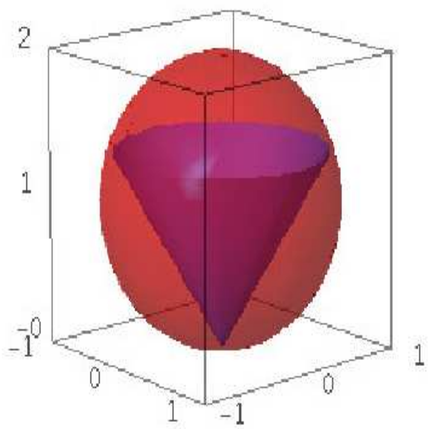

Figure 7.10: An inscribed cone, height y and base radius x, in a sphere.

HINT. Volume of cone = $\frac{1}{3}\pi x^2 y$. But $x^2 = BC \times CD = y(2r - y)$; therefore the function to be tested is $f(y) = \frac{\pi}{3} y^2 (2r - y)$.

Ans. Altitude of cone = $\frac{4}{3}r$.

5. Find the altitude of the cylinder of maximum volume that can be inscribed in a given right cone (see Figure 7.11).

HINT. Let $AU = r$ and $BC = h$. Volume of cylinder = $\pi x^2 y$. But from similar triangles ABC and DBG, $r/x = h/(h - y)$, so $x = \frac{r(h-y)}{h}$. Hence the function to be tested is $f(y) = \frac{r^2}{h^2} y(h - y)^2$.

Ans. Altitude = $\frac{1}{3}h$.

6. Divide a into two parts such that their product is a maximum.

Ans. Each part $= \frac{a}{2}$.

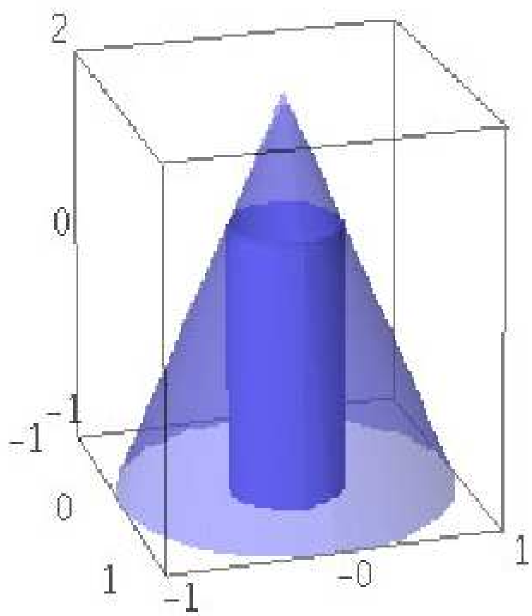

Figure 7.11: An inscribed cylinder in a cone.

7. Divide 10 into two such parts that the sum of the double of one and square of the other may be a minimum.

   Ans. 9 and 1.

8. Find the number that exceeds its square by the greatest possible quantity.

   Ans. $\frac{1}{2}$.

9. What number added to its reciprocal gives the least possible sum?

   Ans. 1.

10. Assuming that the stiffness of a beam of rectangular cross section varies

directly as the breadth and the cube of the depth, what must be the breadth of the stiffest beam that can be cut from a log 16 inches in diameter?

Ans. Breadth $= 8$ inches.

11. A water tank is to be constructed with a square base and open top, and is to hold 64 cubic yards. If the cost of the sides is \$ 1 a square yard, and of the bottom \$ 2 a square yard, what are the dimensions when the cost is a minimum? What is the minimum cost?

    Ans. Side of base = 4 yd., height = 4 yd., cost \$ 96.

12. A rectangular tract of land is to be bought for the purpose of laying out a quarter-mile track with straightaway sides and semicircular ends. In addition a strip 35 yards wide along each straightaway is to be bought for grand stands, training quarters, etc. If the land costs \$ 200 an acre, what will be the maximum cost of the land required?

    Ans. \$ 856.

13. A torpedo boat is anchored 9 miles from the nearest point of a beach, and it is desired to send a messenger in the shortest possible time to a military camp situated 15 miles from that point along the shore. If he can walk 5 miles an hour but row only 4 miles an hour, required the place he must land.

    Ans. 3 miles from the camp.

14. A gas holder is a cylindrical vessel closed at the top and open at the bottom, where it sinks into the water. What should be its proportions for a given volume to require the least material (this would also give least weight)?

    Ans. Diameter = double the height.

15. What should be the dimensions and weight of a gas holder of $8,000,000$ cubic feet capacity, built in the most economical manner out of sheet iron $\frac{1}{16}$ of an inch thick and weighing $\frac{5}{2}$ lb. per sq. ft.?

    Ans. Height = 137 ft., diameter = 273 ft., weight = 220 tons.

16. A sheet of paper is to contain 18 sq. in. of printed matter. The margins at the top and bottom are to be 2 inches each and at the sides 1 inch each. Determine the dimensions of the sheet which will require the least amount of paper.

    Ans. 5 in. by 10 in.

17. A paper-box manufacturer has in stock a quantity of cardboard 30 inches by 14 inches. Out of this material he wishes to make open-top boxes by cutting equal squares out of each corner and then folding up to form the sides. Find the side of the square that should be cut out in order to give the boxes maximum volume.

    Ans. 3 inches.

18. A roofer wishes to make an open gutter of maximum capacity whose bottom and sides are each 4 inches wide and whose sides have the same slope. What should be the width across the top?

    Ans. 8 inches. 4

19. Assuming that the energy expended in driving a steamboat through the water varies as the cube of her velocity, find her most economical rate per hour when steaming against a current running c miles per hour.

    HINT. Let v = most economical speed; then av^3 = energy expended each hour, a being a constant depending upon the particular conditions, and $v - c$ = actual distance advanced per hour. Hence $\frac{av^3}{v-c}$ is the energy expended per mile of distance advanced, and it is therefore the function whose minimum is wanted.

20. Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base. Show that when the canvas is laid out flat it will be a circle with a sector of $152^0 9' = 2.6555...$ cut out. A bell tent 10 ft. high should then have a base of diameter 14 ft. and would require 272 sq. ft. of canvas.

21. A cylindrical steam boiler is to be constructed having a capacity of 1000 cu. ft. The material for the side costs \$ 2 a square foot, and for the ends \$ 3 a square foot. Find radius when the cost is the least.

    Ans. $\frac{1}{\sqrt[3]{3\pi}}$ ft.

22. In the corner of a field bounded by two perpendicular roads a spring is situated 6 rods from one road and 8 rods from the other.

    (a) How should a straight road be run by this spring and across the corner so as to cut off as little of the field as possible?

    (b) What would be the length of the shortest road that could be run across?

    Ans. (a) 12 and 16 rods from corner. (b) $(6^{\frac{2}{3}} + 8^{\frac{2}{3}})^{\frac{3}{2}}$ rods.

23. Show that a square is the rectangle of maximum perimeter that can be inscribed in a given circle.

24. Two poles of height a and b feet are standing upright and are c feet apart. Find the point on the line joining their bases such that the sum of the squares of the distances from this point to the tops of the poles is a minimum. (Ans. Midway between the poles.) When will the sum of these distances be a minimum?

25. A conical tank with open top is to be built to contain V cubic feet. Determine the shape if the material used is a minimum.

26. An isosceles triangle has a base 12 in. long and altitude 10 in. Find the rectangle of maximum area that can be inscribed in it, one side of the rectangle coinciding with the base of the triangle.

27. Divide the number 4 into two such parts that the sum of the cube of one part and three times the square of the other shall have a maximum value.

28. Divide the number a into two parts such that the product of one part by the fourth power of the other part shall be a maximum.

29. A can buoy in the form of a double cone is to be made from two equal circular iron plates of radius r. Find the radius of the base of the cone when the buoy has the greatest displacement (maximum volume).

    Ans. $r\sqrt{\frac{2}{3}}$.

30. Into a full conical wineglass of depth a and generating angle a there is carefully dropped a sphere of such size as to cause the greatest overflow. Show that the radius of the sphere is $\frac{\alpha \sin\alpha}{\sin\alpha\cos 2\alpha}$.

31. A wall 27 ft. high is 8 ft. from a house. Find the length of the shortest ladder that will reach the house if one end rests on the ground outside of the wall.

    Ans. $13\sqrt{13}$.

    Here's how to solve this using Sage : Let h be the height above ground at which the ladder hits the house and let d be the distance from the wall that the ladder hits the ground on the other side of the wall. By similar triangles,

$h/27 = (8+d)/d = 1 + \frac{8}{d}$, so $d + 8 = 8\frac{h}{h-27}$. The length of the ladder is, by the Pythagorean theorem, $f(h) = \sqrt{h^2 + (8+d)^2} = \sqrt{h^2 + (8\frac{h}{h-27})^2}$.

Sage

```
sage: h = var("h")
sage: f(h) = sqrt(h^2+(8*h/(h-27))^2)
sage: f1(h) = diff(f(h),h)
sage: f2(h) = diff(f(h),h,2)
sage: crit_pts = solve(f1(h) == 0,h); crit_pts
[h == 21 - 6*sqrt(3)*I, h == 6*sqrt(3)*I + 21, h == 39, h == 0]
sage: h0 = crit_pts[2].rhs(); h0
39
sage: f(h0)
13*sqrt(13)
sage: f2(h0)
3/(4*sqrt(13))
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This says $f(h)$ has four critical points, but only one of which is meaningful, $h_0 = 39$. At this point, $f(h)$ is a minimum.

32. A vessel is anchored 3 miles offshore, and opposite a point 5 miles further along the shore another vessel is anchored 9 miles from the shore. A boat from the first vessel is to land a passenger on the shore and then proceed to the other vessel. What is the shortest course of the boat?

    Ans. 13 miles.

33. A steel girder 25 ft. long is moved on rollers along a passageway 12.8 ft. wide and into a corridor at right angles to the passageway. Neglecting the width of the girder, how wide must the corridor be?

    Ans. 5.4 ft.

34. A miner wishes to dig a tunnel from a point A to a point B 300 feet below and 500 feet to the east of A. Below the level of A it is bed rock and above A is soft earth. If the cost of tunneling through earth is \$ 1 and through rock \$ 3 per linear foot, find the minimum cost of a tunnel.

    Ans. \$ 1348.53.

35. A carpenter has 108 sq. ft. of lumber with which to build a box with a square base and open top. Find the dimensions of the largest possible box he can make.

    Ans. $6 \times 6 \times 3$.

36. Find the right triangle of maximum area that can be constructed on a line of length h as hypotenuse.

    Ans. $\frac{h}{\sqrt{2}}$ = length of both legs.

37. What is the isosceles triangle of maximum area that can be inscribed in a given circle?

    Ans. An equilateral triangle.

38. Find the altitude of the maximum rectangle that can be inscribed in a right triangle with base b and altitude h.

    Ans. Altitude = $\frac{h}{2}$.

39. Find the dimensions of the rectangle of maximum area that can be inscribed in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

    Ans. $a\sqrt{2} \times b\sqrt{2}$; area = $2ab$.

40. Find the altitude of the right cylinder of maximum volume that can be inscribed in a sphere of radius r.

    Ans. Altitude of cylinder = $\frac{2r}{\sqrt{3}}$.

41. Find the altitude of the right cylinder of maximum convex (curved) surface that can be inscribed in a given sphere.

    Ans. Altitude of cylinder = $r\sqrt{2}$.

42. What are the dimensions of the right hexagonal prism of minimum surface whose volume is 36 cubic feet?

    Ans. Altitude = $2\sqrt{3}$; side of hexagon = 2.

43. Find the altitude of the right cone of minimum volume circumscribed about a given sphere.

    Ans. Altitude = $4r$, and volume = $2\times$ vol. of sphere.

44. A right cone of maximum volume is inscribed in a given right cone, the vertex of the inside cone being at the center of the base of the given cone. Show that the altitude of the inside cone is one third the altitude of the given cone.

45. Given a point on the axis of the parabola $y^2 = 2px$ at a distance a from the vertex; find the abscissa of the point of the curve nearest to it.

    Ans. $x = a - p$.

46. What is the length of the shortest line that can be drawn tangent to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and meeting the coordinate axes?

    Ans. $a + b$.

47. A Norman window consists of a rectangle surmounted by a semicircle. Given the perimeter, required the height and breadth of the window when the quantity of light admitted is a maximum.

    Ans. Radius of circle = height of rectangle.

48. A tapestry 7 feet in height is hung on a wall so that its lower edge is 9 feet above an observer's eye. At what distance from the wall should he stand in order to obtain the most favorable view? (HINT. The vertical angle subtended by the tapestry in the eye of the observer must be at a maximum.)

    Ans. 12 feet.

49. What are the most economical proportions of a tin can which shall have a given capacity, making allowance for waste? (HINT. There is no waste in cutting out tin for the side of the can, but for top and bottom a hexagon of tin circumscribing the circular pieces required is used up. NOTE 1. If no allowance is made for waste, then height = diameter. NOTE 2. We know that the shape of a bee cell is hexagonal, giving a certain capacity for honey with the greatest possible economy of wax.)

    Ans. Height = $\frac{2\sqrt{3}}{\pi}\times$ diameter of base.

50. An open cylindrical trough is constructed by bending a given sheet of tin at breadth $2a$. Find the radius of the cylinder of which the trough forms a part when the capacity of the trough is a maximum.

    Ans. Rad. = $\frac{2a}{\pi}$; i.e. it must be bent in the form of a semicircle.

51. A weight W is to be raised by means of a lever with the force F at one end and the point of support at the other. If the weight is suspended from a point at a distance a from the point of support, and the weight of the beam is w pounds per linear foot, what should be the length of the lever in order that the force required to lift it shall be a minimum?

    Ans. $x = \sqrt{\frac{2aW}{w}}$ feet.

52. An electric arc light is to be placed directly over the center of a circular plot of grass 100 feet in diameter. Assuming that the intensity of light varies directly as the sine of the angle under which it strikes an illuminated surface, and inversely as the square of its distance from the surface, how high should the light he hung in order that the best possible light shall fall on a walk along the circumference of the plot?

    Ans. $\frac{50}{\sqrt{2}}$ feet

53. The lower corner of a leaf, whose width is a, is folded over so as just to reach the inner edge of the page.

    (a) Find the width of the part folded over when the length of the crease is a minimum.

    (b) Find the width when the area folded over is a minimum.

    Ans. (a) $\frac{3}{4}a$; (b) $\frac{2}{3}a$.

54. A rectangular stockade is to be built which must have a certain area. If a stone wall already constructed is available for one of the sides, find the dimensions which would make the cost of construction the least.

    Ans. Side parallel to wall = twice the length of each end.

55. When the resistance of air is taken into account, the inclination of a pendulum to the vertical may be given by the formula $\theta = ae^{-kt}\cos(nt + \eta)$. Show that the greatest elongations occur at equal intervals $\frac{\pi}{n}$ of time.

56. It is required to measure a certain unknown magnitude x with precision. Suppose that n equally careful observations of the magnitude are made, giving the results $a_1, a_2, a_3, \dots, a_n$. The errors of these observations are evidently $x - a_1, x - a_2, x - a_3, \cdots, x - a_n$, some of which are positive and some negative. It has been agreed that the most probable value of x is such that it renders the sum of the squares of the errors, namely $(x - a_1)^2 + (x -$

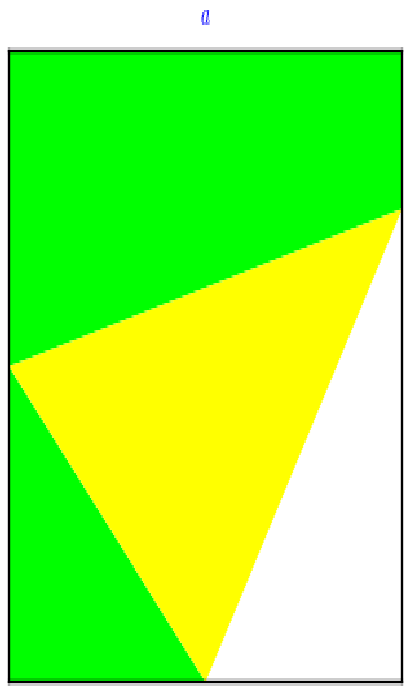

Figure 7.12: A leafed page of width a.

$a_2)^2 + (x - a_3)^2 + \cdots + (x - a_n)^2$, a minimum. Show that this gives the arithmetical mean of the observations as the most probable value of x.

(This is related to the method of least squares, discovered by Gauss, a commonly used technique in statistical applications.)

57. The bending moment at x of a beam of length ℓ, uniformly loaded, is given by the formula $M = \frac{1}{2}w\ell x - \frac{1}{2}wx^2$, where w = load per unit length. Show that the maximum bending moment is at the center of the beam.

58. If the total waste per mile in an electric conductor is $W = c^2r + \frac{t^2}{r}$, where c = current in amperes (a constant), r = resistance in ohms per mile, and t = a constant depending on the interest on the investment and the depreciation

of the plant, what is the relation between c, r, and t when the waste is a minimum?

Ans. $cr = t$.

59. A submarine telegraph cable consists of a core of copper wires with a covering made of non-conducting material. If x denote the ratio of the radius of the core to the thickness of the covering, it is known that the speed of signaling varies as

$$x^2 \log \frac{1}{x}.$$

Show that the greatest speed is attained when $x = \frac{1}{\sqrt{e}}$.

60. Assuming that the power given out by a voltaic cell is given by the formula

$$P = \frac{E^2 R}{(r+R)^2},$$

when E = constant electro-motive force, r = constant internal resistance, R = external resistance, prove that P is a maximum when $r = R$.

61. The force exerted by a circular electric current of radius a on a small magnet whose axis coincides with the axis of the circle varies as

$$\frac{x}{(a^2 + x^2)^{\frac{5}{2}}}.$$

where x = distance of magnet from plane of circle. Prove that the force is a maximum when $x = \frac{a}{2}$.

62. We have two sources of heat at A and B, which we visualize on the real line (with B to the right or A), with intensities a and b respectively. The total intensity of heat at a point P between A and B at a distance of x from A is given by the formula $I = \frac{a}{x^2} + \frac{b}{(d-x)^2}$. Show that the temperature at P will be the lowest when $\frac{d-x}{x} = \frac{b^{1/3}}{a^{1/3}}$. that is, the distances BP and AP have the same ratio as the cube roots of the corresponding heat intensities. The distance of P from A is $x = \frac{a^{\frac{1}{3}} d}{a^{\frac{1}{3}} + b^{\frac{1}{3}}}$.

63. The range of a projectile in a vacuum is given by the formula $R = \frac{v_0^2 \sin 2\phi}{g}$, where v_0 = initial velocity, g = acceleration due to gravity, ϕ = angle of projection with the horizontal. Find the angle of projection which gives the greatest range for a given initial velocity.

    Ans. $\phi = 45^o = \pi/4$.

64. The total time of flight of the projectile in the last problem is given by the formula $T = \frac{2v_0 \sin \phi}{g}$. At what angle should it be projected in order to make the time of flight a maximum?

    Ans. $\phi = 90^o = \pi/2$.

65. The time it takes a ball to roll down an inclined plane with angle ϕ (with respect to the x-axis) is given by the formula $T = 2\sqrt{\frac{2}{g \sin 2\phi}}$. Neglecting friction, etc., what must be the value of ϕ to make the quickest descent?

    Ans. $\phi = 45^o = \pi/4$.

66. Examine the function $(x-1)^2(x+1)^3$ for maximum and minimum values. Use the first method.

    Solution. $f(x) = (x-1)^2(x+1)^3$.

    First step. $f'(x) = 2(x-1)(x+1)^3 + 3(x-1)^2(x+1)^2 = (x-1)(x+1)^2(5x-1)$.

    Second step. $(x-1)(x+1)^2(5x-1) = 0$, $x = 1, -1, \frac{1}{5}$, which are critical values.

    Third step. $f'(x) = 5(x-1)(x+1)^2(x - \frac{1}{5})$.

    Fourth step. Examine first for critical value $x = 1$.

    When $x < 1$, $f'(x) = 5(-)(+)2(+) = -$. When $x > 1$, $f'(x) = 5(+)(+)2(+) = +$. Therefore, when $x = 1$ the function has a minimum value $f(l) = 0$. Examine now for the critical value $x = \frac{1}{5}$. When $x < \frac{1}{5}$, $f'(x) = 5(-)(+)^2(-) = +$. When $x > \frac{1}{5}$, $f'(x) = 5(-)(+)^2(+) = -$. Therefore, when $x = \frac{1}{5}$ the function has a maximum value $f(\frac{1}{5}) = 1.11$. Examine lastly for the critical value $x = -1$. When $x < -1$, $f'(x) = 5(-)(-)2(-) = +$. When $x > -1$, $f'(x) = 5(-)(+)2(-) = +$. Therefore, when $x = -1$ the function has neither a maximum nor a minimum value.

Examine the following functions for maximum and minimum values:

67. $(x-3)^2(x-2)$.

Ans. $x=\frac{7}{3}$, gives max. = $\frac{4}{27}$; $x=3$, gives min. = 0.

68. $(x-1)^3(x-2)^2$.

Ans. $x=\frac{8}{5}$, gives max. = 0.03456; $x=2$, gives min. = 0; $x=1$, gives neither.

69. $(x-4)^5(x+2)^4$.

Ans. $x=-2$, gives max.; $x=\frac{2}{3}$ gives min; $x=4$, gives neither.

70. $(x-2)^5(2x+1)^4$.

Ans. $x=-\frac{1}{2}$, gives max.; $x=\frac{11}{18}$, gives min.; $x=2$, gives neither.

71. $(x+1)^{\frac{2}{3}}(x-5)^2$.

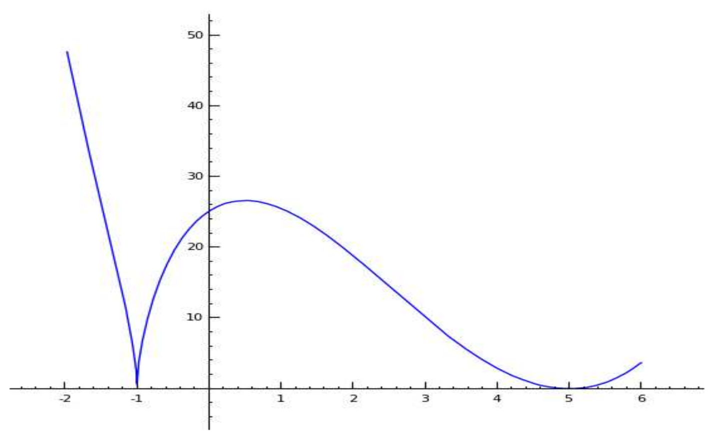

Figure 7.13: Sage plot of $y=(x+1)^{\frac{2}{3}}(x-5)^2$.

Ans. $x=\frac{1}{2}$, gives max.; $x=-1$ and 5, give min.

72. $(2x-a)^{\frac{1}{3}}(x-a)^{\frac{2}{3}}$.

Ans. $x=\frac{2a}{3}$, gives max.; $x=1$ and $-\frac{1}{3}$, gives min.; $x=\frac{a}{2}$, gives neither.

73. $x(x-1)^2(x+1)^3$.

Ans. $x=\frac{1}{2}$, gives max.; $x=1$ and $-\frac{1}{3}$, gives min.; $x=-1$, gives neither.

74. $x(a+x)^2(a-x)^3$

    Ans. $x=-a$ and $\frac{a}{3}$, give max.; $x=-\frac{a}{2}$; $x=a$, gives neither.

75. $b+c(x-a)^{\frac{2}{3}}$.

    Ans. $x=a$, gives min. = b.

76. $a-b(x-c)^{\frac{1}{3}}$.

    Ans. No max. or min.

77. $\frac{x^2-7x+6}{x-10}$.

    Ans. $x=4$, gives max. $x=16$, gives min.

78. $\frac{(a-x)^3}{a-2x}$.

    Ans. $x=\frac{a}{4}$, gives min.

79. $\frac{1-x+x^2}{1+x-x^2}$.

    Ans. $x=\frac{1}{2}$, gives min.

80. $\frac{x^2-3x+2}{x^2+3x+2}$.

    Ans. $x=\sqrt{2}$, gives min. = $12\sqrt{2}-17$; $x=-\sqrt{2}$, gives max. = $-12\sqrt{2}-17$; $x=-1,-2$, give neither.

81. $\frac{(x-a)(b-x)}{x^2}$.

    $x=\frac{2ab}{a+b}$, gives max. = $\frac{(a-b)^2}{4ab}$.

82. $\frac{a^2}{x}+\frac{b^2}{a-x}$.

    Ans. $x=\frac{a^2}{a-b}$, gives min.; $x=\frac{a^2}{a+b}$, gives max.

83. Examine x^3-3x^2-9x+5 for maxima and minima, Use the second method, §7.6.

    Solution. $f(x)=x^3-3x^2-9x+5$.

    First step. $f'(x)=3x^2-6x-9$.

    Second step, $3x^2-6x-9=0$; hence the critical values are $x=-1$ and 3.

    Third step. $f''(x)=6x-6$.

Fourth step. $f''(-1) = -12$.

Therefore, $f(-1) = 10$ = maximum value. $f''(3) = +12$. Therefore, $f(3) = -22$ = minimum value.

84. Examine $\sin^2 x \cos x$ for maximum and minimum values.

    Solution. $f(x) = \sin^2 x \cos x$.

    First step. $f'(x) = 2 \sin x \cos^2 x - \sin^3 x$.

    Second step. $2 \sin x \cos^2 x - \sin^3 x = 0$; hence the critical values are $x = n\pi$ and $x = n\pi \pm \arctan(-\sqrt{2}) = n\pi \pm \alpha$.

    Third step. $f''(x) = \cos x(2 \cos^2 x - 7 \sin^2 x)$.

    Fourth step. $f''(0) = +$. Therefore, $f(0) = 0$ = minimum value. $f''(\pi) = -$. Therefore,$f(\pi) = 0$ = maximum value. $f''(\alpha) = -$. Therefore, $f(\alpha)$ maximum value. $f''(\pi - \alpha) = +$. Therefore,$f(\pi - \alpha)$ minimum value.

Examine the following functions for maximum and minimum values:

87. $3x^3 - 9x^2 - 27x + 30$.

    Ans. $x = -1$, gives max. = 45; $x = 3$, gives min. = -51.

88. $2x^3 - 21x^2 + 36x - 20$.

    Ans. $x = 1$, gives max. = -3; $x = 6$, gives min. = -128.

89. $\frac{x^3}{3} - 21x^2 + 3x + 1$.

    Ans. $x = 1$, gives max. = $\frac{7}{3}$; $x = 3$, gives min. = 1.

90. $2x^3 - 15x^2 + 36x + 10$.

    Ans. $x = 2$, gives max. = 38; $x = 3$, gives min. = 37.

91. $x^3 - 9x^2 + 15x - 3$.

    Ans. $x = 1$, gives max. = 4; $x = 5$, gives min. = -28.

92. $x^3 - 3x^2 + 6x + 10$.

    Ans. No max. or min.

93. $x^5 - 5x^4 + 5x^3 + 1$. $x = 1$, gives max. = 2; $x = 3$, gives min. = -26; $x = 0$, gives neither.

94. $3x^5 - 125x^2 + 2160x$.

$x = -4$ and 3, give max.; $x = -3$ and 4, give min.

95. $2x^3 - 3x^2 - 12x + 4$.

96. $2x^3 - 21x^2 + 36x - 20$.

97. $x^4 - 2x^2 + 10$.

98. $x^4 - 4$.

99. $x^3 - 8$.

100. $4 - x^6$.

101. $\sin x(1 + \cos x)$.

Ans. $x = 2n\pi + \frac{\pi}{3}$, give max. $= \frac{3}{4}\sqrt{3}$; $x = 2n\pi - \frac{\pi}{3}$, give min. $= \frac{3}{4}\sqrt{3}$; $x = n\pi$, give neither.

102. $\frac{x}{\log x}$.

Ans. $x = e$, gives min. = e; $x = 1$, gives neither.

103. $\log \cos x$.

Ans. $x = n\pi$, gives max.

104. $ae^{kx} + be^{-kx}$.

Ans. $x = \frac{1}{k} \log \sqrt{\frac{b}{a}}$, gives min. $= 2\sqrt{ab}$.

105. x^x.

$x = \frac{1}{e}$, gives min.

106. $x^{\frac{1}{x}}$.

Ans. $x = e$, gives max.

107. $\cos x + \sin x$.

Ans. $x = \frac{\pi}{4}$, gives max. = $\sqrt{2}$. $x = \frac{5\pi}{4}$, gives min. = $-\sqrt{2}$.

108. $\sin 2x - x$.

Ans. $x = \frac{\pi}{6}$, gives max.; $x = -\frac{\pi}{6}$, gives min.

109. $x + \tan\, x$.

Ans. No max. or min.

110. $\sin^3 x \cos\, x$.

Ans. $x = n\pi + \frac{\pi}{3}$, gives max. = $\frac{3}{16}\sqrt{3}$; $x = n\pi - \frac{\pi}{3}$, gives min. = $-\frac{3}{16}\sqrt{3}$; $x = n\pi$, gives neither.

111. $x\ \cos\, x$.

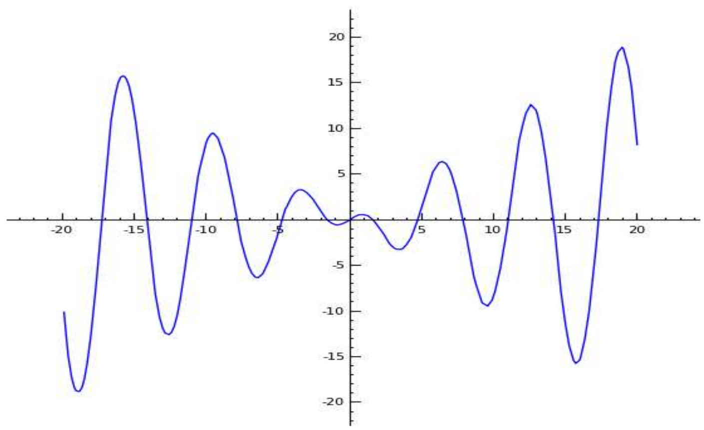

Figure 7.14: Sage plot of $y = x\cos(x)$.

Ans. x such that $x \sin\, x = \cos\, x$, gives max/min.

112. $\sin\, x + \cos\, 2x$.

Ans. $\arcsin \frac{1}{4}$, gives max.; $x = \frac{\pi}{2}$, gives min.

113. $2\tan\, x - \tan^2 x$.

Ans. $x = \frac{\pi}{4}$, gives max.

114. $\frac{\sin x}{1+\tan x}$.

Ans. $x = \frac{\pi}{4}$, gives max.

115. $\frac{x}{1+x\tan x}$.

$x = \cos\, x$, gives max.; $x = -\cos\, x$, gives min.

7.8 Points of inflection

Definition 7.8.1. Consider the graph $y = f(x)$ is a twice continuously differentiable function. *Points of inflection* separate concave upwards sections of the graph from concave downwards sections. They may also be defined as points where

(a) $\frac{d^2y}{dx^2} = 0$ and $\frac{d^2y}{dx^2}$ changes sign,

or

(b) $\frac{d^2x}{dy^2} = 0$ and $\frac{d^2x}{dy^2}$ changes sign.

Thus, if a curve $y = f(x)$ changes from concave upwards to concave downwards at a point, or the from concave down to concave up, then such a point is called a *point of inflection*.

From the discussion of §7.6, it follows at once that where the curve is concave up, $f''(x) = +$, and where the curve is concave down, $f''(x) = -$. In order to change sign it must pass through the value zero[6]; hence we have:

Lemma 7.8.1. At points of inflection, $f''(x) = 0$.

Solving the equation resulting from Lemma 7.8.1 gives the x-coordinate(s) of the point(s) of inflection. To determine the direction of curving or direction of bending in the vicinity of a point of inflection, test $f''(x)$ for values of x, first slightly less and then slightly more than the x-coordinate at that point.

If $f''(x)$ changes sign, we have a point of inflection, and the signs obtained determine if the curve is concave upwards or concave downwards in the neighborhood of each point of inflection.

The student should observe that near a point where the curve is concave upwards the curve lies above the tangent, and at a point where the curve is concave downwards the curve lies below the tangent. At a point of inflection the tangent evidently crosses the curve.

Following is a *rule for finding points of inflection* of the curve whose equation is $y = f(x)$. This rule includes also directions for examining the direction of curvature of the curve in the neighborhood of each point of inflection.

- FIRST STEP. Find $f''(x)$.

- SECOND STEP. Set $f''(x) = 0$, and solve the resulting equation for real roots.

[6] It is assumed that $f'(x)$ and $f''(x)$ are continuous. The solution of Exercise 2, §7.8, shows how to discuss a case where $f'(x)$ and $f''(x)$ are both infinite.

- THIRD STEP. Write $f''(x)$ in factor form.
- FOURTH STEP. Test $f''(x)$ for values of x, first a trifle less and then a trifle greater than each root found in the second step. If $f''(x)$ changes sign, we have a point of inflection.

When $f''(x) = +$, the curve is concave upwards[7].
When $f''(x) = -$, the curve is concave downwards.

7.9 Examples

Examine the following curves for points of inflection and direction of bending.

1. $y = 3x^4 - 4x^3 + 1$.

   Solution. $f(x) = 3x^4 - 4x^3 + 1$.

   First step. $f''(x) = 36x^2 - 24x$.

   Second step. $36x^2 - 24x = 0$, $x = \frac{2}{3}$ and $x = 0$, critical values.

   Third step. $f''(x) = 36x(x - \frac{2}{3})$.

   Fourth step. When $x < 0$, $f''(x) = +$; and when $x > 0$, $f''(x) = -$. Therefore, the curve is concave upwards to the left and concave downwards to the right of $x = 0$. When $x < \frac{2}{3}$, $f''(x) = -$; and when $x > \frac{2}{3}$, $f''(x) = +$. Therefore, the curve is concave downwards to the left and concave upwards to the right of $x = \frac{2}{3}$.

   The curve is evidently concave upwards everywhere to the left of $x = 0$, concave downwards between $(0, 1)$ and $(\frac{2}{3}, \frac{11}{27})$, and concave upwards everywhere to the right of $(\frac{2}{3}, \frac{11}{27})$.

2. $(y - 2)^3 = (x - 4)$.

   Solution. $y = 2 + (x - 4)^{-\frac{1}{3}}$.

   First step. $\frac{dy}{dx} = \frac{1}{3}(x - 4)^{-\frac{2}{3}}$.

   Second step. When $x = 4$, both first and second derivatives are infinite.

   Third step. When $x < 4$, $\frac{d^2y}{dx^2} = +$; but when $x > 4$, $\frac{d^2y}{dx^2} = -$.

[7] This may be easily remembered if we say that a vessel shaped like the curve where it is concave upwards will "hold (+) water", and where it is concave downwards will "spill (−) water."

We may therefore conclude that the tangent at $(4,2)$ is perpendicular to the x-axis, that to the left of $(4,2)$ the curve is concave upwards, and to the right of $(4,2)$ it is concave downwards. Therefore $(4,2)$ must be considered a point of inflection.

3. $y = x^2$.

   Ans. Concave upwards everywhere.

4. $y = 5 - 2x - x^2$.

   Ans. Concave downwards everywhere.

5. $y = x^3$.

   Ans. Concave downwards to the left and concave upwards to the right of $(0,0)$.

6. $y = x^3 - 3x^2 - 9x + 9$.

   Ans. Concave downwards to the left and concave upwards to the right of $(1,-2)$.

7. $y = a + (x-b)^3$.

   Ans. Concave downwards to the left and concave upwards to the right of (b,a).

8. $a^2y = \frac{x^3}{3} - ax^2 + 2a^3$.

   Ans. Concave downwards to the left and concave upwards to the right of $(a, \frac{4a}{3})$.

9. $y = x^4$.

   Ans. Concave upwards everywhere.

10. $y = x^4 - 12x^3 + 48x^2 - 50$.

    Ans. Concave upwards to the left of $x = 2$, concave downwards between $x = 2$ and $x = 4$, concave upwards to the right of $x = 4$.

11. $y = \sin\ x$.

    Ans. Points of inflection are $x = n\pi$, n being any integer.

12. $y = \tan x$.

    Ans. Points of inflection are x = n, n being any integer.

13. Show that no conic section can have a point of inflection.

14. Show that the graphs of e^x and $\log x$ have no points of inflection.

7.10 Curve plotting

The elementary method of plotting a curve whose equation is given in rectangular coordinates, and one with which the student is already familiar, is to (a) solve its equation for y (or x), (b) take several arbitrary values of x (or y), tabulate the corresponding values of y (or x), (c) plot the respective points, and (d) draw a smooth curve through them. The result is an approximation to the required curve. This process is laborious at best, and in case the equation of the curve is of a degree higher than the second, the solved form of such an equation may be unsuitable for the purpose of computation, or else it may fail altogether, since it is not always possible to solve the equation for y or x.

The general form of a curve is usually all that is desired, and calculus furnishes us with useful methods for determining the shape of a curve with very little computation.

The first derivative gives us the slope of the curve at any point; the second derivative determines the intervals within which the curve is concave upward or concave downward, and the points of inflection separate these intervals; the maximum points are the high points and the minimum points are the low points on the curve. As a guide in his work the student may follow the following rule.

Rule for plotting curves in rectangular coordinates.

- FIRST STEP. Find the first derivative; place it equal to zero; solving gives the abscissas of maximum and minimum points.
- SECOND STEP. Find the second derivative; place it equal to zero; solving gives the abscissas of the points of inflection.
- THIRD STEP. Calculate the corresponding ordinates of the points whose abscissas were found in the first two steps. Calculate as many more points as may be necessary to give a good idea of the shape of the curve. Fill out a table such as is shown in the example worked out.

- FOURTH STEP. Plot the points determined and sketch in the curve to correspond with the results shown in the table.

If the calculated values of the ordinates are large, it is best to reduce the scale on the y-axis so that the general behavior of the curve will be shown within the limits of the paper used. Coordinate plotting (graph) paper should be employed.

7.11 Exercises

Trace the following curves, making use of the above rule. Also find the equations of the tangent and normal at each point of inflection.

1. $y = x^3 - 9x^2 + 24x - 7$.

   Solution. Use the above rule.

   First step. $y' = 3x^2 - 18x + 24$, $3x^2 - 18x + 24 = 0$, $x = 2, 4$.

   Second step. $y'' = 6x - 18$, $6x - 18 = 0$, $x = 3$.

   Third step.

x	y	y'	y''	Remarks	Direction of Curve
0	-7	+	-		concave down
2	13	0	-	max.	concave down
3	11	-	0	pt. of infl.	concave up
4	9	0	+	min.	concave up
6	29	+	+		concave up

   Fourth step. Plot the points and sketch the curve. To find the equations of the tangent and normal to the curve at the point of inflection $(3, 11)$, use formulas (5.1), ((5.2). This gives $3x + y = 20$ for the tangent and $3y - x = 30$ for the normal.

2. $y = x^3 - 6x^2 - 36x + 5$.

   Ans. Max. $(-2, 45)$; min. $(6, -211)$; pt. of infl. $(2, -83)$; tan. $y + 48x - 13 = 0$; nor. $48y - x + 3986 = 0$.

   We shall solve this using Sage .

   Sage

```
sage: x = var("x")
```

```
sage: f = x^3 - 6*x^2 - 36*x + 5
sage: f1 = diff(f(x),x); f1
3*x^2 - 12*x - 36
sage: crit_pts = solve(f1(x) == 0, x); crit_pts
[x == 6, x == -2]
sage: f2 = diff(f(x),x,2); f2(x)
6*x - 12
sage: x0 = crit_pts[0].rhs(); x0
6
sage: x1 = crit_pts[1].rhs(); x1
-2
sage: f(x0); f2(x0)
-211
24
sage: f(x1); f2(x1)
45
-24
sage: infl_pts = solve(f2(x) == 0, x); infl_pts
[x == 2]
sage: p = plot(f, -5, 10)
sage: show(p)
```

3. $y = x^4 - 2x^2 + 10$.

   Ans. Max. $(0, 10)$; min. $(\pm 1, 9)$; pt. of infl. $\left(\pm\frac{1}{\sqrt{3}}, \frac{85}{9}\right)$.

4. $y = \frac{1}{2}x^4 - 3x^2 + 2$.

   Ans. Max. $(0, 2)$; min. $(\pm\sqrt{3}, -\frac{5}{2})$; pt. of infl. $(\pm 1, -\frac{1}{2})$.

5. $y = \frac{6x}{1+x^2}$.

   Ans. Max. $(1, 3)$; min. $(-1, -3)$; pt. of infl. $(0, 0)$, $\left(\pm\sqrt{3}, \pm\frac{3\sqrt{3}}{2}\right)$.

6. $y = 12x - x^3$.

   Ans. Max. $(2, 16)$; min. $(-2, -16)$; pt. of infl. $(0, 0)$.

7. $4y + x^3 - 3x^2 + 4 = 0$.

   Ans. Max. $(2, 0)$; min. $(0, -1)$.

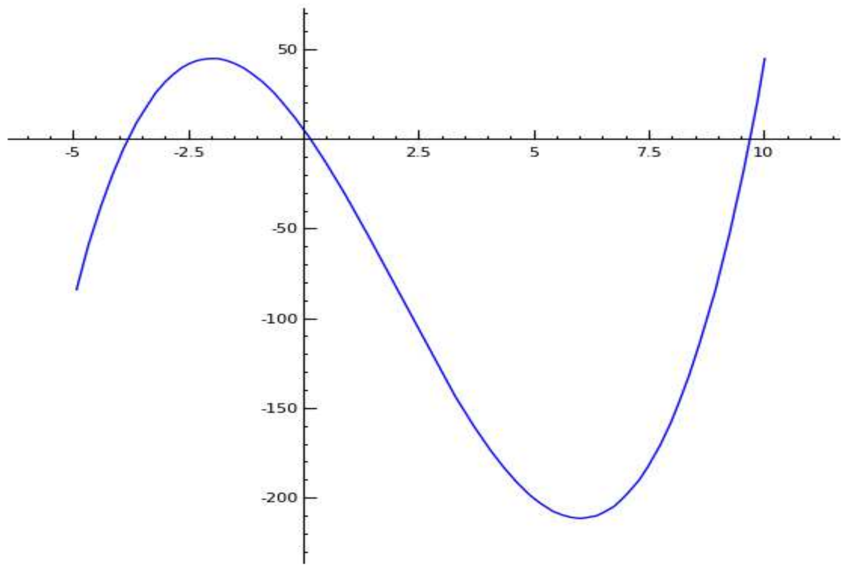

Figure 7.15: Plot for Exercise 8.11-2, $y = x^3 - 6x^2 - 36x + 5$.

8. $y = x^3 - 3x^2 - 9x + 9$.
9. $2y + x^3 - 9x + 6 = 0$.
10. $y = x^3 - 6x^2 - 15x + 2$.
11. $y(1 + x^2) = x$.
12. $y = \frac{8a^3}{x^2+4a^2}$.
13. $y = e^{-x^2}$.
14. $y = \frac{4+x}{x^2}$.
15. $y = (x + l)^{\frac{2}{3}}(x - 5)^2$.
16. $y = \frac{x+2}{x^3}$.
17. $y = x^3 - 3x^2 - 24x$.
18. $y = 18 + 36x - 3x^2 - 2x^3$.

19. $y = x - 2\cos x$.
20. $y = 3x - x^3$.
21. $y = x^3 - 9x^2 + 15x - 3$.
22. $x^2 y = 4 + x$.
23. $4y = x^4 - 6x^2 + 5$.
24. $y = \frac{x^3}{x^2+3a^2}$.
25. $y = \sin x + \frac{x}{2}$.
26. $y = \frac{x^2+4}{x}$.
27. $y = 5x - 2x^2 - \frac{1}{3}x^3$.
28. $y = \frac{1+x^2}{2x}$.
29. $y = x - 2\sin x$.
30. $y = \log \cos x$.
31. $y = \log(1 + x^2)$.

CHAPTER

EIGHT

Application to arclength and rates

8.1 Introduction

Thus far we have represented the derivative of $y = f(x)$ by the notation

$$\frac{dy}{dx} = f'(x).$$

We have taken special pains to impress on the student that the symbol

$$\frac{dy}{dx}$$

was to be considered not as an ordinary fraction with dy as numerator and dx as denominator, but as a single symbol denoting the limit of the quotient

$$\frac{\Delta y}{\Delta x}$$

as Δx approaches the limit zero.

Problems do occur, however, where it is very convenient to be able to give a meaning to dx and dy separately, and it is especially useful in applications using integral calculus. How this may be done is explained in the first part of this chapter.

In the second part (starting with §8.6), we apply what we know about the derivative to functions of time t. If $f(t)$ is some quantity (for example, distance) changing with time then we can regard $f'(t)$ as the rate of change of f (for example, velocity). The method of solving "related rates" problems will be explained in the second part of this chapter.

8.2 Definitions

If $f'(x)$ is the derivative of $f(x)$ for a particular value of x, and Δx is an arbitrarily chosen[1] increment of x, then the ***differential*** of $f(x)$, denoted by the symbol $df(x)$, is defined by the equation

$$df(x) = f'(x)\Delta x. \tag{8.1}$$

If now $f(x) = x$, then $f'(x) = 1$, and (8.1) reduces to $dx = \Delta x$, showing that when x is the independent variable, the differential of x ($= dx$) is identical with Δx. Hence, if $y = f(x)$, (8.1) may in general be written in the form

$$dy = f'(x)\, dx. \tag{8.2}$$

The differential of a function equals its derivative multiplied by the differential of the independent variable. Observe that, since dx may be given any arbitrary value whatever, dx is independent of x. Hence, dy is a function of two independent variables x and dx.

Let us illustrate what this means geometrically.

Let $f'(x)$ be the derivative of $y = f(x)$ at P, then

$$dy = f'(x)dx = \tan\tau \cdot dx.$$

Therefore dy, or $df(x)$, is the increment of the y-coordinate on the tangent line to the curve $y = f(x)$ corresponding to replacing x by $x + dx$.

This gives the following interpretation of the derivative as a fraction.

If an arbitrarily chosen increment of the independent variable x for a point (x, y) on the curve $y = f(x)$ is denoted by dx, then in the derivative

$$\frac{dy}{dx} = f'(x) = \tan\tau,$$

dy denotes the corresponding increment of the y-coordinate drawn to the tangent.

[1] The term "arbitrarily chosen" essentially means that the variable Δx is independent from the variable x.

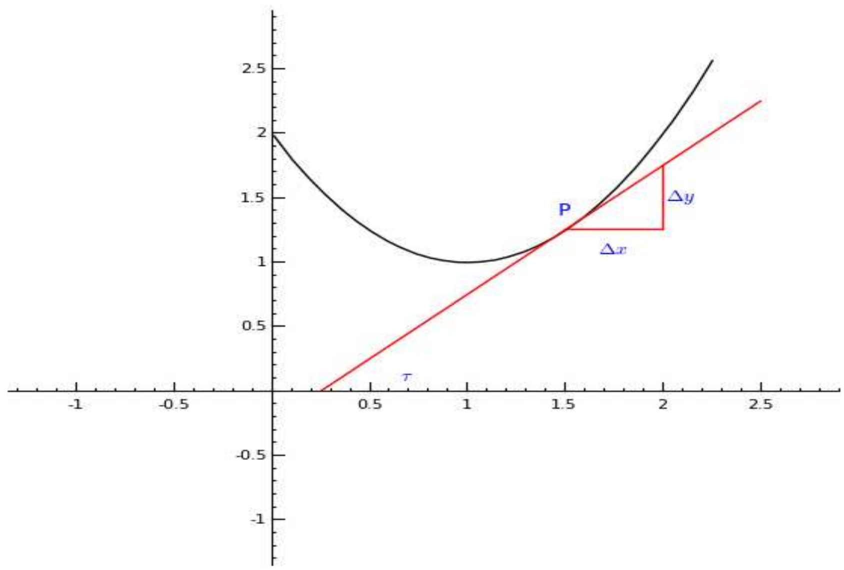

Figure 8.1: The differential of a function.

8.3 Derivative of the arclength in rectangular coordinates

Let s be the arclength[2] of the part of the curve $y = f(x)$ from a given point A on the curve to some 'variable" point P.

Denote the increment of s (= arc PQ in Figure 8.2) by Δs. The definition of the arclength depends on the assumption that, as Q approaches P,

$$\lim \left(\frac{\text{chord}PQ}{\text{arc}PQ} \right) = 1.$$

In the limit of the ratio of chord PQ and a second infinitesimal, chord PQ may be replaced by arc PQ (= Δs).

From Figure 8.2, we have

$$(\text{chord } PQ)^2 = (\Delta x)^2 + (\Delta y)^2. \tag{8.3}$$

[2]Defined in integral calculus. For now, we simply assume that there is a function $s = s(x)$ such that if you go along the curve from a given point A (such as the point $(0, f(0))$) to a point $P = (x, y)$ then $s(x)$ describes the arclength.

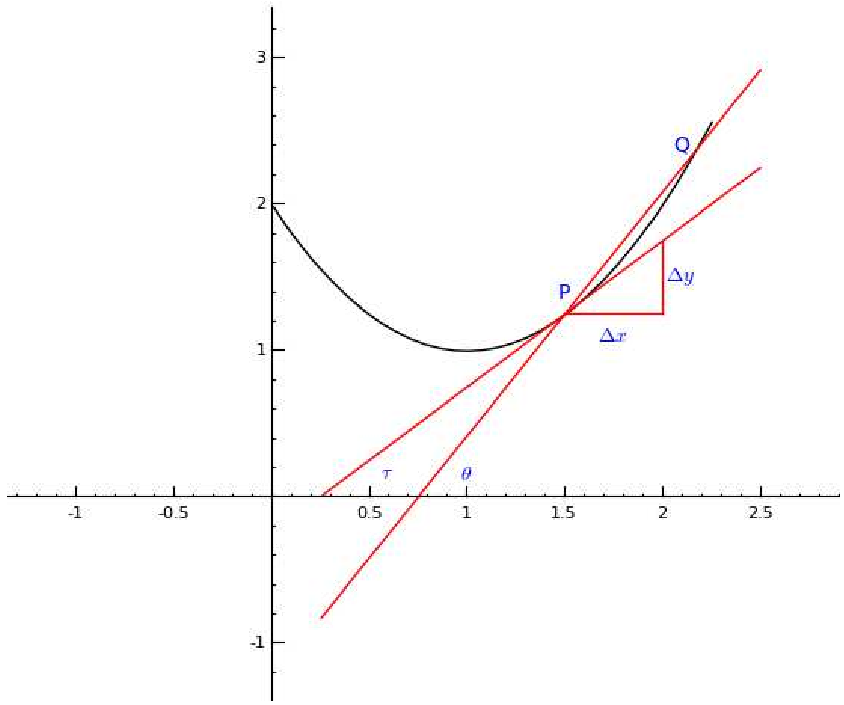

Figure 8.2: The differential of the arclength.

Dividing through by $(\Delta x)^2$, we get

$$\left(\frac{\text{chord}PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2.$$

Now let Q approach P in the limit. Then $\Delta x \to 0$ and we have

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

(Since $\lim_{\Delta x \to 0}\left(\frac{chordPQ}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{\Delta s}{\Delta x}\right) = \frac{ds}{dx}$.) Therefore,

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{8.4}$$

Similarly, if we divide (8.3) by $(\Delta y)^2$ and pass to the limit, we get

$$\frac{ds}{dy} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}.$$

Also, from the above figure,

$$\cos\theta = \frac{\Delta x}{\text{chord}PQ}, \quad \sin\theta = \frac{\Delta y}{\text{chord}PQ}.$$

Now as Q approaches P as a limiting position $\theta \to \tau$, and we get

$$\cos\,\tau = \frac{dx}{ds}, \quad \sin\tau = \frac{dy}{ds}. \tag{8.5}$$

(Since $\lim \frac{\Delta x}{\text{chord}PQ} = \lim \frac{\Delta x}{\Delta x} = \frac{dx}{ds}$, and $\lim \frac{\Delta y}{\text{chord}PQ} = \lim \frac{\Delta y}{\Delta s} = \frac{dy}{ds}$.) Using the notation of differentials, these formulas may be written

$$ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx \tag{8.6}$$

and

$$ds = \left[\left(\frac{dx}{dy}\right)^2 + 1\right]^{\frac{1}{2}} dy, \tag{8.7}$$

respectively. Substituting the value of ds from (8.6) in (8.5),

$$\cos\tau = \frac{1}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}}, \quad \sin\tau = \frac{\frac{dy}{dx}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}}, \tag{8.8}$$

the same relations given by (8.5).

8.4 Derivative of the arclength in polar coordinates

In the discussion below we shall employ the same figure and the same notation used in §5.7.

$$(\text{chord } PQ)^2 = (PR)^2 + (RQ)^2 = (\rho\sin\Delta\theta)^2 + (\rho + \Delta\rho - \rho\cos\Delta\theta)^2.$$

Dividing throughout by $(\Delta\theta)^2$, we get

$$\left(\frac{\text{chord}PQ}{\Delta\theta}\right)^2 = \rho^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left(\frac{\Delta\rho}{\Delta\theta} + \rho\cdot\frac{1-\cos\Delta\theta}{\Delta\theta}\right)^2.$$

Passing to the limit as $\Delta\theta$ diminishes towards zero, we get[3]

$$\left(\frac{ds}{d\theta}\right)^2 = \rho^2 + \left(\frac{d\rho}{d\theta}\right)^2,$$

$$\frac{ds}{d\theta} = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}. \tag{8.9}$$

In the notation of differentials this becomes

$$ds = \left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{1}{2}} d\theta. \tag{8.10}$$

These relations between ρ and the differentials ds, $d\rho$, and $d\theta$ are correctly represented by a right triangle whose hypotenuse is ds and whose sides are $d\rho$ and $\rho d\theta$. Then

$$ds = \sqrt{(\rho d\theta)^2 + (d\rho)^2},$$

and dividing by $d\theta$ gives (8.9). Denoting by ψ the angle between $d\rho$ and ds, we get at once

$$\tan\psi = \rho\frac{d\theta}{d\rho},$$

which is the same as ((5.12).

Example 8.4.1. Find the differential of the arc of the circle $x^2 + y^2 = r^2$.

Solution. Differentiating, $\frac{dy}{dx} = -\frac{x}{y}$.

To find ds in terms of x we substitute in (8.6), giving

$$ds = \left[1 + \frac{x^2}{y^2}\right]^{\frac{1}{2}} dx = \left[\frac{y^2 + x^2}{y^2}\right]^{\frac{1}{2}} dx = \left[\frac{r^2}{y^2}\right]^{\frac{1}{2}} dx = \frac{r\,dy}{\sqrt{r^2 - x^2}}.$$

[3]Recall: $\lim_{\Delta\theta\to 0}\frac{\text{chord}PQ}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{\Delta s}{\Delta\theta} = \frac{ds}{d\theta}$, by §8.3;$\lim_{\Delta\to 0}\frac{\sin\Delta\theta}{\Delta\theta} = 1$, by §2.10; $\lim_{\Delta\theta\to 0}\frac{1-\cos\Delta\theta}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{2\sin^2\frac{\Delta\theta}{2}}{\Delta\theta} = \lim_{\Delta\theta\to 0}\sin\frac{\Delta\theta}{2}\cdot\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} = 0\cdot 1 = 0$, by §2.10 and 39 in §12.1.

To find ds in terms of y we substitute in (8.7), giving

$$ds = \left[1 + \frac{y^2}{x^2}\right]^{\frac{1}{2}} dy = \left[\frac{x^2 + y^2}{x^2}\right]^{\frac{1}{2}} dy = \left[\frac{r^2}{x^2}\right]^{\frac{1}{2}} = \frac{rdy}{\sqrt{r^2 - y^2}}.$$

Example 8.4.2. Find the differential of the arclength of the cardioid $\rho = a(l - \cos\theta)$ in terms of θ.

Solution. Differentiating, $\frac{d\rho}{d\theta} = a \sin\theta$.

Substituting in (8.10), gives

$$ds = [a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{\frac{1}{2}} d\theta = a[2-2\cos\theta]^{\frac{1}{2}} d\theta = a\left[4\sin^2\frac{\theta}{2}\right]^{\frac{1}{2}} d\theta = 2a\sin\frac{\theta}{2} d\theta.$$

8.5 Exercises

Find the differential of arclength in each of the following curves:

1. $y^2 = 4x$.

   Ans. $ds = \sqrt{\frac{1+x}{x}} dx$.

2. $y = ax^2$.

   Ans. $ds = \sqrt{1 + 4a^2x^2} dx$.

3. $y = x^3$.

   Ans. $ds = \sqrt{1 + 9x^4} dx$.

4. $y^3 = x^2$.

   Ans. $ds = \frac{1}{2}\sqrt{4 + 9y} dy$.

5. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

   Ans. $ds = \sqrt[3]{\frac{a}{y}} dy$.

6. $b^2x^2 + a^2y^2 = a^2b^2$.

   Ans. $ds = \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}} dx$.

7. $e^y \cos\, x = 1$.

   Ans. $ds = \sec\, x\, dx$.

8. $\rho = a\cos\theta$.

   Ans. $ds = a\, d\theta$.

9. $\rho^2 = a^2 \cos\, 2\theta$.

   Ans. $ds = \sqrt{\sec 2\theta} d\theta$.

10. $\rho = ae^{\theta \cot\, a}$.

    Ans. $ds = \rho \csc\, a \cdot d\theta$.

11. $\rho = a\theta$.

    Ans. $ds = a^\theta \sqrt{1 + \log^2 a} d\theta$.

12. $\rho = a\theta$.

    Ans. $ds = \frac{1}{a}\sqrt{a^2 + \rho^2} d\rho$.

13.

$(a)\ \ x^2 - y^2 = a^2$.	$(h)\ \ x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.
$(b)\ \ x^2 = 4ay$.	$(i)\ \ y^2 = ax^3$.
$(c)\ \ y = e^x + e^{-x}$.	$(j)\ \ y = \log\, x$.
$(d)\ \ xy = a$.	$(k)\ \ 4x = y^3$.
$(e)\ \ y = \log\sec\, x$.	$(l)\ \ \rho = a\sec^2 \frac{\theta}{2}$.
$(f)\ \ \rho = 2a\tan\theta\sin\theta$.	$(m)\ \ \rho = 1 + \sin\theta$.
$(g)\ \ \rho = a\sec^3 \frac{\theta}{3}$.	$(n)\ \ \rho\theta = a$.

8.6 The derivative considered as the ratio of two rates

Let

$$y = f(x)$$

be the equation of a curve generated by a moving point P. Its coordinates x and y may then be considered as functions of the time, as explained in §5.13. Differentiating with respect to t, by the chain rule (Formula XXV in §4.1), we have

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}. \tag{8.11}$$

At any instant the time rate of change of y (or the function) equals its derivative multiplied by the time rate change of the independent variable.

Or, write (8.11) in the form

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = f'(x) = \frac{dy}{dx}.$$

The derivative measures the ratio of the time rate of change of y to that of x.

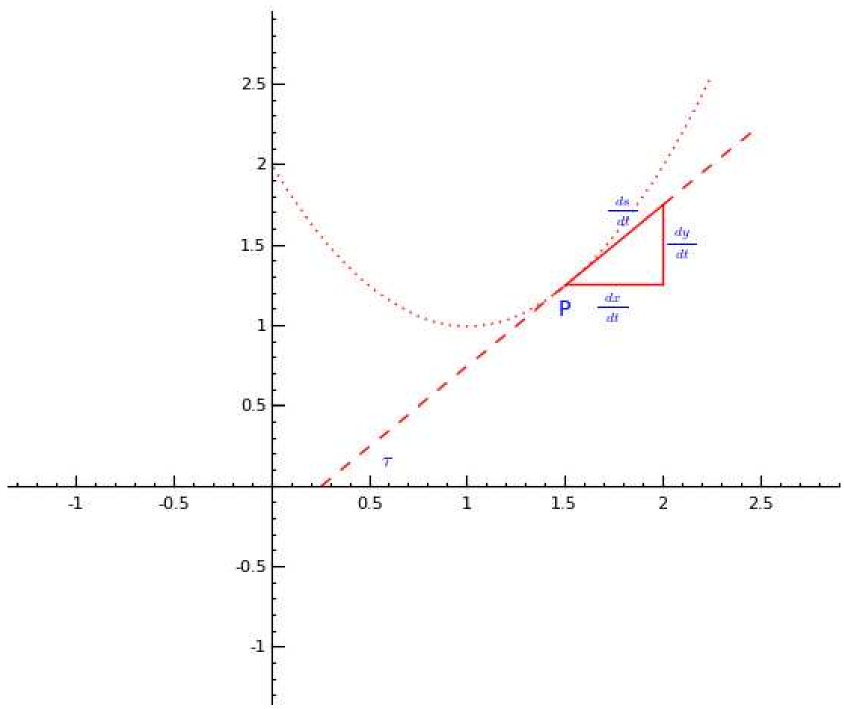

Figure 8.3: Geometric visualization of the derivative the arclength.

$\frac{ds}{dt}$ being the time rate of change of length of arc, we have from (5.26),

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dt}{dt}\right)^2}. \tag{8.12}$$

which is the relation indicated by Figure 8.3.

As a guide in solving rate problems use the following rule.

- FIRST STEP. Draw a figure illustrating the problem. Denote by x, y, z, etc., the quantities which vary with the time.
- SECOND STEP. Obtain a relation between the variables involved which will hold true at any instant.
- THIRD STEP. Differentiate with respect to the time.
- FOURTH STEP. Make a list of the given and required quantities.
- FIFTH STEP. Substitute the known quantities in the result found by differentiating (third step), and solve for the unknown.

8.7 Exercises

1. A man is walking at the rate of 5 miles per hour towards the foot of a tower 60 ft. high. At what rate is he approaching the top when he is 80 ft. from the foot of the tower?

   Solution. Apply the above rule.

   First step. Draw the figure. Let x = distance of the man from the foot and y = his distance from the top of the tower at any instant.

   Second step. Since we have a right triangle, $y^2 = x^2 + 3600$.

   Third step. Differentiating, we get $2y\frac{dy}{dt} = 2x\frac{dx}{dt}$, or, $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$, meaning that at any instant whatever (Rate of change of y) = $\left(\frac{x}{y}\right)$ (rate of change of x).

   Fourth step.

$$\begin{aligned} x &= 80,\ \tfrac{dx}{dt} = 5 \text{ miles/hour}, \\ &= 5 \times 5280 ft/hour, \\ y &= \sqrt{x^2+3600} \\ &= 100. \\ \tfrac{dy}{dt} &=? \end{aligned}$$

   Fifth step. Substituting back in the above $\frac{dy}{dt} = \frac{80}{100} \times 5 \times 5280$ ft/hour = 4 miles/hour.

2. A point moves on the parabola $6y = x^2$ in such a way that when $x = 6$, the x-coordinate is increasing at the rate of 2 ft. per second. At what rates are the y-coordinate and arclength increasing at the same instant?

   Solution. First step. Plot the parabola.

   Second step. $6y = x^2$.

   Third step. $6\frac{dy}{dt} = 2x\frac{dx}{dt}$, or, $\frac{dy}{dt} = \frac{x}{3} \cdot \frac{dx}{dt}$. This means that at any point on the parabola (Rate of change of y-coordinate) = $\left(\frac{x}{3}\right)$ (rate of change of abcissa).

   Fourth step. $\frac{dx}{dt} = 2$ ft. per second, $x = 6$, $\frac{dy}{dt} =?$, $y = \frac{x^2}{6} = 6$, $\frac{ds}{dt} =?$

   Fifth step. Substituting back in the above, $\frac{dy}{dt} = \frac{6}{3} \times 2 = 4$ ft. per second.

   From the first result we note that at the point $(6, 6)$ the y-coordinate changes twice as rapidly as the x-coordinate.

   If we consider the point $(-6, 6)$ instead, the result is $\frac{dy}{dt} = -4$ ft. per second, the minus sign indicating that the y-coordinate is decreasing as the x-coordinate increases.

   We shall now solve this using Sage .

   Sage

```
sage: t = var("t")
sage: x = function("x",t)
sage: y = function("y",t)
sage: eqn = 6*y - x^2
sage: solve(diff(eqn,t) == 0, diff(y(t), t, 1))
[diff(y(t), t, 1) == x(t)*diff(x(t), t, 1)/3]
sage: s = sqrt(x^2+y^2)
sage: diff(s,t)
(2*y(t)*diff(y(t), t, 1)
  + 2*x(t)*diff(x(t), t, 1))/(2*sqrt(y(t)^2 + x(t)^2))
```

   This tells us that $\frac{dy}{dt} = \frac{x}{3} \cdot \frac{dx}{dt}$ and

$$\frac{ds}{dt} = \frac{y(t)y'(t) + x(t)x'(t)}{\sqrt{x(t)^2 + y(t)^2}}.$$

   Substituting $\frac{dx}{dt} = 2$, $x = 6$, gives $\frac{dy}{dt} = 4$. In addition, if $y = 6$ then this gives $\frac{ds}{dt} = 36/\sqrt{72} = 3\sqrt{2}$.

3. A circular plate of metal expands by heat so that its radius increases uniformly at the rate of 0.01 inch per second. At what rate is the surface increasing when the radius is two inches?

   Solution. Let x = radius and y = area of plate. Then $y = \pi x^2$, $\frac{dy}{dt} = 2\pi x \frac{dx}{dt}$, That is; at any instant the area of the plate is increasing in square inches $2\pi x$ times as fast as the radius is increasing in linear inches. $x = 2$, $\frac{dx}{dt} = 0.01$, $\frac{dy}{dt} = ?$. Substituting in the above, $\frac{dy}{dt} = 2\pi \times 2 \times 0.01 = 0.04\pi$ sq. in. per sec.

4. A street light is hung 12 ft. directly above a straight horizontal walk on which a boy 5 ft. in height is walking. How fast is the boy's shadow lengthening when he is walking away from the light at the rate of 168 ft. per minute?

   Solution. Let x = distance of boy from a point directly under light L, and y = length of boy's shadow. By similar triangle, $y/(y+x) = 5/12$, or $y = \frac{5}{7}x$. Differentiating, $\frac{dy}{dt} = \frac{5}{7}\frac{dx}{dt}$; i.e. the shadow is lengthening $\frac{5}{7}$ as fast as the boy is walking, or 120 ft. per minute.

5. In a parabola $y^2 = 12x$, if x increases uniformly at the rate of 2 in. per second, at what rate is y increasing when $x = 3$ in. ?

   Ans. 2 in. per sec.

6. At what point on the parabola of the last example do the x-coordinate and y-coordinate increase at the same rate?

   Ans. $(3, 6)$.

7. In the function $y = 2x^3 + 6$, what is the value of x at the point where y increases 24 times as fast as x?

   Ans. $x = \pm 2$.

8. The y-coordinate of a point describing the curve $x^2 + y^2 = 25$ is decreasing at the rate of $3/2$ in. per second. How rapidly is the x-coordinate changing when the y-coordinate is 4 inches?

   Ans. $\frac{dx}{dt} = 2$ in. per sec.

9. Find the values of x at the points where the rate of change of $x^3 - 12x^2 + 45x - 13$ is zero.

   Ans. $x = 3$ and 5.

10. At what point on the ellipse $16x^2 + 9y^2 = 400$ does y decrease at the same rate that x increases?

    Ans. $(3, \frac{16}{3})$.

11. Where in the first quadrant does the arclength increase twice as fast as the y-coordinate?

    Ans. At $60^o = \pi/3$.

A point generates each of the following curves (problems 12-16). Find the rate at which the arclength is increasing in each case:

12. $y^2 = 2x$; $\frac{dx}{dt} = 2$, $x = 2$.

    Ans. $\frac{ds}{dt} = \sqrt{5}$.

13. $xy = 6$; $\frac{dy}{dt} = 2$, $y = 3$.

    Ans. $\frac{ds}{dt} = \frac{2}{3}\sqrt{13}$.

14. $x^2 + 4y^2 = 20$; $\frac{dx}{dt} = -1$, $y = 1$.

    Ans. $\frac{ds}{dt} = \sqrt{2}$.

15. $y = x^3$; $\frac{dx}{dt} = 3$, $x = -3$.

16. $y^2 = x^3$; $\frac{dy}{dt} = 4$, $y = 8$.

17. The side of an equilateral triangle is 24 inches long, and is increasing at the rate of 3 inches per hour. How fast is the area increasing?

    Ans. $36\sqrt{3}$ sq. in. per hour.

18. Find the rate of change of the area of a square when the side b is increasing at the rate of a units per second.

    Ans. $2ab$ sq. units per sec.

19. (a) The,volume of a spherical soap bubble increases how many times as fast as the radius? (b) When its radius is 4 in. and increasing at the rate of $1/2$ in. per second, how fast is the volume increasing?

    Ans. (a) $4\pi r^2$ times as fast; (b) 32π cu. in. per sec.

    How fast is the surface increasing in the last case?

20. One end of a ladder 50 ft. long is leaning against a perpendicular wall standing on a horizontal plane. Supposing the foot of the ladder to be pulled away from the wall at the rate of 3 ft. per minute; (a) how fast is the top of the ladder descending when the foot is 14 ft. from the wall? (b) when will the top and bottom of the ladder move at the same rate? (c) when is the top of the ladder descending at the rate of 4 ft. per minute?

    Ans. (a) $\frac{7}{78}$ ft. per min.; (b) when $25\sqrt{2}$ ft. from wall; (c) when 40 ft. from wall.

21. A barge whose deck is 12 ft. below the level of a dock is drawn up to it by means of a cable attached to a ring in the floor of the dock, the cable being hauled in by a windlass on deck at the rate of 8 ft. per minute. How fast is the barge moving towards the dock when 16 ft. away?

    Ans. 10 ft. per minute.

22. An elevated car is 40 ft. immediately above a surface car, their tracks intersecting at right angles. If the speed of the elevated car is 16 miles per hour and of the surface car 8 miles per hour, at what rate are the cars separating 5 minutes after they meet?

    Ans. 17.9 miles per hour.

23. One ship was sailing south at the rate of 6 miles per hour; another east at the rate of 8 miles per hour. At 4 P.M. the second crossed the track of the first where the first was two hours before; (a) how was the distance between the ships changing at 3 P.M.? (b) how at 5 P.M.? (c) when was the distance between them not changing?

    Ans. (a) Diminishing 2.8 miles per hour; (b) increasing 8.73 miles per hour; (c) $3:17$ P.M.

24. Assuming the volume of the wood in a tree to be proportional to the cube of its diameter, and that the latter increases uniformly year by year when growing, show that the rate of growth when the diameter is 3 ft. is 36 times as great as when the diameter is 6 inches.

25. A railroad train is running 15 miles an hour past a station 800 ft. long, the track having the form of the parabola $y^2 = 600x$, and situated as shown in Figure 8.4.

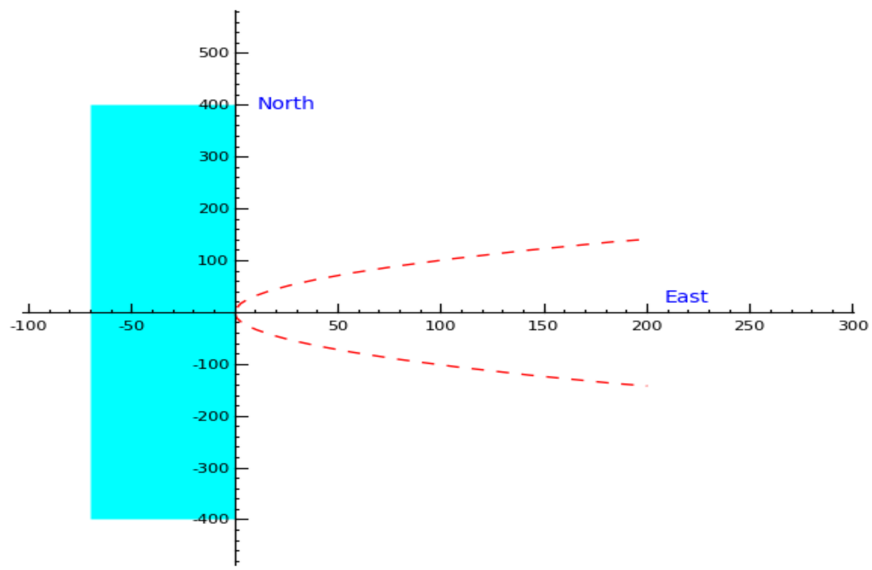

Figure 8.4: Train station and the train's trajectory.

If the sun is just rising in the east, find how fast the shadow S of the locomotive L is moving along the wall of the station at the instant it reaches the end of the wall.

Solution. $y^2 = 600x$, $2y\frac{dy}{dt} = 600\frac{dx}{dt}$, or $\frac{dx}{dt} = \frac{y}{300}\frac{dy}{dt}$. Substituting this value of $\frac{dx}{dt}$ in $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$, we get $\left(\frac{ds}{dt}\right)^2 = \left(\frac{y}{300}\frac{dy}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$. Now $\frac{ds}{dt} = 15$ miles per hour = 22 ft. per sec., $y = 400$ and $\frac{dy}{dt} =$?. Substituting back in the above, we get $(22)^2 = \left(\frac{16}{9} + 1\right)\left(\frac{dy}{dt}\right)^2$, or, $\frac{dy}{dt} = 13\frac{1}{5}$ ft. per second.

26. An express train and a balloon start from the same point at the same instant. The former travels 50 miles an hour and the latter rises at the rate of 10 miles an hour. How fast are they separating?

    Ans. 51 miles an hour.

27. A man 6 ft. tall walks away from a lamp-post 10 ft. high at the rate of 4 miles an hour. How fast does the shadow of his head move?

    Ans. 10 miles an hour.

28. The rays of the sun make an angle of $30^o = \pi/6$ with the horizon. A ball is thrown vertically upward to a height of 64 ft. How fast is the shadow of the ball moving along the ground just before it strikes the ground?

    Ans. 110.8 ft. per sec.

29. A ship is anchored in 18 ft. of water. The cable passes over a sheave on the bow 6 ft. above the surface of the water. If the cable is taken in at the rate of 1 ft. a second, how fast is the ship moving when there are 30 ft. of cable out?

    Ans. $\frac{5}{3}$ ft. per sec.

30. A man is hoisting a chest to a window 50 ft. up by means of a block and tackle. If he pulls in the rope at the rate of 10 ft. a minute while walking away from the building at the rate of 5 ft. a minute, how fast is the chest rising at the end of the second minute?

    Ans. 10.98 ft. per min.

31. Water flows from a faucet into a hemispherical basin of diameter 14 inches at the rate of 2 cu. in. per second. How fast is the water rising (a) when the water is halfway to the top? (b) just as it runs over? (The volume of a spherical segment = $\frac{1}{2}\pi r^2 h + \frac{1}{6}\pi h^3$, where h = altitude of segment.)

32. Sand is being poured on the ground from the orifice of an elevated pipe, and forms a pile which has always the shape of a right circular cone whose height is equal to the radius of the base. If sand is falling at the rate of 6 cu. ft. per sec., how fast is the height of the pile increasing when the height is 5 ft.?

33. An aeroplane is 528 ft. directly above an automobile and starts east at the rate of 20 miles an hour at the same instant the automobile starts east at the rate of 40 miles an hour. How fast are they separating?

34. A revolving light sending out a bundle of parallel rays is at a distance of t a mile from the shore and makes 1 revolution a minute. Find how fast the light is traveling along the straight beach when at a distance of 1 mile from the nearest point of the shore.

    Ans. 15.7 miles per min.

35. A kite is 150 ft. high and 200 ft. of string are out. If the kite starts drifting away horizontally at the rate of 4 miles an hour, how fast is the string being paid out at the start?

    Ans. 2.64 miles an hour.

36. A solution is poured into a conical filter of base radius 6 cm. and height 24 cm. at the rate of 2 cu. cm. a second, and filters out at the rate of 1 cu. cm. a second. How fast is the level of the solution rising when (a) one third of the way up? (b) at the top?

    Ans. (a) 0.079 cm. per sec.; (b) 0.009 cm. per sec.

37. A horse runs 10 miles per hour on a circular track in the center of which is a street light. How fast will his shadow move along a straight board fence (tangent to the track at the starting point) when he has completed one eighth of the circuit?

    Ans. 20 miles per hour.

38. The edges of a cube are 24 inches and are increasing at the rate of 0.02 in. per minute. At what rate is (a) the volume increasing? (b) the area increasing?

39. The edges of a regular tetrahedron are 10 inches and are increasing at the rate of 0.3 in. per hour. At what rate is (a) the volume increasing? (b) the area increasing?

40. An electric light hangs 40 ft. from a stone wall. A man is walking 12 ft. per second on a straight path 10 ft. from the light and perpendicular to the wall. How fast is the man's shadow moving when he is 30 ft. from the wall?

    Ans. 48 ft. per sec.

41. The approach to a drawbridge has a gate whose two arms rotate about the same axis as shown in the figure. The arm over the driveway is 4 yards long and the arm over the footwalk is 3 yards long. Both arms rotate at the rate of 5 radians per minute. At what rate is the distance between the extremities of the arms changing when they make an angle of $45^o = \pi/4$ with the horizontal?

    Ans. 24 yd. per min.

42. A conical funnel of radius 3 inches and of the same depth is filled with a solution which filters at the rate of 1 cu. in. per minute. How fast is the surface falling when it is 1 inch from the top of the funnel?

    Ans. $\frac{1}{4\pi}$ in. per mm.

43. An angle is increasing at a constant rate. Show that the tangent and sine are increasing at the same rate when the angle is zero, and that the tangent increases eight times as fast as the sine when the angle is $60^o = \pi/3$.

CHAPTER

NINE

Change of variable

If $y = f(x)$ is a function of x and x is a function of some other variable t then $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc., can be expressed in terms of $\frac{dy}{dt}$, $\frac{dx}{dt}$, $\frac{d^2y}{dt^2}$, etc.. This chapter is devoted to explaining the techniques to find the formulas necessary for making such a change of variables.

9.1 Interchange of dependent and independent variables

If $y = f(x)$ is a one-to-one function of x then it can be "inverted" so that $x = f^{-1}(y)$ is a function of y. It is sometimes desirable to transform an expression involving derivatives of y with respect to x into an equivalent expression involving derivatives of x with respect to y. Our examples will show that in many cases such a change transforms the given expression into a much simpler one. Or perhaps x is given as an explicit function of y in a problem, and it is found more convenient to use a formula involving $\frac{dx}{dy}$, $\frac{d^2x}{dy^2}$, etc., than one involving $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc. We shall now find the formulas necessary for making such transformations.

Given $y = f(x)$, then from item 4.28 in §4.1, we have

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \quad \frac{dx}{dy} \neq 0 \tag{9.1}$$

giving $\frac{dy}{dx}$ in terms of $\frac{dx}{dy}$. Also, by 4.27 in §4.1,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right)\frac{dy}{dx},$$

or

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)\frac{dy}{dx}. \tag{9.2}$$

But $\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right) = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}$; and $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ from (9.1). Substituting these in (9.2), we get

$$\frac{d^2y}{dx^2} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}, \tag{9.3}$$

giving $\frac{d^2y}{dx^2}$ in terms of $\frac{dx}{dy}$ and $\frac{d^2x}{dy^2}$. Similarly,

$$\frac{d^3y}{dx^3} = -\frac{\frac{d^3x}{dy^3}\frac{dx}{dy} - 3\left(\frac{d^2x}{dy^2}\right)^2}{\left(\frac{dx}{dy}\right)^5}, \tag{9.4}$$

and so on for higher derivatives. This transformation is called changing the independent variable from x to y.

Example 9.1.1. Change the independent variable from x to y in the equation

$$3\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx}\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^2 = 0.$$

Solution. Substituting from (9.1), (9.3), (9.4),

$$3\left(-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}\right)^2 - \left(\frac{1}{\frac{dx}{dy}}\right)\left(-\frac{\frac{d^3x}{dy^3}\frac{dx}{dy} - 3\left(\frac{d^2x}{dy^2}\right)^2}{\left(\frac{dx}{dy}\right)^5}\right) - \left(-\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}\right)\left(\frac{1}{\frac{dx}{dy}}\right)^2 = 0.$$

Reducing, we get

$$\frac{d^3x}{dy^3} + \frac{d^2x}{dy^2} = 0,$$

a much simpler equation.

9.2 Change of the dependent variable

Let

$$y = f(x),$$

and suppose at the same time y is a function of z, say

$$y = g(z).$$

We may then express $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc., in terms of $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$, etc., as follows

In general, z is a function of y, and since y is a function of x, it is evident that z is a function of x. Hence by 4.27 of §4.1, we have

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \psi'(z)\frac{dz}{dx}.$$

Also $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(g'(z)\frac{dz}{dx}\right) = \frac{dz}{dx}\frac{d}{dx}g'(z) + g'(z)\frac{d^2z}{dx^2}$. But $\frac{d}{dx}g'(z) = \frac{d}{dz}g'(z)\frac{dz}{dx} = g''(z)\frac{dz}{dx}$. Therefore,

$$\frac{d^2y}{dx^2} = g''(z)\left(\frac{dz}{dx}\right)^2 + g'(z)\frac{d^2z}{dx^2}.$$

Similarly for higher derivatives. This transformation is called *changing the dependent variable* from y to z, the independent variable remaining x throughout. We will now illustrate this process by means of an example.

Example 9.2.1. Having given the equation

$$\frac{d^2y}{dx^2} = 1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2,$$

change the dependent variable from y to z by means of the relation

$$y = \tan z.$$

Solution. From the above,

$$\frac{dy}{dx} = \sec^2(z)\frac{dz}{dx}, \frac{d^2y}{dx^2} = \sec^2(z)\frac{d^2z}{dx^2} + 2\sec^2(z)\tan(z)\left(\frac{dz}{dx}\right)^2,$$

Substituting,

$$\sec^2(z)\frac{d^2z}{dx^2} + 2\sec^2(z)\tan(z)\left(\frac{dz}{dx}\right)^2 = 1 + \frac{2(1+\tan z)}{1+\tan^2 z}\left(\sec^2 z\frac{dz}{dx}\right)^2,$$

and reducing, we get $\frac{d^2z}{dx^2} - 2\left(\frac{dz}{dx}\right)^2 = \cos^2 z$.

9.3 Change of the independent variable

Let y be a function of x, and at the same time let x (and hence also y) be a function of a new variable t. It is required to express

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \text{etc.},$$

in terms of new derivatives having t as the independent variable. By 4.27 §4.1, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, or

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}. \tag{9.5}$$

This is another formulation of the so-called *chain rule*. Also

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

But differentiating $\frac{dy}{dx}$ with respect to t,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^2}.$$

Therefore

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dx^2}}{\left(\frac{dx}{dt}\right)^3}, \tag{9.6}$$

and so on for higher derivatives. This transformation is called changing the independent variable from x to t. It is usually better to work out examples by the methods illustrated above rather than by using the formulas deduced.

Example 9.3.1. Change the independent variable from x to t in the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

where $x = e^t$.

Solution. $\frac{dx}{dt} = e^t$, therefore

$$\frac{dt}{dx} = e^{-t} \, .$$

Also $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$; therefore $\frac{dy}{dx} = e^{-t}\frac{dy}{dt}$. Also $\frac{d^2y}{dx^2} = e^{-t}\frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{dy}{dt}e^{-t}\frac{dt}{dx} = e^{-t}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} - \frac{dy}{dt}e^{-t}\frac{dt}{dx}$. Substituting into the last result $\frac{dt}{dx} = e^{-t}$,

$$\frac{d^2y}{dx^2} = e^{-2t}\frac{d^2y}{dt^2} - \frac{dy}{dt}e^{-2t}.$$

Substituting these into the differential equation,

$$e^{2t}\left(e^{-2t}\frac{d^2y}{dt^2} - \frac{dy}{dt}e^{-2t}\right) + e^t\left(e^{-t}\frac{dy}{dt}\right) + y = 0,$$

and reducing, we get $\frac{d^2y}{dt^2} + y = 0$.

Since the formulas deduced in the Differential Calculus generally involve derivatives of y with respect to x, such formulas as the chain rule are especially useful when the parametric equations of a curve are given. Such examples were given in §5.5, and many others will be employed in what follows.

9.4 Change of independent and dependent variables

It is often desirable to change both independent and dependent variables simultaneously. An important case is that arising in the transformation from rectangular to polar coordinates. Since

$$x = \rho\cos\theta, \text{ and } y = \rho\sin\theta,$$

the equation

$$f(x, y) = 0$$

becomes by substitution an equation between ρ and θ, defining ρ as a function of θ. Hence ρ, x, y are all functions of θ.

Example 9.4.1. Transform the formula for the radius of curvature (11.5),

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

into polar coordinates.

Solution. Since in (9.5) and (9.6), t is any variable on which x and y depend, we may in this case let $t = \theta$, giving $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$, and

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\frac{d^2x}{d\theta^2}}{\left(\frac{dx}{d\theta}\right)^3}.$$

Substituting these into R, we get

$$R = \left[\frac{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}{\left(\frac{dx}{d\theta}\right)^2}\right]^{\frac{3}{2}} \div \frac{\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\frac{d^2x}{d\theta^2}}{\left(\frac{dx}{d\theta}\right)^3},$$

or

$$R = \frac{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\frac{d^2x}{d\theta^2}}. \tag{9.7}$$

But since $x = \rho\cos\theta$ and $y = \rho\sin\theta$, we have

$$\frac{dx}{d\theta} = -\rho\sin\theta + \cos\theta\frac{d\rho}{d\theta};$$

$$\frac{dy}{d\theta} = \rho\cos\theta + \sin\theta\frac{d\rho}{d\theta};$$

$$\frac{d^2x}{d\theta^2} = -\rho\cos\theta - 2\sin\theta\frac{d\rho}{d\theta} + \cos\theta\frac{d^2\rho}{d\theta^2};$$

$$\frac{d^2y}{d\theta^2} = -\rho\sin\theta + 2\cos\theta\frac{d\rho}{d\theta} + \sin\theta\frac{d^2\rho}{d\theta^2}.$$

Substituting these in (9.7) and reducing,

$$R = \frac{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2 + 2\left(\frac{d\rho}{d\theta}\right)^2 - \rho\frac{d^2\rho}{d\theta^2}}.$$

9.5 Exercises

Change the independent variable from x to y in the following equations.

1. $R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$

   Ans. $R = -\frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}.$

2. $\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2 = 0.$

   Ans. $\frac{d^2x}{dy^2} - 2y\frac{dx}{dy} = 0.$

3. $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0.$

   Ans. $x\frac{d^2y}{dx^2} - 1 + \left(\frac{dx}{dy}\right)^2 = 0.$

4. $\left(3a\frac{dy}{dx} + 2\right)\left(\frac{d^2y}{dx^2}\right)^2 = \left(a\frac{dy}{dx} + 1\right)\frac{dy}{dx}\frac{d^3y}{dx^3}.$

   Ans. $\left(\frac{d^2x}{dy^2}\right)^2 = \left(\frac{dx}{dy} + a\right)\frac{d^3x}{dy^3}.$

Change the dependent variable from y to z in the following equations:

5. $(1+y)^2\left(\frac{d^3y}{dx^3} - 2y\right) + \left(\frac{dy}{dx}\right)^2 = 2(1+y)\frac{dy}{dx}\frac{d^2y}{dx^2}, y = z^2 + 2z.$

   Ans. $(z+1)\frac{d^3x}{dx^3} = \frac{dz}{dx}\frac{d^2z}{dx^2} + z^2 + 2z.$

6. $\frac{d^2y}{dx^2} = 1 + \frac{2(1+y)}{1+y^2}\left(\frac{dy}{dx}\right)^2, y = \tan z.$

   Ans. $\frac{d^2z}{dx^2} - 2\left(\frac{dz}{dx}\right)^2 = \cos^2 z.$

7. $y^2\frac{d^3y}{dx^3} - \left(3y\frac{dy}{dx} + 2xy^2\right)\frac{d^2y}{dx^2} + \left\{2\left(\frac{dy}{dx}\right)^2 2xy\frac{dy}{dx} + 3x^2y^2\right\}\frac{dy}{dx} + x^3y^3 = 0, y = e^z$.

   Ans. $\frac{d^3z}{dx^3} - 2x\frac{d^2z}{dx^2} + 3x^2\frac{dz}{dx} + x^3 = 0$.

Change the independent variable in the following eight equations:

8. $\frac{d^2y}{dx^2} - \frac{x}{1-x^2}\frac{dy}{dx} + \frac{y}{1-x^2} = 0, \quad x = \cos\, t$.

   Ans. $\frac{d^2y}{dt^2} + y = 0$.

9. $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$, $x = \cos\, z$.

   Ans. $\frac{d^2y}{dz^2} = 0$.

10. $(1-y^2)\frac{d^2u}{dy^2} - y\frac{du}{dy} + a^2u = 0, \quad y = \sin\, x$.

    Ans. $\frac{d^2u}{dx^2} + a^2u = 0$.

11. $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \frac{a^2}{x^2}y = 0, \quad x = \frac{1}{z}$.

    Ans. $\frac{d^2y}{dz^2} + a^2y = 0$.

12. $x^3\frac{d^3v}{dx^3} + 3x^2\frac{d^2v}{dx^2} + x\frac{dv}{dx} + v = 0, \quad x = e^t$.

    Ans. $\frac{d^3v}{dx^3} + v = 0$.

13. $\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{y}{(1+x^2)^2} = 0, \quad x = \tan\theta$.

    Ans. $\frac{d^2y}{d\theta^2} + y = 0$.

14. $\frac{d^2u}{ds^2} + su\frac{du}{ds} + \sec^2 s = 0$.

    Ans. $s = \arctan\, t$.

15. $x^4\frac{d^2y}{dx^2} + a^2y = 0, \quad x = \frac{1}{z}$.

    Ans. $\frac{d^2y}{dz^2} + \frac{2}{z}\frac{dy}{dz} + a^2y = 0$.

In the following seven examples the equations are given in parametric form. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in each case:

16. $x = 7 + t^2$, $y = 3 + t^2 - 3t^4$.

    Ans. $\frac{dy}{dx} = 1 - 6t^2$, $\frac{d^2y}{dx^2} = -6$.

    We shall solve this using Sage .

    Sage

```
sage: t = var("t")
sage: x = 7 + t^2
sage: y = 3 + t^2 - 3*t^4
sage: f = (x, y)
sage: p = parametric_plot(f, 0, 1)
sage: D_x_of_y = diff(y,t)/diff(x,t); D_x_of_y
(2*t - 12*t^3)/(2*t)
sage: solve(D_x_of_y == 0,t)
[t == -1/sqrt(6), t == 1/sqrt(6)]
sage: t0 = solve(D_x_of_y == 0,t)[1].rhs()
sage: (x(t0),y(t0))
(43/6, 37/12)
sage: D_xx = (diff(y,t,t)*diff(x,t)-diff(x,t,t)*diff(y,t))/diff(x,t)^2
sage: D_xx
(2*t*(2 - 36*t^2) - 2*(2*t - 12*t^3))/(4*t^2)
sage: D_xx(t0)
-12/sqrt(6)
```

    This tells us that the critical point is at $(43/6, 37/12) = (7.166.., 3.0833..)$, which is a maximum. The plot in Figure 9.1 illustrates this.

17. $x = \cot\, t$, $y = \sin^3 t$.

    Ans. $\frac{dy}{dx} = -3\sin^4 t\cos t$, $\frac{d^2y}{dx^2} = 3\sin^5 t(4 - 5\sin^2 t)$.

18. $x = a(\cos\, t + \sin\, t)$, $y = a(\sin\, t - t\cos\, t)$.

    Ans. $\frac{dy}{dx} = \tan t$, $\frac{d^2y}{dx^2} = \frac{1}{at\cos^3 t}$.

19. $x = \frac{1-t}{1+t}$, $y = \frac{2t}{1+t}$.

20. $x = 2t$, $y = 2 - t^2$.

21. $x = 1 - t^2$, $y = t^3$.

22. $x = a\cos\, t$, $y = b\sin\, t$.

23. Transform $\frac{x\frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$ by assuming $x = \rho\cos\theta$, $y = \rho\sin\theta$.

    Ans. $\frac{\rho^2}{\sqrt{\rho\left(\frac{d\rho}{d\theta}\right)^2}}$.

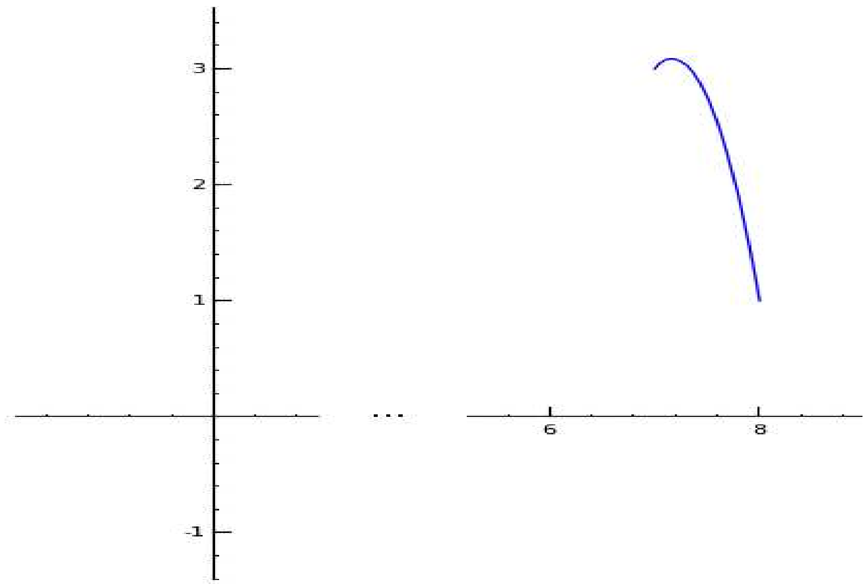

Figure 9.1: Plot for Exercise 11.5-16, $x = 7 + t^2$, $y = 3 + t^2 - 3t^4$.

24. Let $f(x, y) = 0$ be the equation of a curve. Find an expression for its slope $\left(\frac{dy}{dx}\right)$ in terms of polar coordinates.

    Ans. $\frac{dy}{dx} = \frac{\rho\cos\theta + \sin\theta\frac{d\rho}{d\theta}}{-\rho\sin\theta + \cos\theta\frac{d\rho}{d\theta}}$.

CHAPTER

TEN

Applications of higher derivatives

We have seen how the first derivative can be applied to solving max-min problems and related rate problems. In this chapter, we present some applications of higher derivatives. Below, we introduce the mean value theorem, L'Hôpital's rule for limits of "indeterminant forms," and Taylor series approximations.

10.1 Rolle's Theorem

Let $y = f(x)$ be a continuous single-valued function of x, vanishing for $x = a$ and $x = b$, and suppose that $f'(x)$ changes continuously when x varies from a to b. The function will then be represented graphically by a continuous curve starting at a point on the x-axis and ending at another point on the x-axis, as in Figure 10.1. Geometric intuition tells us that for at least one value of x between a and b the tangent is parallel to the x-axis (as at P); that is, the slope is zero.
This illustrates

Rolle's Theorem: *If $f(x)$ vanishes when $x = a$ and $x = b$, and $f(x)$ and $f'(x)$ are continuous for all values of x from $x = a$ to $x = b$, then $f'(x)$ will be zero for at least one value of x between a and b.*

This theorem is obviously true, because as x increases from a to b, $f(x)$ cannot always increase or always decrease as x increases, since $f(a) = 0$ and $f(b) = 0$. Hence for at least one value of x between a and b, $f(x)$ must cease to increase and begin to decrease, or else cease to decrease and begin to increase; and for that

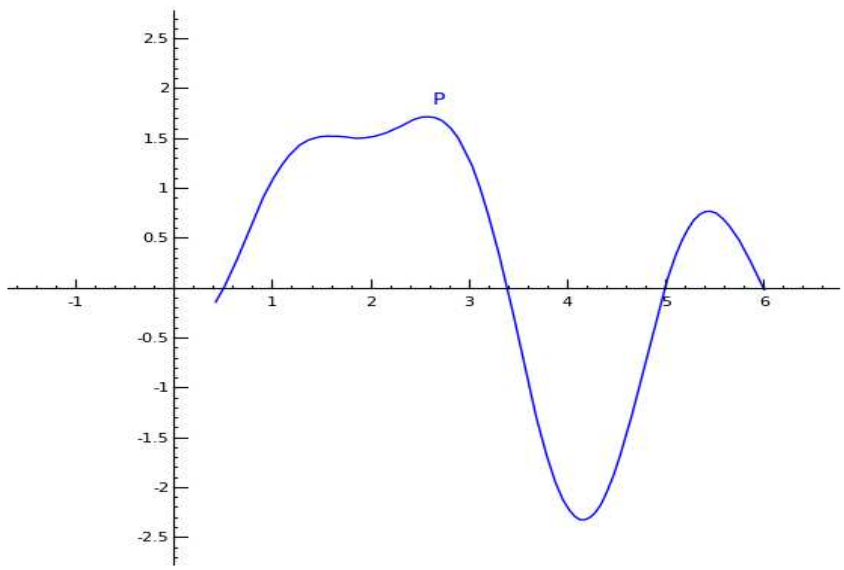

Figure 10.1: Geometrically illustrating Rolle's theorem.

particular value of x the first derivative must be zero (see §7.3).

That Rolle's Theorem does not apply when $f(x)$ or $f'(x)$ are discontinuous is illustrated in Figure 10.2.

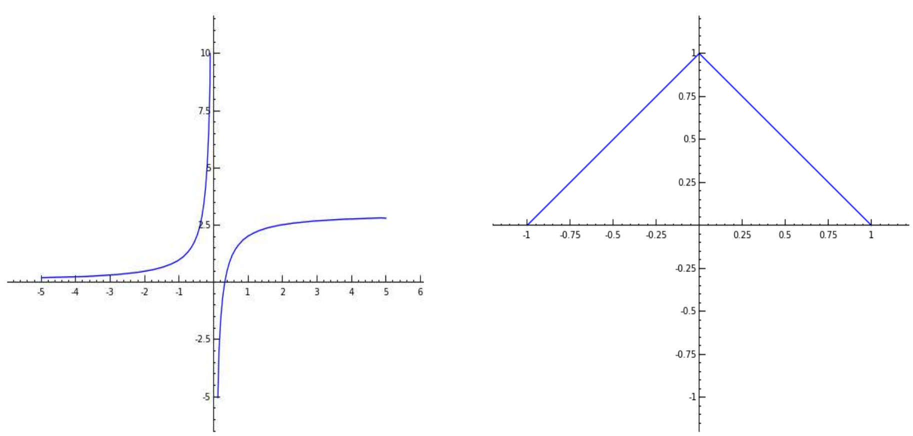

Figure 10.2: Counterexamples to Rolle's theorem.

Figure 10.2 shows

(a) the graph of a function which is discontinuous (= ∞) for $x = c$, a value lying between a and b, and

(b) a continuous function whose first derivative is discontinuous (= ∞) for such an intermediate value $x = c$.

In either case it is seen that at no point on the graph between $x = a$ and $x = b$ does the tangent (or curve) be,come parallel to the x-axis.

10.2 The mean value theorem

Consider the quantity Q defined by the equation

$$\frac{f(b) - f(a)}{b - a} = Q, \tag{10.1}$$

or

$$f(b) - f(a) - (b - a)Q = 0. \tag{10.2}$$

Let $F(x)$ be a function formed by replacing b by x in the left-hand member of (10.2); that is,

$$F(x) = f(x) - f(a) - (x - a)Q. \tag{10.3}$$

From (10.2), $F(b) = 0$, and from (10.3), $F(a) = 0$; therefore, by Rolle's Theorem (see §10.1), $F'(x)$ must be zero for at least one value of x between a and b, say for x_1. But by differentiating (10.3) we get

$$F'(x) = f'(x) - Q.$$

Therefore, since $F'(x_1) = 0$, then also $f'(x_1) - Q = 0$, and $Q = f'(x_1)$. Substituting this value of Q in (10.1), we get the ean value theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(x_1),\ a < x_1 < b \tag{10.4}$$

where in general all we know about x_1 is that it lies between a and b.

The mean value theorem interpreted geometrically.

Let the curve in the figure be the locus of $y = f(x)$.

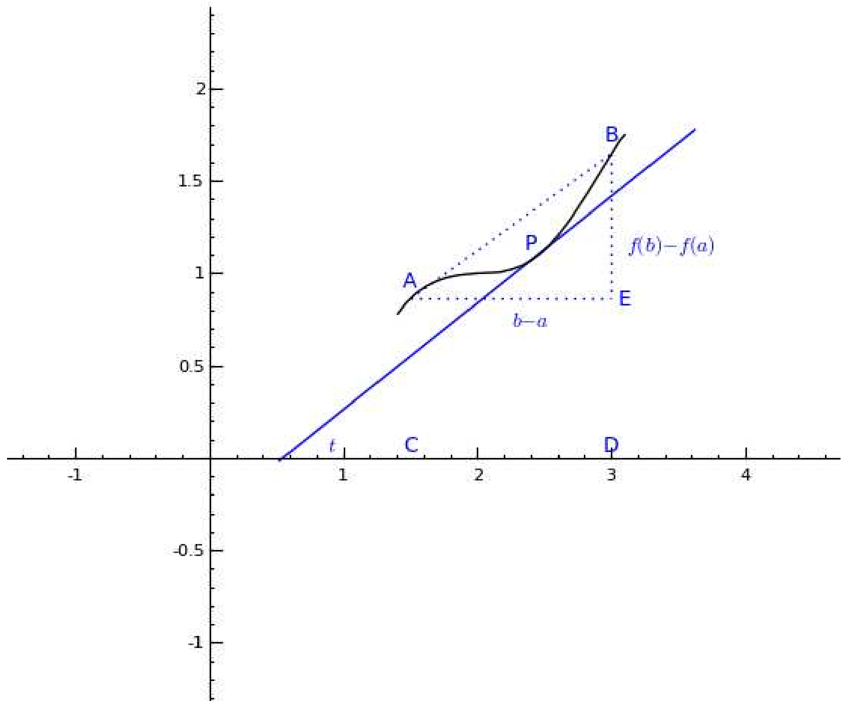

Figure 10.3: Geometric illustration of the mean value theorem.

In the notation of Figure 10.3, take $OC = a$ and $OD = b$; then $f(a) = CA$ and $f(b) = DB$, giving $AE = b - a$ and $EB = f(b) - f(a)$. Therefore the slope of the chord AB is

$$\tan EAB = \frac{EB}{AE} = \frac{f(b) - f(a)}{b - a}.$$

There is at least one point on the curve between A and B (as P) where the tangent (or curve) is parallel to the chord AB. If the abscissa of P is x_1 the slope at P is

$$\tan\ t = f'(x_1) = \tan\ EAB.$$

Equating these last two equations, we get

$$\frac{f(b) - f(a)}{b - a} = f'(x_1),$$

which is the mean value theorem.

The student should draw curves (as the one in §10.1), to show that there may be more than one such point in the interval; and curves to illustrate, on the other hand, that the theorem may not be true if $f(x)$ becomes discontinuous for any value of x between a and b (see Figure 10.2 (a)), or if $f'(x)$ becomes discontinuous (see Figure 10.2 (b)).

Clearing (10.4) of fractions, we may also write the theorem in the form

$$f(b) = f(a) + (b-a)f'(x_1). \tag{10.5}$$

Let $b = a + \Delta a$; then $b - a = \Delta a$, and since x_1 is a number lying between a and b, we may write

$$x_1 = a + \theta \cdot \Delta a,$$

where θ is a positive proper fraction. Substituting in (10.4), we get another form of the mean value theorem.

$$f(a+\Delta a) - f(a) = \Delta a f'(a + \theta \cdot \Delta a), \quad 0 < \theta < 1. \tag{10.6}$$

10.3 The extended mean value theorem

Following the method of the last section, let R be defined by the equation

$$f(b) - f(a) - (b-a)f'(a) - \frac{1}{2}(x-a)^2 R = 0. \tag{10.7}$$

Let $F(x)$ be a function formed by replacing b by x in the left-hand member of (10.1); that is,

$$F(x) = f(x) - f(a) - (x-a)f'(a) - \frac{1}{2}(x-a)^2 R. \tag{10.8}$$

From (10.7), $F(b) = 0$; and from (10.8), $F(a) = 0$; therefore, by Rolle's Theorem, at least one value of x between a and b, say x_1 will cause $F'(x)$ to vanish. Hence, since

$$F'(x) = f'(x) - f'(a) - (x-a)R,$$

we get

$$F'(x_1) = f'(x_1) - f'(a) - (x_1 - a)R = 0.$$

Since $F'(x_1) = 0$ and $F'(a) = 0$, it is evident that $F'(x)$ also satisfies the conditions of Rolle's Theorem, so that its derivative, namely $F''(x)$, must vanish for at least one value of x between a and x_1, say x_2, and therefore x_2 also lies between a and b. But $F''(x) = f''(x) - R$; therefore $F''(x_2) = f''(x_2) - R = 0$, and $R = f''(x_2)$. Substituting this result in (10.7), we get

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f''(x_2), \quad a < x_2 < b.$$

In the same manner, if we define S by means of the equation

$$f(b) - f(a) - (b-a)f'(a) - \frac{1}{2!}(b-a)^2 f''(a) - \frac{1}{3!}(b-a)^2 f''(a)S = 0,$$

we can derive the equation

$$\begin{aligned} f(b) = f(a) \quad & +(b-a)f'(a) + \tfrac{1}{2!}(b-a)^2 f''(a) \\ & +\tfrac{1}{3!}(b-a)^3 f'''(x_3), a < x_3 < b, \end{aligned} \tag{10.9}$$

where x_3 lies between a and b. By continuing this process we get the general result,

$$\begin{aligned} f(b) = f(a) \quad & +\tfrac{(b-a)}{1!} f'(a) + \tfrac{(b-a)^2}{2!} f''(a) \\ & +\tfrac{(b-a)^3}{3!} f'''(a) + \cdots + \tfrac{(b-a)^{(n-1)}}{(n-1)!} f^{(n-1)}(a) \\ & +\tfrac{(b-a)^n}{n!} f^{(n)}(x_1), \quad a < x_1 < b, \end{aligned}$$

where x_1 lies between a and b. This equation is called the extended mean value theorem[1].

Applications of this theorem will be presented in §10.15 below.

10.4 Exercises

Examine the following functions for maximum and minimum values, using the methods above.

1. $y = 3x^4 - 4x^3 + 1$

   Ans. $x = 1$ is a min., $y = 0$; $x = 0$ gives neither.

[1] Also called Taylor's formula.

2. $y = x^3 - 6x^2 + 12x + 48$

   Ans. $x = 2$ gives neither.

3. $y = (x-1)^2(x+1)^3$

   Ans. $x = 1$ is a min., $y = 0$; $x = 1/5$ is a max; $x = -1$ gives neither.

4. Investigate $y = x^5 - 5x^4 + 5x^3 - 1$ at $x = 1$ and $x = 3$.

5. Investigate $y = x^3 - 3x^2 + 3x + 7$ at $x = 1$.

6. Show the if the first derivative of $f(x)$ which does not vanish at $x = a$ is of odd order n then $f(x)$ is increasing or decreasing at $x = a$, according to whether $f^{(n)}(a)$ is positive or negative.

10.5 Maxima and minima treated analytically

By making use of the results of the last two sections we can now give a general discussion of maxima and minima of functions of a single independent variable.

Given the function $f(x)$. Let h be a positive number as small as we please; then the definitions given in §7.4, may be stated as follows: If, for all values of x different from a in the interval $[a-h, a+h]$,

$$f(x) - f(a) = \text{ a negative number,} \tag{10.10}$$

then $f(x)$ is said to be a *maximum* when $x = a$. If, on the other hand,

$$f(x) - f(a) = \text{ a positive number,} \tag{10.11}$$

then $f(x)$ is said to be a *minimum* when $x = a$. Consider the following cases:

I Let $f'(a) \neq 0$. From (10.5), [§10.2], replacing b by x and transposing f(a),

$$f(x) - f(a) = (x-a)f'(x_1), \quad a < x_1 < x, \tag{10.12}$$

Since $f'(a) \neq 0$, and $f'(x)$ is assumed as continuous, h may be chosen so small that $f'(x)$ will have the same sign as $f'(a)$ for all values of x in the interval $[a-h, a+h]$. Therefore $f'(x_1)$ has the same sign as $f'(a)$ (Chap. 2). But $x - a$ changes sign according as x is less or greater than a. Therefore, from (10.12), the difference $f(x) - f(a)$ will also change sign, and, by

(10.10) and (10.11), $f(a)$ will be neither a maximum nor a minimum. This result agrees with the discussion in §7.4, where it was shown that for all values of x for which $f(x)$ is a maximum or a minimum, the first derivative $f'(x)$ must vanish.

II Let $f'(a)=0$, and $f''(a)\neq 0$. From (10.12), replacing b by x and transposing $f(a)$,

$$f(x)-f(a)=\frac{(x-a)^2}{2!}f''(x_2), \quad a<x_2<x. \tag{10.13}$$

Since $f''(a)\neq 0$, and $f''(x)$ is assumed as continuous, we may choose our interval $[a-h,a+h]$ so small that $f''(x_2)$ will have the same sign as $f''(a)$ (Chap. 2). Also $(x-a)^2$ does not change sign. Therefore the second member of (10.13) will not change sign, and the difference $f(x)-f(a)$ will have the same sign for all values of x in the interval $[a-h,a+h]$, and, moreover, this sign will be the same as the sign of $f''(a)$. It therefore follows from our definitions (10.10) and (10.11) that

$$f(a) \text{ is a maximum if } f'(a)=0 \text{ and } f''(a)= \text{ a negative number}; \tag{10.14}$$

$$f(a) \text{ is a minimum if } f'(a)=0 \text{ and } f''(a)= \text{ a positive number} \tag{10.15}$$

These conditions are the same as (7.3) and (7.4), [§7.6].

III Let $f'(a)=f''(a)=0$, and $f'''(a)\neq 0$. From (10.9), [§10.3], replacing b by x and transposing $f(a)$,

$$f(x)-f(a)=\frac{1}{3!}(x-a)^3 f'''(x_3), \quad a<x_3<x. \tag{10.16}$$

As before, $f'''(x_3)$ will have the same sign as $f'''(a)$. But $(x-a)^3$ changes its sign from $-$ to $+$ as x increases through a. Therefore the difference $f(x)-f(a)$ must change sign, and $f(a)$ is neither a maximum nor a minimum.

IV Let $f'(a)=f''(a)=\cdots=f^{(n-l)}(a)=0$, and $f^{(n)}(a)\neq 0$. By continuing the process as illustrated in I, II, and III, it is seen that if the first derivative of $f(x)$ which does not vanish for $x=a$ is of even order ($=n$), then[2]

[2]As in §7.4, a critical value $x=a$ is found by placing the first derivative equal to zero and solving the resulting equation for real roots.

$$f(a) \text{ is a maximum if } f^{(n)}(a) = \text{ a negative number;} \tag{10.17}$$

$$f(a) \text{ is a minimum if } f^{(n)}(a) = \text{ a positive number.} \tag{10.18}$$

If the first derivative of $f(x)$ which does not vanish for $x = a$ is of odd order, then $f(a)$ will be neither a maximum nor a minimum.

Example 10.5.1. Examine $x^3 - 9x^2 + 24x - 7$ for maximum and minimum values.
Solution. $f(x) = x^3 - 9x^2 + 24x - 7$. $f'(x) = 3x^2 - 18x + 24$. Solving $3x^2 - 18x + 24 = 0$ gives the critical values $x = 2$ and $x = 4$. Thus $f'(2) = 0$, and $f'(4) = 0$. Differentiating again, $f''(x) = 6x - 18$. Since $f''(2) = -6$, we know from (10.17) that $f(2) = 13$ is a maximum. Since $f''(4) = +6$, we know from (10.18) that $f(4) = 9$ is a minimum.

Example 10.5.2. Examine $e^x + 2\cos(x) + e^{-x}$ for maximum and minimum values.
Solution. $f(x) = e^x + 2\cos(x) + e^{-x}$, $f'(x) = e^x - 2\sin x - e^{-x} = 0$, for $x = 0$ (and $x = 0$ is the *only* root of the equation $e^x - 2\sin x - e^{-x} = 0$), $f''(x) = e^x - 2\cos(x) + e^{-x} = 0$, for $x = 0$, $f'''(x) = e^x + 2\sin x - e^{-x} = 0$, for $x = 0$, $f^{(4)}(x) = e^x + 2\cos(x) + e^{-x} = 4$, for $x = 0$. Hence from (10.18), $f(0) = 4$ is a minimum.

10.6 Exercises

Examine the following functions for maximum and minimum values, using the method of the last section.

1. $3x^4 - 4x^3 + 1$.

   Ans. $x = 1$ gives min. $= 0$; $x = 0$ gives neither.

2. $x^3 - 6x^2 + 12x + 48$.

   Ans. $x = 2$ gives neither.

3. $(x - 1)^2(x + 1)^3$.

   Ans. $x = 1$ gives min. $= 0$; $x = \frac{1}{5}$ gives max.; $x = -1$ gives neither.

4. Investigate $x^6 - 5x^4 + 5x^3 - 1$, at $x = 1$ and $x = 3$.

5. Investigate $x^3 - 3x^2 + 3x + 7$, at $x = 1$.

6. Show that if the first derivative of $f(x)$ which does not vanish for $x = a$ is of odd order ($= n$), then $f(x)$ is an increasing or decreasing function when $x = a$, according as $f^{(n)}(a)$ is positive or negative.

10.7 Indeterminate forms

Some singularities are easy to diagnose. Consider the function $\frac{\cos x}{x}$ at the point $x = 0$ (see Figure 10.4). The function evaluates to $\frac{1}{0}$ and is thus discontinuous at that point. Since the numerator and denominator are continuous functions and the denominator vanishes while the numerator does not, the left and right limits as $x \to 0$ do not exist. Thus the function has an infinite discontinuity at the point $x = 0$.

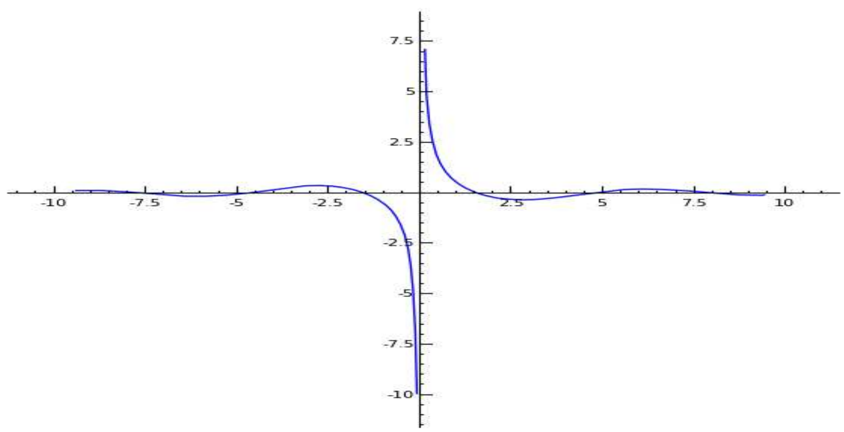

Figure 10.4: $\frac{\cos(x)}{x}$.

More generally, a function which is composed of continuous functions and evaluates to $\frac{a}{0}$ at a point where $a \neq 0$ must have an infinite discontinuity there.

Other singularities require more analysis to diagnose. Consider the functions $\frac{\sin x}{x}$, $\frac{\sin x}{|x|}$ and $\frac{\sin x}{1-\cos x}$ at the point $x = 0$. All three functions evaluate to $\frac{0}{0}$ at that point, but have different kinds of singularities. The first has a removable discontinuity, the second has a finite discontinuity and the third has an infinite discontinuity. See Figure 10.5.

An expression that evaluates (for a particular value of the independent variable) to $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 1^∞, 0^0 or ∞^0 is called an *indeterminate form*. A function

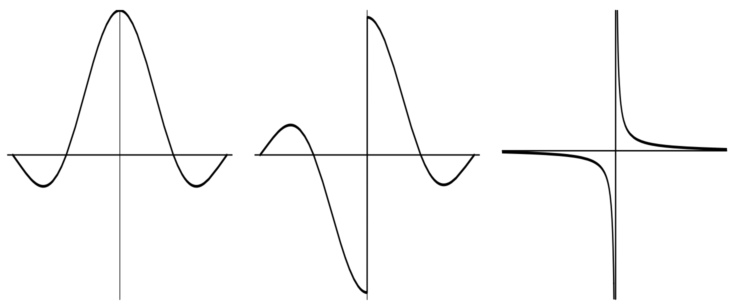

Figure 10.5: The functions $\frac{\sin x}{x}$, $\frac{\sin x}{|x|}$, $\frac{\sin x}{1-\cos x}$, resp..

$h(x)$ which takes an indeterminate form at $x = a$ is not defined for $x = a$ by the given analytical expression. For example, suppose we have

$$h(x) = \frac{f(x)}{g(x)},$$

where at $x = a$,

$$f(a) = 0, \text{ and } g(a) = 0.$$

For this value of x our function is not defined and we may therefore assign to it any value we please. It is usually desirable to assign to the function a value that will make it continuous when $x = a$ whenever it is possible to do so. L'Hôpital's rule, given in (10.19) below, helps us determine this value of $h(a)$ which makes h continuous at $x = a$.

10.8 Evaluation of a function taking on an indeterminate form

If when $x = a$ the function $f(x)$ assumes an indeterminate form, then

$$\lim_{x=a} f(x)$$

is taken as the value of $f(x)$ for $x = a$. The calculation of this limiting value is called *evaluating the indeterminate form*.

The assumption of this limiting value makes $f(x)$ continuous for $x = a$. This agrees with the theorem under Case II [§2.6], and also with our practice in Chapter 2, where several functions assuming the indeterminate form $\frac{0}{0}$ were evaluated.

Example 10.8.1. For $x = 2$ the function $\frac{x^2-4}{x-2}$ assumes the form $\frac{0}{0}$ but

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$$

Hence 4 is taken as the value of the function for $x = 2$. Let us now illustrate graphically the fact that if we assume 4 as the value of the function for $x = 2$, then the function is continuous for $x = 2$. Let $y = \frac{x^2-4}{x-2}$ This equation may also be written in the form $y(x - 2) = (x - 2)(x + 2)$; or $(x - 2)(y - x - 2) = 0$. Placing each factor separately equal to zero, we have $x = 2$, and $y = x + 2$. Also, when $x = 2$, we get $y = 4$.

In plotting, the loci of these equations are found to be two lines. Since there are infinitely many points on a line, it is clear that when $x = 2$, the value of y (or the function) may be taken as any number whatever. When x is different from 2, it is seen from the graph of the function that the corresponding value of y (or the function) is always found from $y = x + 2$, which we saw was also the limiting value of y (or the function) for $x = 2$. It is evident from geometrical considerations that if we assume 4 as the value of the function for $x = 2$, then the function is continuous for $x = 2$.

Similarly, several of the examples given in Chapter 2 illustrate how the limiting values of many functions assuming indeterminate forms may be found by employing suitable algebraic or trigonometric transformations, and how in general these limiting values make the corresponding functions continuous at the points in question. The most general methods, however, for evaluating indeterminate forms depend on differentiation.

10.9 Evaluation of the indeterminate form $\frac{0}{0}$

Given a function of the form $\frac{f(x)}{g(x)}$ such that $f(a) = 0$ and $g(a) = 0$; that is, the function takes on the indeterminate form $\frac{0}{0}$ when a is substituted for x. It is then required to find

$$\lim_{x \to a} \frac{f(x)}{g(x)}.$$

(See Figure 10.6.) Since, by hypothesis, $f(a) = 0$ and $g(a) = 0$, these graphs intersect at $(a, 0)$.

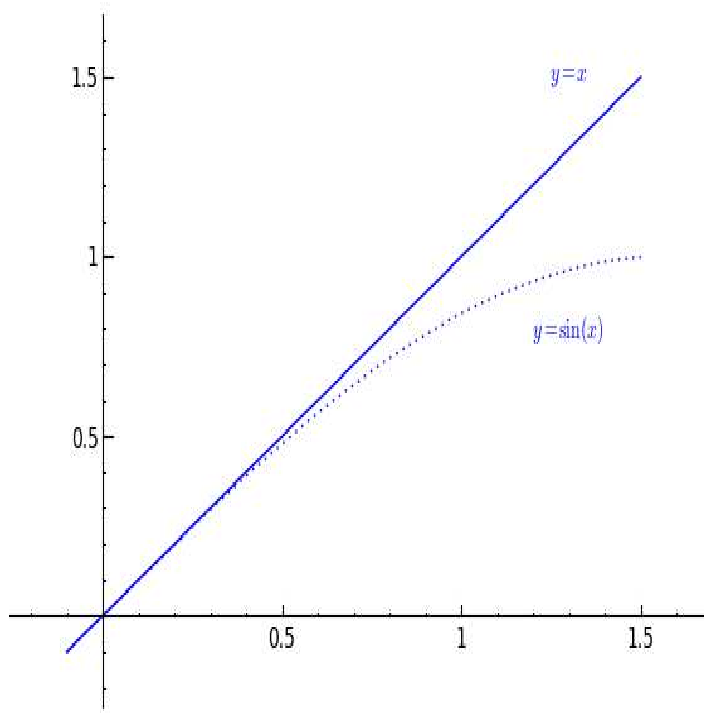

Figure 10.6: The graphs of the functions $f(x) = x$ and $g(x) = \sin(x)$.

Applying the mean value theorem to each of these functions (replacing b by x), we get $f(x) = f(a)+(x-a)f'(x_1)$, $a < x_l < x$, and $g(x) = g(a)+(x-a)g'(x_2)$. $a < x_2 < x$. Since $f(a) = 0$ and $g(a) = 0$, we get, after canceling out $(x - a)$,

$$\frac{f(x)}{g(x)} = \frac{f'(x_1)}{g'(x_2)}.$$

Now let $x \longrightarrow a$; then $x_l \longrightarrow a$, $x_2 \longrightarrow a$, and $\lim_{x\to a} f'(x_1) = f'(a), \lim_{x\to a} g'(x_2) = g'(a)$. Therefore,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}, \tag{10.19}$$

provided $g'(a) \neq 0$. This is a special case of the so-called

L'Hospital's Rule[3]: Let $f(x)$ and $g(x)$ be differentiable and $f(a) = g(a) = 0$. Further, let $g(x)$ be nonzero in a punctured neighborhood of $x = a$, (for some small δ, $g(x) \neq 0$ for $x \in \{0 < |x - a| < \delta\}$). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

The rule is named after the 17th-century Frenchman Guillaume de l'Hospital, who published the rule in his book **l'Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes** (translation: Analysis of the infinitely small to understand curves), the first book about differential calculus. This book was first published in the late 1600's and basically consisted of the lectures of his teacher Johann Bernoulli. In particular, this rule is, in fact, due to Johann Bernoulli (1667 - 1748).

Example 10.9.1. Consider the three functions $\frac{\sin x}{x}$, $\frac{\sin x}{|x|}$ and $\frac{\sin x}{1-\cos x}$ at the point $x = 0$.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

Thus $\frac{\sin x}{x}$ has a removable discontinuity at $x = 0$.

$$\lim_{x\to 0^+} \frac{\sin x}{|x|} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0^-} \frac{\sin x}{|x|} = \lim_{x\to 0^-} \frac{\sin x}{-x} = -1$$

Thus $\frac{\sin x}{|x|}$ has a finite discontinuity at $x = 0$.

$$\lim_{x\to 0} \frac{\sin x}{1-\cos x} = \lim_{x\to 0} \frac{\cos x}{\sin x} = \frac{1}{0} = \infty$$

Thus $\frac{\sin x}{1-\cos x}$ has an infinite discontinuity at $x = 0$.

Example 10.9.2. We use Sage to compute $\lim_{x\to 0} \frac{\cos(x)-1}{x^2}$.

Sage

```
sage: limit((cos(x)-1)/x^2,x=0)
-1/2
sage: limit((-sin(x))/(2*x),x=0)
```

[3] Also written L'Hôpital and pronounced "low-peh-tall".

```
-1/2
sage: limit((-cos(x))/(2),x=0)
-1/2
```

This verifies

$$\lim_{x\to 0} \frac{\cos(x)-1}{x^2} = \lim_{x\to 0} \frac{-\sin(x)}{2x} = \lim_{x\to 0} \frac{-\cos(x)}{2} = -1/2.$$

Example 10.9.3. Let a and d be nonzero.

$$\begin{aligned} \lim_{x\to\infty} \frac{ax^2+bx+c}{dx^2+ex+f} &= \lim_{x\to\infty} \frac{2ax+b}{2dx+e} \\ &= \lim_{x\to\infty} \frac{2a}{2d} \\ &= \frac{a}{d} \end{aligned}$$

Example 10.9.4. Consider

$$\lim_{x\to 0} \frac{\cos x - 1}{x \sin x}.$$

This limit is an indeterminate of the form $\frac{0}{0}$. Applying L'Hospital's rule we see that limit is equal to

$$\lim_{x\to 0} \frac{-\sin x}{x\cos x + \sin x}.$$

This limit is again an indeterminate of the form $\frac{0}{0}$. We apply L'Hospital's rule again.

$$\lim_{x\to 0} \frac{-\cos x}{-x\sin x + 2\cos x} = -\frac{1}{2}$$

Thus the value of the original limit is $-\frac{1}{2}$. We could also obtain this result by expanding the functions in Taylor series.

$$\begin{aligned}
\lim_{x\to 0}\frac{\cos x - 1}{x\sin x} &= \lim_{x\to 0}\frac{\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots\right)-1}{x\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\right)}\\
&= \lim_{x\to 0}\frac{-\frac{x^2}{2}+\frac{x^4}{24}-\cdots}{x^2-\frac{x^4}{6}+\frac{x^6}{120}-\cdots}\\
&= \lim_{x\to 0}\frac{-\frac{1}{2}+\frac{x^2}{24}-\cdots}{1-\frac{x^2}{6}+\frac{x^4}{120}-\cdots}\\
&= -\frac{1}{2}
\end{aligned}$$

Example 10.9.5. We use Sage to compute $\lim_{x\to 0}\frac{\cos(x)-1}{x^2}$.

Sage

```
sage: limit((cos(x)-1)/x^2,x=0)
-1/2
sage: limit((-sin(x))/(2*x),x=0)
-1/2
sage: limit((-cos(x))/(2),x=0)
-1/2
```

This verifies

$$\lim_{x\to 0}\frac{\cos(x)-1}{x^2} = \lim_{x\to 0}\frac{-\sin(x)}{2x} = \lim_{x\to 0}\frac{-\cos(x)}{2} = -1/2.$$

10.9.1 Rule for evaluating the indeterminate form $\frac{0}{0}$

Differentiate the numerator for a new numerator and differentiate the denominator for a new denominator[4]. The value of this new fraction for the assigned value[5] of the variable will be the limiting value of the original fraction.

[4]A warning to the student: don't make the mistake of differentiating the whole expression as a fraction using the quotient rule!

[5]If $a = \inf$, the substitution $x = \frac{1}{z}$ reduces the problem to the evaluation of the limit for $z = 0$. Thus $\lim_{x\to\inf}\frac{f(x)}{g(x)} = \lim_{z\to 0}\frac{-f'\left(\frac{1}{z}\right)\frac{1}{z^2}}{-g'\left(\frac{1}{z}\right)\frac{1}{z^2}} = \lim_{z\to 0}\frac{f'\left(\frac{1}{z}\right)}{g'\left(\frac{1}{z}\right)} = \lim_{x\to\inf}\frac{f'(x)}{g'(x)}$. Therefore the rule holds in this case also.

In case it so happens that $f'(a) = 0$ and $g'(a) = 0$, that is, the first derivatives also vanish for $x = a$, then we still have the indeterminate form $\frac{0}{0}$, and the theorem can be applied anew to the ratio $\frac{f'(x)}{g'(x)}$ giving us $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f''(a)}{g''(a)}$. When also $f''(a) = 0$ and $g''(a) = 0$, we get in the same manner $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'''(a)}{g'''(a)}$, and so on.

It may be necessary to repeat this process several times.

Example 10.9.6. Evaluate $\frac{f(x)}{g(x)} = \frac{x^3-3x+2}{x^3-x^2-x-1}$ when $x = 1$.
Solution.

$$\frac{f(1)}{g(1)} = \left.\frac{x^3 - 3x + 2}{x^3 - x^2 + 1}\right]_{x=1} = \frac{1 - 3 + 2}{1 - 1 - 1 + 1} = \frac{0}{0}.$$

Therefore, this is an indeterminate form.

$$\frac{f'(1)}{g'(1)} = \left.\frac{3x^2 - 3}{3x^2 - 2x - 1}\right]_{x=1} = \frac{3 - 3}{3 - 2 - 1} = \frac{0}{0}.$$

Therefore, this is an indeterminate form.

$$\frac{f''(1)}{g''(1)} = \left.\frac{6x}{6x - 2}\right]_{x=1} = \frac{6}{6 - 2} = \frac{3}{2}. \text{ Ans.}$$

Example 10.9.7. Evaluate $\lim_{x\to 0} \frac{e^x-e^{-x}-2x}{x-\sin x}$.
Solution.

$$\frac{f(0)}{g(0} = \left.\frac{e^x - e^{-x} - 2x}{x - \sin x}\right]_{x=0} = \frac{1 - 1 - 0}{0 - 0} = \frac{0}{0}.$$

Therefore, this is an indeterminate form.

$$\frac{f'(0)}{g'(0)} = \left.\frac{e^x - e^{-x} - 2}{1 - \cos x}\right]_{x=0} = \frac{1 + 1 - 2}{1 - 1} = \frac{0}{0}.$$

Therefore, this is an indeterminate form.

$$\frac{f''(0)}{g''(0)} = \left.\frac{e^x - e^{-x}}{\sin x}\right]_{x=0} = \frac{1 - 1}{0} = \frac{0}{0}.$$

Therefore, this is an indeterminate form.

$$\frac{f'''(0)}{g'''(0)} = \left.\frac{e^x - e^{-x}}{\cos x}\right]_{x=0} = \frac{1 + 1}{1} = 2. \text{ Ans.}$$

10.9.2 Exercises

Evaluate the following by differentiation[6].

1. $\lim_{x\to 4} \frac{x^2-16}{x^2+x-20}$.

   Ans. $\frac{8}{9}$.

2. $\lim_{x\to 1} \frac{x-1}{x^n-1}$.

   Ans. $\frac{1}{n}$.

3. $\lim_{x\to 1} \frac{\log x}{x-1}$.

   Ans. 1.

4. $\lim_{x\to 0} \frac{e^x-e^{-x}}{\sin x}$.

   Ans. 2.

5. $\lim_{x\to 0} \frac{\tan x-x}{x-\sin x}$.

   Ans. 2.

6. $\lim_{x\to \frac{\pi}{2}} \frac{\log \sin x}{(\pi-2x)^2}$.

   Ans. $-\frac{1}{8}$.

7. $\lim_{x\to 0} \frac{a^x-b^x}{x}$.

   Ans. $\log \frac{a}{b}$.

8. $\lim_{r\to a} \frac{r^3-ar^2-a^2r+a^3}{r^2-a^2}$.

   Ans. 0.

9. $\lim_{\theta\to 0} \frac{\theta-\arcsin \theta}{\sin^3 \theta}$.

   Ans. $-\frac{1}{6}$.

10. $\lim_{x\to \phi} \frac{\sin x-\sin \phi}{x-\phi}$.

    Ans. $\cos \phi$.

[6]After differentiating, the student should in every case reduce the resulting expression to its simplest possible form before substituting the value of the variable.

11. $\lim_{y\to 0} \frac{e^y+\sin y-1}{\log(1+y)}$.

    Ans. 2.

12. $\lim_{\theta\to 0} \frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$.

    Ans. 1.

13. $\lim_{\phi\to\frac{\pi}{4}} \frac{\sec^2\phi-2\tan\phi}{1+\cos 4\phi}$.

    Ans. $\frac{1}{2}$.

14. $\lim_{z\to a} \frac{az-z^2}{a^4-2a^3z+2az^3-z^4}$.

    Ans. $+\infty$.

15. $\lim_{x\to 2} \frac{(e^x-e^2)^2}{(x-4)e^x+e^2x}$.

    Ans. $6e^4$.

16. $\lim_{x\to 1} \frac{x^2+x-2}{x^2-1}$.

17. $\lim_{x\to -2} \frac{x^3+8}{x^5+32}$.

18. $\lim_{x\to 0} \frac{\sin 2x}{x}$.

19. $\lim_{x\to 0} \frac{x-\sin x}{x^3}$.

20. $\lim_{x\to 1} \frac{\log\cos(x-1)}{1-\sin\frac{\pi x}{2}}$.

21. $\lim_{x\to 0} \frac{\tan x-\sin x}{\sin^3 x}$.

10.10 Evaluation of the indeterminate form $\frac{\infty}{\infty}$

In order to compute

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

when $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, that is, when for $x = a$ the function $\frac{f(x)}{g(x)}$ assumes the indeterminate form $\frac{\infty}{\infty}$, we follow the same rule as that given in §10.9 for evaluating the indeterminate form $\frac{0}{0}$.

Rule for evaluating the indeterminate form $\frac{\infty}{\infty}$: Differentiate the numerator for a new numerator and the denominator for a new denominator. The value of this new fraction for the assigned value of the variable will be the limiting value of the original fraction.

A rigorous proof of this rule is beyond the scope of this book and is left for more advanced treatises.

Example 10.10.1. Evaluate $\frac{\log x}{\csc x}$ for $x = 0$.

Solution.

$$\frac{f(0)}{g(0)} = \left.\frac{\log(x)}{\csc(x)}\right]_{x=0} = \frac{-\infty}{\infty}.$$

Therefore, this is an indeterminate form.

$$\frac{f'(0)}{g'(0)} = \left.\frac{\frac{1}{x}}{-\csc x \cot x}\right]_{x=0} = -\left.\frac{\sin^2 x}{x \cos x}\right]_{x=0} = \frac{0}{0}.$$

Therefore, this is an indeterminate form.

$$\frac{f''(0)}{g''(0)} = -\left.\frac{2 \sin x \cos x}{\cos x - x \sin x}\right]_{x=0} = -\frac{0}{1} = 0. \text{ Ans.}$$

Example 10.10.2. Let a and d be nonzero.

$$\begin{aligned} \lim_{x\to\infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} &= \lim_{x\to\infty} \frac{2ax + b}{2dx + e} \\ &= \lim_{x\to\infty} \frac{2a}{2d} \\ &= \frac{a}{d} \end{aligned}$$

10.11 Evaluation of the indeterminate form $0 \cdot \infty$

If a function $f(x) \cdot \phi(x)$ takes on the indeterminate form $0 \cdot \infty$ for $x = a$, we write the given function

$$f(x) \cdot \phi(x) = \frac{f(x)}{\frac{1}{\phi(x)}} \left(\text{or} = \frac{\phi(x)}{\frac{1}{f(x)}}\right)$$

so as to cause it to take on one of the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, thus bringing it under §10.9 or §10.10.

Example 10.11.1. Evaluate $\sec(3x)\cos(5x)$ for $x = \frac{\pi}{2}$.
Solution. $\sec 3x \cos 5x]_{x=\frac{\pi}{2}} = \infty \cdot 0$. Therefore, this is an indeterminate form. Substituting $\frac{1}{\cos 3x}$ for $\sec 3x$, the function becomes $\frac{\cos 5x}{\cos 3x} = \frac{f(x)}{g(x)}$.

$$\frac{f(\frac{\pi}{2})}{g(\frac{\pi}{2})} = \frac{\cos 5x}{\cos 3x}\Big]_{x=\frac{\pi}{s}} = \frac{0}{0}.$$

Therefore, this is an indeterminate form.

$$\frac{f'(\frac{\pi}{2})}{g'(\frac{\pi}{2})} = \frac{-\cos x}{-\sin x}\Big]_{x=\frac{\pi}{s}} = \frac{0}{-1} = 0. \text{ Ans.}$$

Example 10.11.2.

$$\begin{aligned}
\lim_{x\to 0}\left(\cot x - \frac{1}{x}\right) &= \lim_{x\to 0}\frac{x\cos x - \sin x}{x\sin x} \\
&= \lim_{x\to 0}\frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x} \\
&= \lim_{x\to 0}\frac{-x\sin x}{\sin x + x\cos x} \\
&= \lim_{x\to 0}\frac{-x\cos x - \sin x}{\cos x + \cos x - x\sin x} \\
&= 0
\end{aligned}$$

Here is the Sage command for this example:

Sage

```
sage: limit(cot(x)-1/x,x=0)
0
sage: limit((- x*cos(x) - sin(x) )/(cos(x) + cos(x) - x*sin(x)),x=0)
0
```

This verifies the answer obtained "by hand" above.

10.12 Evaluation of the indeterminate form $\infty - \infty$

It is possible in general to transform the expression into a fraction which will assume either the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 10.12.1. Evaluate $\sec x - \tan x$ for $x = \frac{\pi}{2}$.

Solution. $\sec x - \tan x\big]_{x=\frac{\pi}{2}} = \infty - \infty$. Therefore, this is an indeterminate form. By Trigonometry,

$\sec x - \tan x = \frac{1}{\cos x} - \frac{\sin x}{\cos x} = \frac{1-\sin x}{\cos x} = \frac{f(x)}{g(x)}$.

$\frac{f(\frac{\pi}{2})}{g(\frac{\pi}{2})} = \frac{1-\sin x}{\cos x}\big]_{x=\frac{\pi}{2}} = \frac{1-1}{0} = \frac{0}{0}$. Therefore, this is an indeterminate form.

$\frac{f'(\frac{\pi}{2})}{g'(\frac{\pi}{2})} = \frac{-\cos x}{-\sin x}\big]_{x=\frac{\pi}{2}} = \frac{0}{-1} = 0$. Ans.

10.12.1 Exercises

Evaluate the following by differentiation[7].

1. $\lim_{x\to\infty} \frac{ax^2+b}{cx^2+d}$.

   Ans. $\frac{a}{c}$.

2. $\lim_{x\to 0} \frac{\cot x}{\log x}$.

   Ans. $-\infty$.

3. $\lim_{x=\infty} \frac{\log x}{x^n}$.

   Ans. 0.

4. $\lim_{x\to\infty} \frac{x^2}{e^x}$.

   Ans. 0.

5. $\lim_{x\to\infty} \frac{e^x}{\log x}$.

   Ans. ∞.

6. $\lim_{x\to 0} x \cot \pi x$.

   Ans. $\frac{1}{\pi}$.

7. $\lim_{y\to\infty} \frac{y}{e^{ay}}$.

   Ans. 0.

8. $\lim_{x\to\frac{\pi}{2}} (\pi - 2x)\tan x$.

   Ans. 2.

[7] In solving the remaining exercises in this chapter, the formulas in §2.12 may be useful, where many special limits are evaluated.

9. $\lim_{x\to\infty} x \sin \frac{a}{x}$.

   Ans. a.

10. $\lim_{x\to 0} x^n \log x$. [n positive.]

    Ans. 0.

11. $\lim_{\theta\to\frac{\pi}{4}} (1 - \tan\theta)\sec 2\theta$.

    Ans. 1.

12. $\lim_{\phi\to a}(a^2 - \phi^2)\tan\frac{\pi\phi}{2a}$.

    Ans. $\frac{4a^2}{\pi}$.

13. $\lim_{x\to 0} \frac{\log\sin 2x}{\log\sin x}$.

    Ans. 1.

14. $\lim_{\theta\to\frac{\pi}{2}} \frac{\tan\theta}{\tan 3\theta}$.

    Ans. 3.

15. $\lim_{\phi\to\frac{\pi}{2}} \frac{\log\left(\phi-\frac{\pi}{2}\right)}{\tan\phi}$.

    Ans. 0.

16. $\lim_{x\to 0} \frac{\log x}{\cot x}$.

    Ans. 0.

17. $\lim_{x\to 0} x \log\sin x$.

    Ans. 0.

18. $\lim_{x\to 1} \left[\frac{2}{x^2-1} - \frac{1}{x-1}\right]$.

    Ans. $-\frac{1}{2}$.

    Sage

```
sage: limit(2/(x^2-1) - 1/(x-1),x=1)
-1/2
```

19. $\lim_{x\to 1} \left[\frac{1}{\log x} - \frac{x}{\log x}\right]$.

    Ans. -1.

Sage

```
sage: limit(1/log(x) - x/log(x),x=1)
-1
```

20. $\lim_{\theta\to\frac{\pi}{2}}[\sec\theta - \tan\theta]$.

    Ans. 0.

21. $\lim_{\phi\to 0}\left[\frac{2}{\sin^2\phi} - \frac{1}{1-\cos\phi}\right]$.

    Ans. $\frac{1}{2}$.

22. $\lim_{y\to 1}\left[\frac{y}{y-1} - \frac{1}{\log y}\right]$.

    Ans. $\frac{1}{2}$.

23. $\lim_{z\to 0}\left[\frac{\pi}{4z} - \frac{\pi}{2z(e^{\pi z}+1)}\right]$.

    Ans. $\frac{\pi^2}{8}$.

10.13 Evaluation of the indeterminate forms 0^0, 1^∞, ∞^0

Given a function of the form

$$f(x)^{\phi(x)}.$$

In order that the function shall take on one of the above three forms, we must have for a certain value of x, $f(x)=0$, $\phi(x)=0$, giving 0^0; or, $f(x)=1$, $\phi(x)=\infty$, giving 1^∞; or, $f(x)=\infty$, $\phi(x)=0$, giving ∞^0. Let $y=f(x)^{\phi(x)}$; taking the logarithm of both sides gives us, $\log y = \phi(x)\log f(x)$. In any of the above cases the logarithm of y (the function) will take on the indeterminate form $0\cdot\infty$.

Evaluating this by the process illustrated in §10.11 gives the limit of the logarithm of the function. This being equal to the logarithm of the limit of the function, the limit of the function is known.

Example 10.13.1. Evaluate x^x when $x = 0$.

Solution. This function assumes the indeterminate form 0^0 for $x = 0$. Let $y = x^x$; then $\log y = x \log x = 0 \cdot (-\infty)$, when $x = 0$. By §10.11,

$$\log y \frac{\log x}{\frac{1}{x}} = \frac{-\infty}{\infty},$$

when $x = 0$. By §10.10,

$$\log y \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -x = 0,$$

when $x = 0$. Since $y = x^x$, this gives $\log_e(x^x) = 0$; i.e., $x^x = 1$. Ans.

Example 10.13.2. Evaluate $(1+x)^{\frac{1}{x}}$ when $x = 0$.

Solution. This function assumes the indeterminate form 1^∞ for $x = 0$. Let $y = (1+x)^{\frac{1}{x}}$; then $\log y = \frac{1}{x}\log(1+x) = \infty \cdot 0$ when $x = 0$. By §10.11, $y = \frac{\log(1+x)}{x} = \frac{0}{0}$, when $x = 0$. By §10.9, $y = \frac{\frac{1}{1+x}}{1} = \frac{1}{1+x} = 1$ when $x = 0$. Since $y = (1+x)^{\frac{1}{1+x}}$, this gives $\log_e(1+x)^{\frac{1}{x}} = 1$; i.e. $(1+x)^{\frac{1}{x}} = e$. Ans.

Example 10.13.3. Evaluate $\cot x \sin x$ for $x = 0$.

Solution. This function assumes the indeterminate form ∞^0 for $x = 0$. Let $y = (\cot x)^{\sin x}$; then $\log y = \sin x \log \cot x = 0 \cdot \infty$ when $x = 0$. By §10.11, $\log y = \frac{\log \cot x}{\csc x} = \frac{\infty}{\infty}$ when $x = 0$. §10.10, $\log y = \frac{\frac{-\csc^2 x}{\cot x}}{-\csc x \cot x} = \frac{\sin x}{\cos^2 x} = 0$, when $x = 0$.

10.14 Exercises

Evaluate the following limits.

1. $\lim_{x\to 1} x^{\frac{1}{1-x}}$.

   Ans. $\frac{1}{e}$.

2. $\lim_{x\to 0} \left(\frac{1}{x}\right)^{\tan x}$.

   Ans. 1.

3. $\lim_{\theta\to\frac{\pi}{2}} (\sin\theta)^{\tan\theta}$.

   Ans. 1.

4. $\lim_{y\to\infty}\left(1+\frac{a}{y}\right)^{y}$.

   Ans. e^a.

5. $\lim_{x\to 0}(1+\sin x)^{\cot x}$.

   Ans. e.

6. $\lim_{x\to\infty}\left(\frac{2}{x}+1\right)^{x}$.

   Ans. e^2.

7. $\lim_{x\to 0}(e^x+x)^{\frac{1}{x}}$.

   Ans. e^2.

8. $\lim_{x\to 0}(\cot x)^{\frac{1}{\log x}}$.

   Ans. $\frac{1}{e}$.

9. $\lim_{z\to 0}(1+nz)^{\frac{1}{z}}$.

   Ans. e^n.

10. $\lim_{\phi\to 1}\left(\tan\frac{\pi\phi}{4}\right)^{\tan\frac{\pi\phi}{2}}$.

    Ans. $\frac{1}{e}$.

11. $\lim_{\theta\to 0}(\cos m\theta)^{\frac{n}{\theta^2}}$.

    Ans. $e^{-\frac{1}{2}nm^2}$.

12. $\lim_{x\to 0}(\cot x)^{x}$.

    Ans. 1.

13. $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$.

    Ans. $e^{\frac{2}{\pi}}$.

14. (a) $\lim_{x\to 0}\frac{x-\sin x}{x^3}$
    (b) $\lim_{x\to 0}\left(\csc x-\frac{1}{x}\right)$
    (c) $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}$
    (d) $\lim_{x\to 0}\left(\csc^2 x-\frac{1}{x^2}\right)$. (First evaluate using L'Hospital's rule then using a Taylor series expansion. You will find that the latter method is more convenient.)

15.
$$\lim_{x\to\infty} x^{a/x}, \qquad \lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^{bx},$$
where a and b are constants.

16. $\lim_{x\to 4} \frac{x^2-16}{x^2+x-20}$

Ans. $8/9$

17. $\lim_{x\to 1} \frac{x-1}{x^n-1}$.

Ans. $1/n$

18. $\lim_{x\to 1} \frac{\log x}{x-1}$.

Ans. 1

19. $\lim_{x\to 0} \frac{e^x-e^{-x}}{\sin(x)}$

Ans. 2

20. $\lim_{x\to \pi/2} \frac{\log\,\sin(x)}{(\pi-2x)^2}$

Ans. $-1/8$

21. $\lim_{x\to 0} \frac{a^x-b^x}{x}$

Ans. $\log(a/b)$

22. $\lim_{x\to 0} \frac{\theta-\arcsin(\theta)}{\theta^2}$

Ans. $-1/6$.

23. $\lim_{x\to \phi} \frac{\sin(x)-\sin(\phi)}{x-\phi}$.

Ans. $\cos(\phi)$.

10.15 Application: Using Taylor's Theorem to approximate functions.

We revisit the ideas in §10.3 above to present some applications.

Theorem 10.15.1. *(***Taylor's Theorem***)* If $f(x)$ is $n+1$ times continuously differentiable in (a,b) then there exists a point $x=\xi\in(a,b)$ such that

$$\begin{aligned} f(b) &= f(a)+(b-a)f'(a)+\tfrac{(b-a)^2}{2!}f''(a)+\cdots \\ &+\tfrac{(b-a)^n}{n!}f^{(n)}(a)+\tfrac{(b-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi). \end{aligned} \tag{10.20}$$

For the case $n=0$, the formula is

$$f(b)=f(a)+(b-a)f'(\xi),$$

which is just a rearrangement of the terms in the theorem of the mean,

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}.$$

One can use Taylor's theorem to approximate functions with polynomials. Consider an infinitely differentiable function $f(x)$ and a point $x=a$. Substituting x for b into Equation 10.20 we obtain,

$$\begin{aligned} f(x) &= f(a)+(x-a)f'(a)+\tfrac{(x-a)^2}{2!}f''(a)+\cdots+\tfrac{(x-a)^n}{n!}f^{(n)}(a) \\ &+\tfrac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi). \end{aligned} \tag{10.21}$$

If the last term in the sum is small then we can approximate our function with an n^{th} order polynomial.

$$f(x)\approx f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots+\frac{(x-a)^n}{n!}f^{(n)}(a)$$

The last term in Equation 10.21 is called the remainder or the error term,

$$R_n=\frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi).$$

Since the function is infinitely differentiable, $f^{(n+1)}(\xi)$ exists and is bounded. Therefore we note that the error must vanish as $x\to0$ because of the $(x-a)^{n+1}$ factor. We therefore suspect that our approximation would be a good one if x is close to a. Also note that $n!$ eventually grows faster than $(x-a)^n$,

$$\lim_{n\to\infty}\frac{(x-a)^n}{n!}=0.$$

So if the derivative term, $f^{(n+1)}(\xi)$, does not grow to quickly, the error for a certain value of x will get smaller with increasing n and the polynomial will become a better approximation of the function. (It is also possible that the derivative factor grows very quickly and the approximation gets worse with increasing n.)

Example 10.15.1. Consider the function $f(x) = e^x$. We want a polynomial approximation of this function near the point $x = 0$. Since the derivative of e^x is e^x, the value of all the derivatives at $x = 0$ is $f^{(n)}(0) = e^0 = 1$. Taylor's theorem thus states that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^{\xi},$$

for some $\xi \in (0, x)$. The first few polynomial approximations of the exponent about the point $x = 0$ are

$$\begin{aligned} f_1(x) &= 1 \\ f_2(x) &= 1 + x \\ f_3(x) &= 1 + x + \frac{x^2}{2} \\ f_4(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \end{aligned}$$

The four approximations are graphed in Figure 10.7.

Note that for the range of x we are looking at, the approximations become more accurate as the number of terms increases.

Here is one way to compute these approximations using Sage :

Sage

```
sage: x = var("x")
sage: y = exp(x)
sage: a = lambda n: diff(y,x,n)(0)/factorial(n)
sage: a(0)
1
sage: a(1)
1
sage: a(2)
1/2
sage: a(3)
1/6
sage: taylor = lambda n: sum([a(i)*x^i for i in range(n)])
sage: taylor(2)
x + 1
```

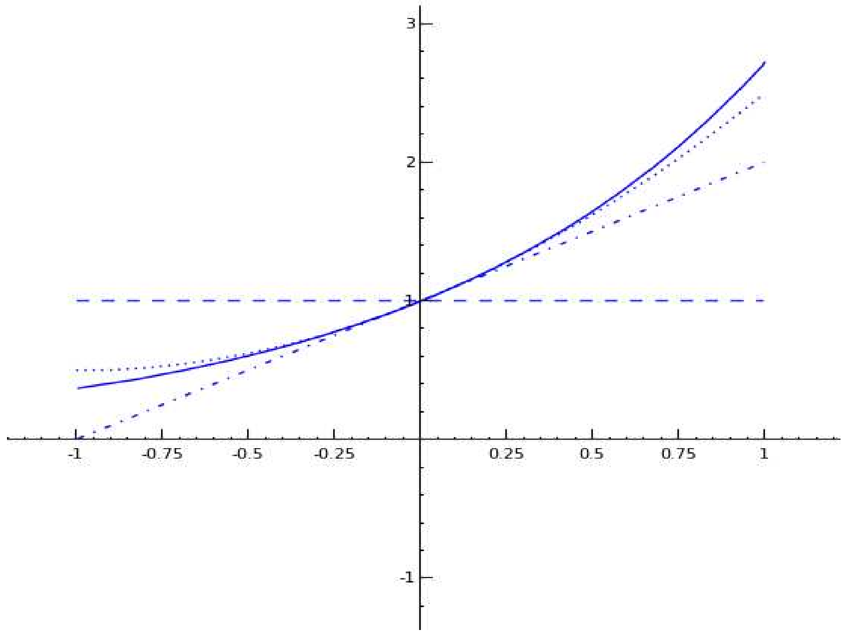

Figure 10.7: Finite Taylor Series Approximations of 1, $1+x$, $1+x+\frac{x^2}{2}$ to e^x.

```
sage: taylor(3)
x^2/2 + x + 1
sage: taylor(4)
x^3/6 + x^2/2 + x + 1
```

Example 10.15.2. Consider the function $f(x) = \cos x$. We want a polynomial approximation of this function near the point $x = 0$. The first few derivatives of f are

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x \end{aligned}$$

It's easy to pick out the pattern here,

$$f^{(n)}(x) = \begin{cases} (-1)^{n/2} \cos x & \text{for even } n, \\ (-1)^{(n+1)/2} \sin x & \text{for odd } n. \end{cases}$$

Since $\cos(0) = 1$ and $\sin(0) = 0$ the n-term approximation of the cosine is,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^{2(n-1)} \frac{x^{2(n-1)}}{(2(n-1))!} + \frac{x^{2n}}{(2n)!} \cos \xi.$$

Here are graphs of the one-, two-, three- and four-term approximations.

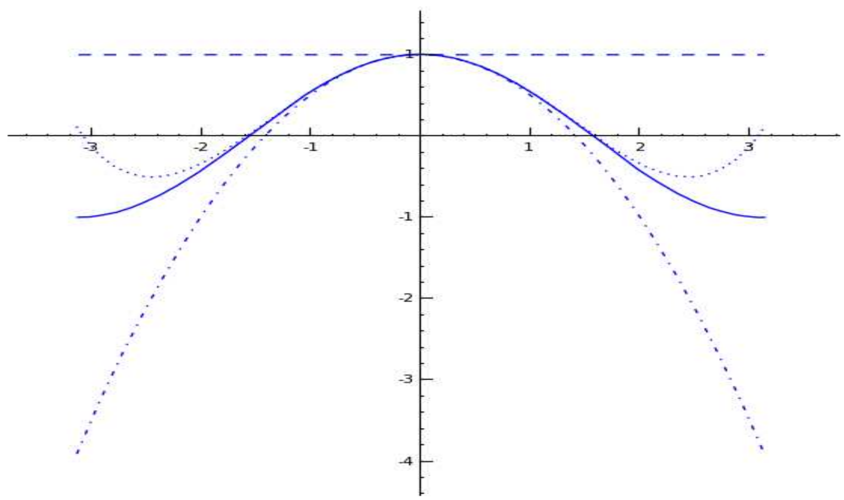

Figure 10.8: Taylor Series Approximations of 1, $1 - \frac{x^2}{2}$, $1 - \frac{x^2}{2} + \frac{x^4}{4!}$ to $\cos x$.

Note that for the range of x we are looking at, the approximations become more accurate as the number of terms increases. Consider the ten term approximation of the cosine about $x = 0$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots - \frac{x^{18}}{18!} + \frac{x^{20}}{20!} \cos \xi.$$

Note that for any value of ξ, $|\cos \xi| \leq 1$. Therefore the absolute value of the error term satisfies,

$$|R| = \left| \frac{x^{20}}{20!} \cos \xi \right| \leq \frac{|x|^{20}}{20!}.$$

Note that the error is very small for $x < 6$, fairly small but non-negligible for $x \approx 7$ and large for $x > 8$. The ten term approximation of the cosine, plotted in Figure 10.9, behaves just we would predict.

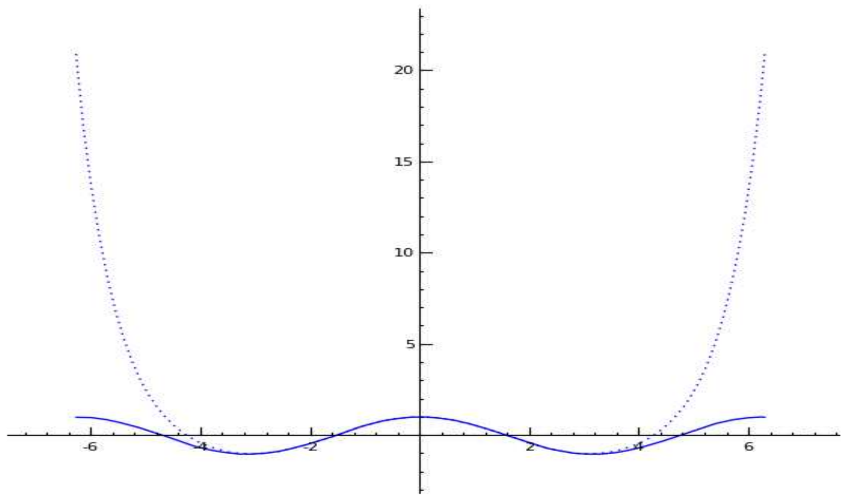

Figure 10.9: Taylor Series Approximation of $1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$ to $\cos x$.

The error is very small until it becomes non-negligible at $x \approx 7$ and large at $x \approx 8$.

Example 10.15.3. Consider the function $f(x) = \ln x$. We want a polynomial approximation of this function near the point $x = 1$. The first few derivatives of f are

$$\begin{aligned} f(x) &= \ln x \\ f'(x) &= \frac{1}{x} \\ f''(x) &= -\frac{1}{x^2} \\ f'''(x) &= \frac{2}{x^3} \\ f^{(4)}(x) &= -\frac{3}{x^4} \end{aligned}$$

The derivatives evaluated at $x = 1$ are

$$f(1) = 0, \qquad f^{(n)}(1) = (-1)^{n-1}(n-1)!, \quad \text{for } n \geq 1.$$

By Taylor's theorem of the mean we have,

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$
$$+ (-1)^{n-1}\frac{(x-1)^n}{n} + (-1)^n\frac{(x-1)^{n+1}}{n+1}\frac{1}{\xi^{n+1}}.$$

Figure 10.10 shows plots of the one-, two-, and three-term approximations.

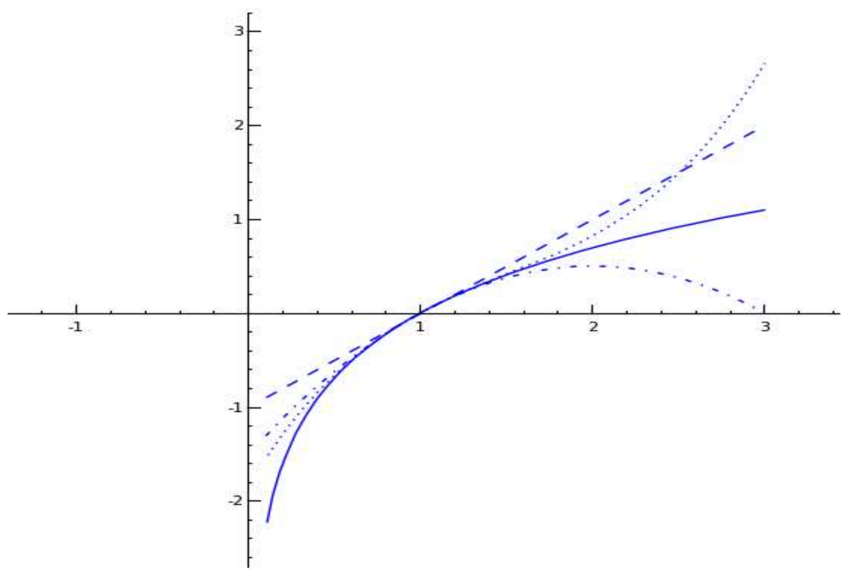

Figure 10.10: Taylor series (about $x = 1$) approximations of $x - 1$, $x - 1 - \frac{(x-1)^2}{2}$, $x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}$ to $\ln x$.

Note that the approximation gets better on the interval $(0, 2)$ and worse outside this interval as the number of terms increases. The Taylor series converges to $\ln x$ only on this interval.

10.16 Example/Application: finite difference methods

This is less of an application of derivatives themselves and more of an explanation of one technique used to numerically approximate derivatives in a computer. Since this is a bit of an advanced topic, rather than explain the theory, we shall just give a detailed example which contains the main ideas.

Example 10.16.1. Suppose you sample a function at the discrete points $n\Delta x$, $n \in \mathbb{Z}$. In Figure 10.11 we sample the function $f(x) = \sin x$ on the interval $[-4, 4]$ with $\Delta x = 1/4$ and plot the data points.

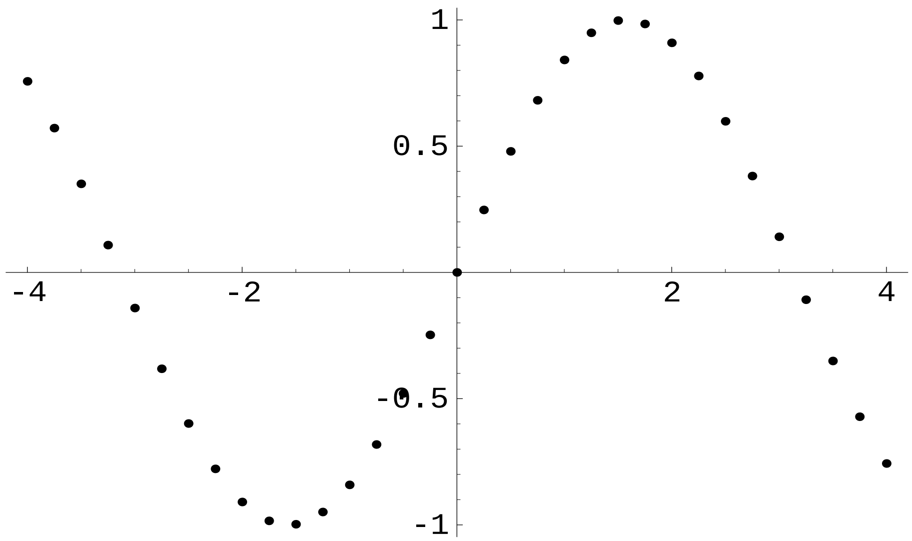

Figure 10.11: Sine function sampling.

We wish to approximate the derivative of the function on the grid points using only the value of the function on those discrete points. From the definition of the derivative, one is lead to the formula

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{10.22}$$

Taylor's theorem states that

$$f(x + \Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(\xi).$$

Substituting this expression into our formula for approximating the derivative we obtain

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(\xi) - f(x)}{\Delta x} = f'(x) + \frac{\Delta x}{2} f''(\xi).$$

Thus we see that the error in our approximation of the first derivative is $\frac{\Delta x}{2} f''(\xi)$. Since the error has a linear factor of Δx, we call this a first order accurate method. Equation 10.22 is called the *forward difference scheme* for calculating the first derivative. Figure 10.12 shows a plot of the value of this scheme for the function $f(x) = \sin x$ and $\Delta x = 1/4$. The first derivative of the function $f'(x) = \cos x$ is shown for comparison.

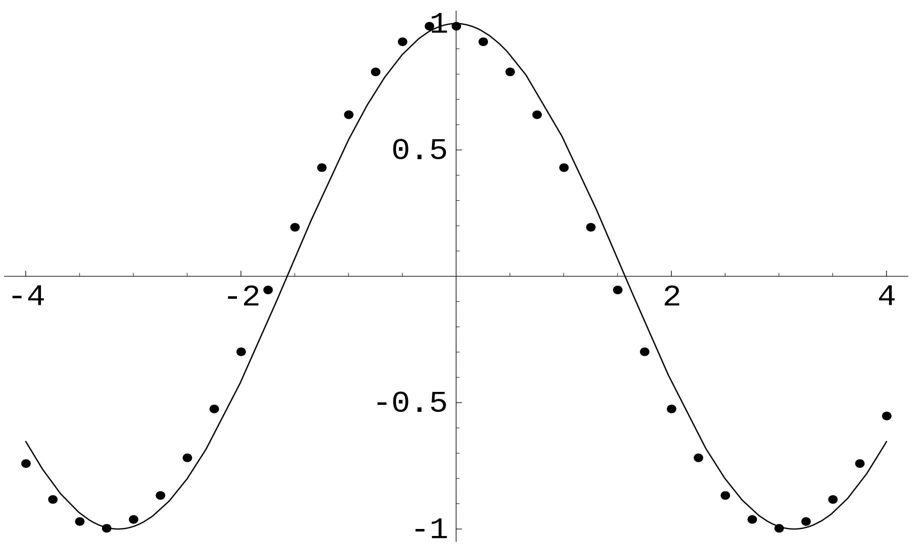

Figure 10.12: Forward Difference Scheme Approximation of the Derivative.

Another scheme for approximating the first derivative is the *centered difference scheme*,

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}.$$

Expanding the numerator using Taylor's theorem,

$$\begin{aligned}
&\frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} \\
&= \frac{f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{6} f'''(\xi) - f(x) + \Delta x f'(x) - \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{6} f'''(\psi)}{2\Delta x} \\
&= f'(x) + \frac{\Delta x^2}{12}(f'''(\xi) + f'''(\psi)).
\end{aligned}$$

The error in the approximation is quadratic in Δx. Therefore this is a second order accurate scheme. Figure 10.13 is a plot of the derivative of the function and the value of this scheme for the function $f(x) = \sin x$ and $\Delta x = 1/4$.

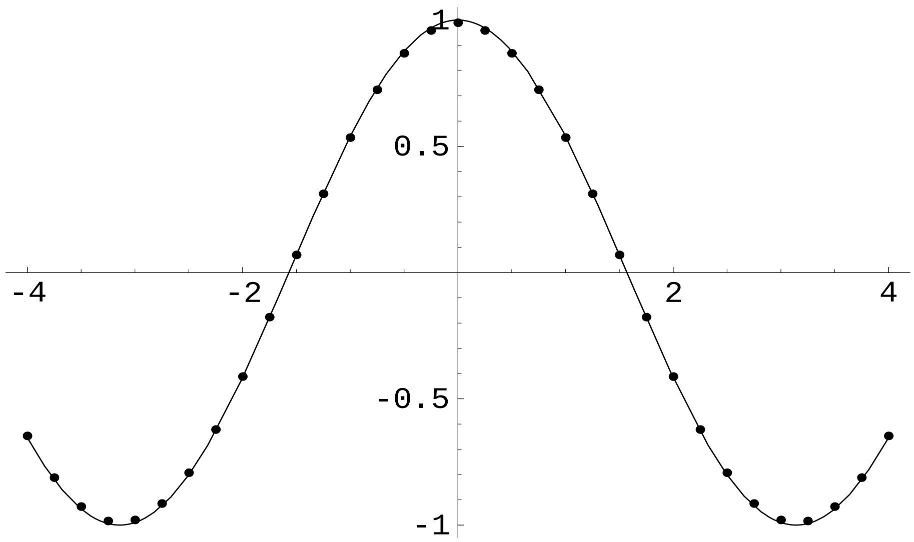

Figure 10.13: Centered Difference Scheme Approximation of the Derivative.

Notice how the centered difference scheme gives a better approximation of the derivative than the forward difference scheme.

CHAPTER

ELEVEN

Curvature

This is a chapter of advanced topics devoted to the elementary differential geometry of curves. Given a curve $y = f(x)$ in the plane, we have studied how well the tangent line at a point $P_0 = (x_0, y_0)$ on the curve approximates the graph near P_0. Analgously, we can study how well the a "tangent circle" at a point $P_0 = (x_0, y_0)$ on the curve approximates the graph near P_0. This "tangent circle" is called the "circle of curvature," its radius the "radius of curvature," and its center the "center of curvature." The topics covered include: the radius of curvature, curvature (which is the inverse of the radius of curvature), circle of curvature, and center of curvature.

11.1 Curvature

The shape of a curve depends very largely upon the rate at which the direction of the tangent changes as the point of contact describes the curve. This rate of change of direction is called *curvature* and is denoted by K. We now proceed to find its analytical expression, first for the simple case of the circle, and then for curves in general.

11.2 Curvature of a circle

Consider a circle of radius R.

In the notation of Figure 11.1, let

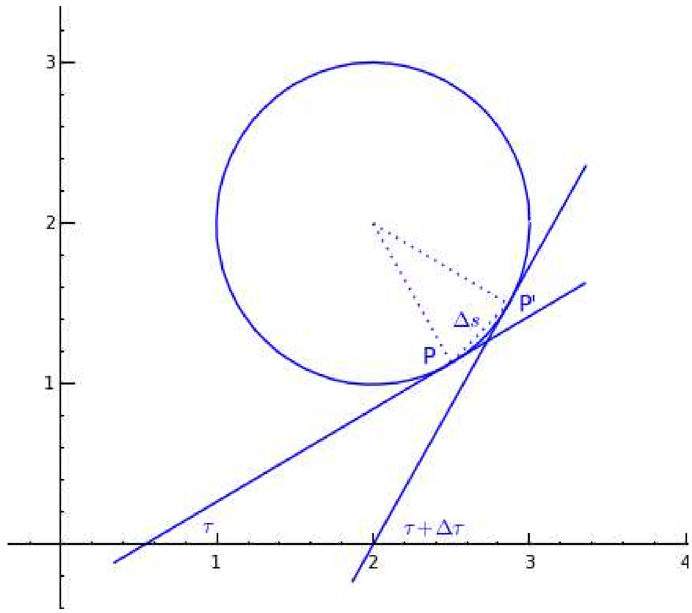

Figure 11.1: The curvature of a circle.

τ = angle that the tangent at P makes with the x-axis,

and

$\tau + \Delta\tau$ = angle made by the tangent at a neighboring point P′.

Then we say $\Delta\tau$ = *total curvature* of arc PP′. If the point P with its tangent be supposed to move along the curve to P′, the total curvature ($= \Delta\tau$) would measure the total change in direction, or rotation, of the tangent; or, what is the same thing, the total change in direction of the arc itself. Denoting by s the length of the arc of the curve measured from some fixed point (as A) to P, and by Δs the length of the arc P P′, then the ratio $\frac{\Delta\tau}{\Delta s}$ measures the average change in direction per unit length of arc[1]. Since, from Figure 11.1, $\Delta s = R \cdot \Delta\tau$, or $\frac{\Delta\tau}{\Delta s} = \frac{1}{R}$, it is evident that this ratio is constant everywhere on the circle. This ratio is, by definition, the curvature of the circle, and we have

$$K = \frac{1}{R}. \tag{11.1}$$

The curvature of a circle equals the reciprocal of its radius.

[1]Thus, if $\Delta\tau = \frac{\pi}{6}$ radians (= 30^o), and $\Delta s = 3$ centimeters, then $\frac{\Delta\tau}{\Delta s} = \frac{\pi}{18}$ radians per centimeter = 10^o per centimeter = average rate of change of direction.

11.3 Curvature at a point

Consider any curve. As in the last section, $\Delta\tau$ = total curvature of the arc PP', and $\frac{\Delta\tau}{\Delta s}$ = average curvature of the arc PP'.

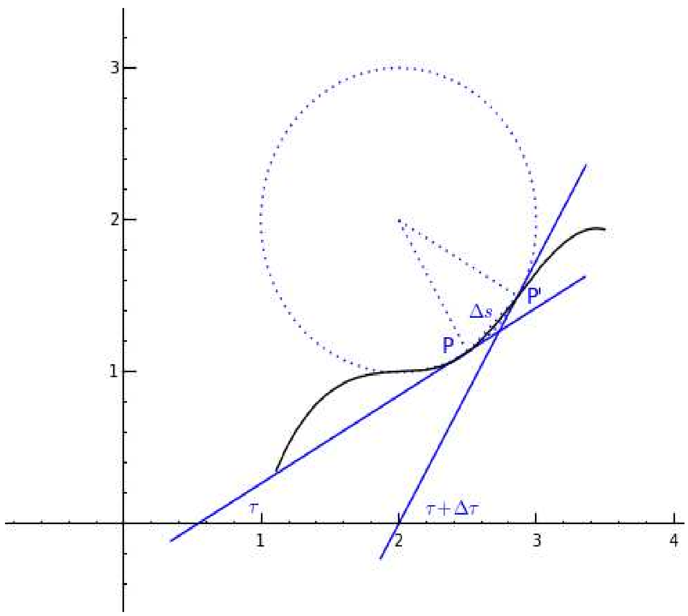

Figure 11.2: Geometry of the curvature at a point.

More important, however, than the notion of the average curvature of an arc is that of curvature at a point. This is obtained as follows. Imagine P to approach P along the curve; then the limiting value of the average curvature $\left(= \frac{\Delta\tau}{\Delta s}\right)$ as P′ approaches P along the curve is defined as the *curvature at P*, that is,

$$\text{Curvature at a point} = \lim_{\Delta s \to 0} \left(\frac{\Delta\tau}{\Delta s}\right) = \frac{d\tau}{ds}.$$

Therefore,

$$K = \frac{d\tau}{ds} = \text{ curvature.} \tag{11.2}$$

Since the angle $\Delta\tau$ is measured in radians and the length of arc Δs in units of length, it follows that the unit of curvature at a point is one radian per unit of length.

11.4 Formulas for curvature

It is evident that if, in the last section, instead of measuring the angles which the tangents made with the x-axis, we had denoted by τ and $\tau + \Delta\tau$ the angles made by the tangents with any arbitrarily fixed line, the different steps would in no wise have been changed, and consequently the results are entirely independent of the system of coordinates used. However, since the equations of the curves we shall consider are all given in either rectangular or polar coordinates, it is necessary to deduce formulas for K in terms of both. We have $\tan\tau = \frac{dy}{dx}$ by §3.9, or $\tau = \arctan\frac{dy}{dx}$. Differentiating with respect to x, using (4.23) in §4.1,

$$\frac{d\tau}{dx} = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}.$$

Also

$$\frac{ds}{dx} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}},$$

by (8.4). Dividing one equation into the other gives

$$\frac{\frac{d\tau}{dx}}{\frac{ds}{dx}} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}.$$

But

$$\frac{\frac{d\tau}{dx}}{\frac{ds}{dx}} = \frac{d\tau}{ds} = K.$$

Hence

$$K = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}. \tag{11.3}$$

If the equation of the curve be given in polar coordinates, K may be found as follows: From (5.13),

$$\tau = \theta + \psi.$$

Differentiating,

$$\frac{d\tau}{d\theta} = 1 + \frac{d\psi}{d\theta}.$$

But

$$\tan\psi = \frac{\rho}{\frac{d\rho}{d\theta}},$$

from (5.12). Therefore,

$$\psi = \arctan\frac{\rho}{\frac{d\rho}{d\theta}}.$$

Differentiating with respect to θ using XX in §4.1 and reducing,

$$\frac{d\psi}{d\theta} = \frac{\left(\frac{d\rho}{d\theta}\right)^2 - \rho\frac{d^2\rho}{d\theta^2}}{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}.$$

Substituting, we get

$$\frac{d\tau}{d\theta} = \frac{\rho^2 - \rho\frac{d^2\rho}{d\theta^2} + 2\left(\frac{d\rho}{d\theta}\right)^2}{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}.$$

Also

$$\frac{ds}{d\theta} = \left[\rho^2\left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{1}{2}},$$

by (8.9). Dividing gives

$$\frac{\frac{d\tau}{d\theta}}{\frac{ds}{d\theta}} = \frac{\rho^2 - \rho\frac{d^2\rho}{d\theta^2} + 2\left(\frac{d\rho}{d\theta}\right)^2}{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}.$$

But

$$\frac{\frac{d\tau}{d\theta}}{\frac{ds}{d\theta}} = \frac{d\tau}{ds} = K.$$

Hence

$$K = \frac{\rho^2 - \rho\frac{d^2\rho}{d\theta^2} + 2\left(\frac{d\rho}{d\theta}\right)^2}{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}. \tag{11.4}$$

Example 11.4.1. Find the curvature of the parabola $y^2 = 4px$ at the left-most end of the chord that passes through the focus and is perpendicular to the y-axis.

Solution. $\frac{dy}{dx} = \frac{2p}{y}$; $\frac{d^2y}{dx^2} = -\frac{2p}{y^2}\frac{dy}{dx} = -\frac{4p^2}{y^3}$. Substituting in (11.3), $K = -\frac{40-p^2}{(y^2+4p^2)^{\frac{3}{2}}}$, giving the curvature at any point. At the left-most end of the chord $(p, 2p)$,

$$K = -\frac{4p^2}{(4p^2+4p^2)^{\frac{3}{2}}} = -\frac{4p^2}{16\sqrt{2}p^3} = -\frac{1}{4\sqrt{2}p}.$$

While in our work it is generally only the numerical value of K that is of importance, yet we can give a geometric meaning to its sign. Throughout our work we have taken the positive sign of the radical $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$. Therefore K will be positive or negative at the same time that $\frac{d^2y}{dx^2}$ is, i.e., (by §7.8), according as the curve is concave upwards or concave downwards.

We shall solve this using Sage .

Sage

```
sage: x = var("x")
sage: p = var("p")
sage: y = sqrt(4*p*x)
sage: K = diff(y,x,2)/(1+diff(y,x)^2)^(3/2)
sage: K
-p^2/(2*(p/x + 1)^(3/2)*(p*x)^(3/2))
```

Taking $x = p$ and simplifying gives the result above.

Sage

```
sage: K.variables()
(p, x)
sage: K(p,p)
-p^2/(4*sqrt(2)*(p^2)^(3/2))
sage: K(p,p).simplify_rational()
```

```
-1/(4*sqrt(2)*sqrt(p^2))
```

Example 11.4.2. Find the curvature of the logarithmic spiral $\rho = e^{a\theta}$ at any point.

Solution. $\frac{d\rho}{d\theta} = ae^{a\theta} = a\rho$; $\frac{d^2\rho}{d\theta^2} = a^2e^{a\theta} = a^2\rho$.

Substituting in (11.4), $K = \frac{1}{\rho\sqrt{1+a^2}}$.

In laying out the curves on a railroad it will not do, on account of the high speed of trains, to pass abruptly from a straight stretch of track to a circular curve. In order to make the change of direction gradual, engineers make use of transition curves to connect the straight part of a track with a circular curve. Arcs of cubical parabolas are generally employed as transition curves.

Now we do this in Sage :

Sage

```
sage: rho = var("rho")
sage: t = var("t")
sage: r = var("r")
sage: a = var("a")
sage: r = exp(a*t)
sage: K = (r^2-r*diff(r,t,2)+2*diff(r,t)^2)/(r^2+diff(r,t)^2)^(3/2)
sage: K
1/sqrt(a^2*e^(2*a*t) + e^(2*a*t))
sage: K.simplify_rational()
e^(-(a*t))/sqrt(a^2 + 1)
```

Example 11.4.3. The transition curve on a railway track has the shape of an arc of the cubical parabola $y = \frac{1}{3}x^3$. At what rate is a car on this track changing its direction (1 mi. = unit of length) when it is passing through (a) the point $(3, 9)$? (b) the point $(2, \frac{8}{3})$? (c) the point $(1, \frac{1}{3})$?

Solution. $\frac{dy}{dx} = x^2$, $\frac{d^2y}{dx^2} = 2x$. Substituting in (11.3), $K = \frac{2x}{(1+x^4)^{\frac{3}{2}}}$. (a) At $(3, 9)$, $K = \frac{6}{(82)^{\frac{3}{2}}}$ radians per mile = $28'$ per mile. (b) At $(2, \frac{8}{3})$, $K = \frac{4}{(17)^{\frac{3}{2}}}$ radians per mile = $3^o16'$ per mile. (c) At $(1, \frac{1}{3})$, $K = \frac{2}{(2)^{\frac{3}{2}}} = \frac{1}{\sqrt{2}}$ radians per mile = $40^o30'$ per mile.

11.5 Radius of curvature

By analogy with the circle (see (11.1)), the radius of curvature of a curve at a point is defined as the reciprocal of the curvature of the curve at that point. Denoting the radius of curvature by R, we have[2]

$$R = \frac{1}{K}.$$

Or, substituting the values of x from (11.3) and (11.4),

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \tag{11.5}$$

and[3]

$$R = \frac{\left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{\frac{3}{2}}}{\rho^2 - \rho\frac{d^2\rho}{d\theta^2} + 2\left(\frac{d\rho}{d\theta}\right)^2}. \tag{11.6}$$

Example 11.5.1. Find the radius of curvature at any point of the catenary $y = \frac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$.

Solution. $\frac{dy}{dx} = \frac{1}{2}(e^{\frac{x}{a}} - e^{-\frac{x}{a}})$; $\frac{d^2y}{dx^2} = \frac{1}{2a}(e^{\frac{x}{a}} - e^{-\frac{x}{a}})$. Substituting in (11.5),

$$\begin{aligned} R &= \frac{\left[1+\left(\frac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{2}\right)^2\right]^{\frac{3}{2}}}{\frac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{2a}} \\ &= \frac{\left(\frac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{2}\right)^3}{\frac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{2a}} = \frac{a(e^{\frac{x}{a}}-e^{-\frac{x}{a}})^2}{4} \\ &= \frac{y^2}{a}. \end{aligned}$$

If the equation of the curve is given in parametric form, find the first and second derivatives of y with respect to x from (9.5) and (9.6), namely:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

and

[2]Hence the radius of curvature will have the same sign as the curvature, that is, $+$ or $-$, according as the curve is concave upwards or concave downwards.

[3]In §9.4, the next equation is derived from the previous one by transforming from rectangular to polar coordinates.

$$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3},$$

and then substitute[4] the results in (11.5).

Example 11.5.2. Find the radius of curvature of the cycloid $x = a(t - \sin t)$, $y = a(t - \cos t)$.

Solution. $\frac{dx}{dt} = a(1 - \cos t)$, $\frac{dy}{dt} = a \sin t$; $\frac{d^2x}{dt^2} = a \sin t$, $\frac{d^2y}{dt^2} = a \cos t$. Substituting the previous example and then in (11.5), we get

$\frac{dy}{dx} = \frac{\sin t}{1-\cos t}$, $\frac{d^2y}{dx^2} = \frac{a(1-\cos t)a\cos t - a\sin t a \sin t}{a^3(1-\cos t)^3} = \frac{1}{a(1-\cos t)^2}$, and $R = \frac{\left[1+\left(\frac{\sin t}{1-\cos t}\right)^2\right]^{\frac{3}{2}}}{-\frac{1}{a(1-\cos t)^2}} = -2a\sqrt{2 - 2\cos t}$.

11.6 Circle of curvature

Consider any point P on the curve C (see Figure 11.3). The tangent drawn to the curve at P has the same slope as the curve itself at P (see §5.1). In an analogous manner we may construct for each point of the curve a circle whose curvature is the same as the curvature of the curve itself at that point. To do this, proceed as follows. Draw the normal to the curve at P on the concave side of the curve.

Move along this normal a distance R from P to a point C. With C as a center, draw the circle passing through P. The curvature of this circle is then $K = \frac{1}{R}$, which also equals the curvature of the curve itself at P.

Definition 11.6.1. (First definition) The circle so constructed is called the *circle of curvature* for the point P on the curve.

In general, the circle of curvature of a curve at a point will cross the curve at that point. This is illustrated in the Figure 11.3.

Just as the tangent at P shows the direction of the curve at P, so the circle of curvature at P aids us very materially in forming a geometric concept of the curvature of the curve at P, the rate of change of direction of the curve and of the circle being the same at P.

The circle of curvature can be defined as the limiting position of a secant circle, a definition analogous to that of the tangent given in §3.9.

[4] Substituting these last two equations in (11.5) gives $R = \frac{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}$.

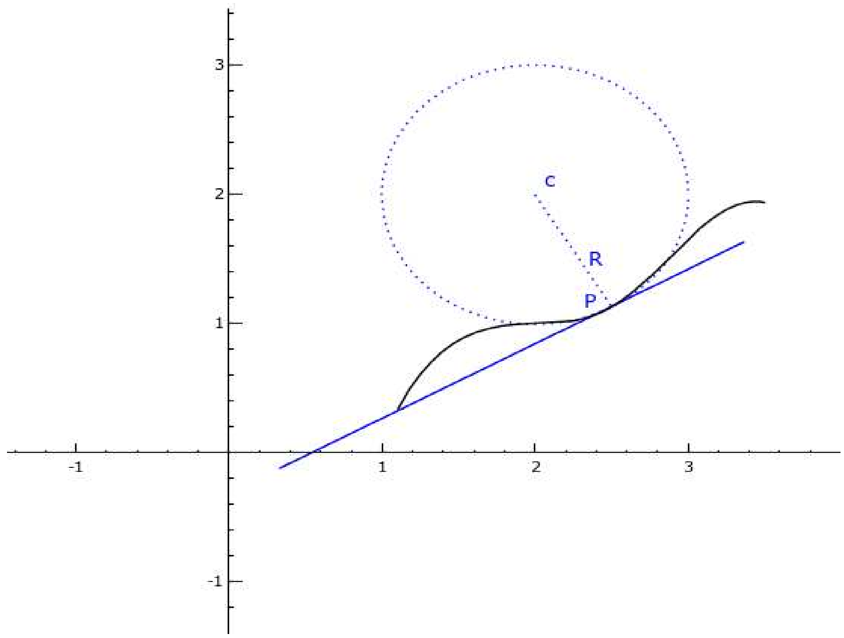

Figure 11.3: The circle of curvature.

Example 11.6.1. Find the radius of curvature at the point $(3,4)$ on the equilateral hyperbola $xy = 12$, and draw the corresponding circle of curvature.

Solution. $\frac{dy}{dx} = -\frac{y}{x}$, $\frac{d^2y}{dx^2} = \frac{2y}{x^2}$. For $(3,4)$, $\frac{dy}{dx} = -\frac{4}{3}$, $\frac{d^2y}{dx^2} = \frac{8}{9}$, so

$$R = \frac{[1+\frac{16}{9}]^{\frac{3}{2}}}{\frac{8}{9}} = \frac{125}{24} = 25\frac{5}{24}.$$

The circle of curvature crosses the curve at two points.

We solve for the circle of curvature using Sage . First, we solve for the intersection of the normal $y - 4 = (-1/m)(x-3)$, where $m = y'(3) = -4/3$, and the circle of radius $R = 125/24$ about $(3,4)$:

Sage

```
sage: x = var("x")
sage: y = 12/x
sage: K = diff(y,x,2)/(1+diff(y,x)^2)^(3/2)
sage: K
24/((144/x^4 + 1)^(3/2)*x^3)
sage: K(3)
24/125
sage: R = 1/K(3)
```

```
sage: m = diff(y,x)(3); m
-4/3
sage: xx = var("xx")
sage: yy = var("yy")
sage: solve((xx-3)^2+(-1/m)^2*(xx-3)^2==R^2, xx)
[xx == -7/6, xx == 43/6]
```

This tells us that the normal line intersects the circle of radius R centered at $(3, 4)$ in 2 points, one of which is at $(43/6, 57/8)$. This is the center of the circle of curvature, so the equation is $(x - 43/6)^2 + (y - 57/8)^2 = R^2$.

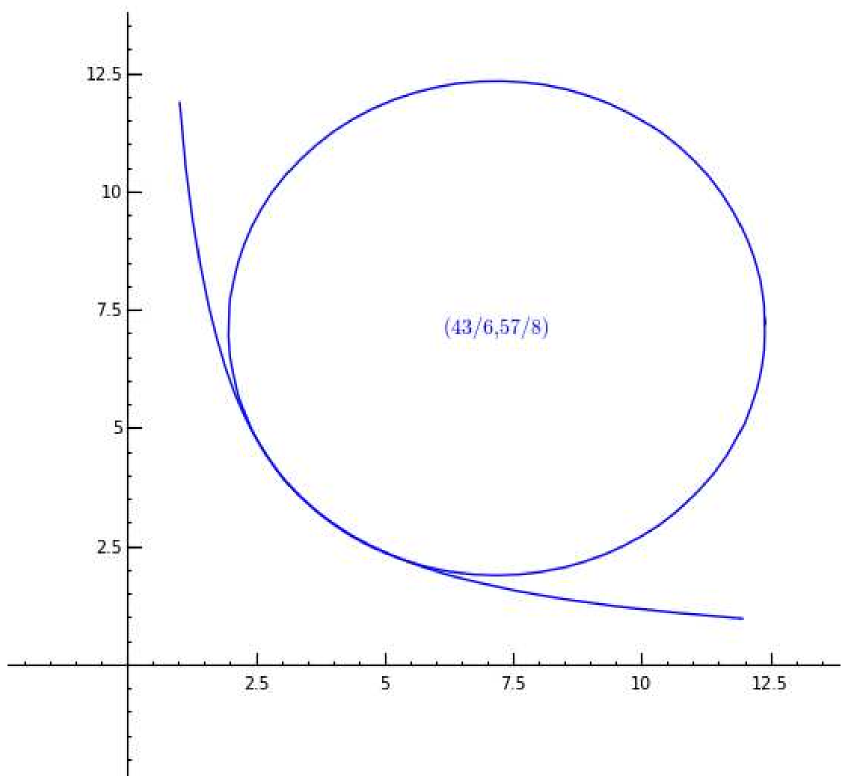

Figure 11.4: The circle of curvature of a hyperbola.

11.7 Exercises

1. Find the radius of curvature for each of the following curves, at the point indicated; draw the curve and the corresponding circle of curvature:

   (a) $b^2x^2 + a^2y^2 = a^2b^2$, $(a, 0)$.
   Ans. $R = \frac{b^2}{a}$.

   (b) $b^2y^2 + a^2y^2 = a^2b^2$, $(0, b)$.
   Ans. $R = \frac{a^2}{b}$.

   (c) $y = x^4 - 4x^3 - 18x^2$, $(0, 0)$.
   Ans. $R = \frac{1}{36}$.

   (d) $16y^2 = 4x^4 - x^6$, $(2, 0)$.
   Ans. $R = 2$.

   (e) $y = x^3$, (x_1, y_1).
   Ans. $R = \frac{(1+9x_1{}^4)^{\frac{3}{2}}}{6x_1}$.

   (f) $y^2 = x^3$, $(4, 8)$.
   Ans. $R = \frac{1}{3}(40)^{\frac{3}{2}}$.

   (g) $y^2 = 8x$, $(\frac{9}{8}, 3)$.
   Ans. $R = \frac{125}{16}$.

   (h) $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$, $(0, b)$.
   Ans. $R = \frac{a^2}{3b}$.

   (i) $x^2 = 4ay$, $(0, 0)$.
   Ans. $R = 2a$.

   (j) $(y - x^2)^2 = x^5$, $(0, 0)$.
   Ans. $R = \frac{1}{2}$.

   (k) $b^2x^2 - a^2y^2 = a^2b^2$, (x_1, y_1).
   Ans. $R = \frac{(b^4x_1{}^2+a^4y_1{}^2)^{\frac{3}{2}}}{a^4b^4}$.

   (ℓ) $e^x = \sin y$, (x_1, y_1).

   (m) $y = \sin x$, $\left(\frac{\pi}{2}, 1\right)$.

   (n) $y = \cos x$, $\left(\frac{\pi}{4}, \sqrt{2}\right)$.

(o) $y = \log\, x$, $x = e$.

(p) $9y = x^3$, $x = 3$.

(q) $4y^2 = x^3$, $x = 4$.

(r) $x^2 - y^2 = a^2$, $y = 0$.

(s) $x^2 + 2y^2 = 9$, $(1, -2)$.

2. Determine the radius of curvature of the curve $a^2y = bx^2 + cx^2y$ at the origin.

   Ans. $R = \frac{a^2}{2b}$.

3. Show that the radius of curvature of the witch $y^2 = \frac{a^2(a-x)}{x}$ at the vertex is $\frac{a}{2}$.

4. Find the radius of curvature of the curve $y = \log \sec\, x$ at the point (x_1, y_1).

   Ans. $R = \sec\, x_1$.

5. Find K at any point on the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$.

   Ans. $K = \frac{a^{\frac{1}{2}}}{2(x+y)^{\frac{3}{2}}}$.

6. Find R at any point on the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

   Ans. $R = 3(axy)^{\frac{1}{3}}$.

7. Find R at any point on the cycloid $x = r \,\text{arcvers}\, \frac{y}{r} - \sqrt{2ry - y^2}$.

   Ans. $R = 2\sqrt{2ry}$.

Find the radius of curvature of the following curves at any point:

8. The circle $\rho = a \sin\theta$.

   Ans. $R = \frac{a}{2}$.

9. The spiral of Archimedes $\rho = a\theta$.

   Ans. $R = \frac{(\rho^2 = a^2)^{\frac{3}{2}}}{\rho^2 + 2a^2}$.

10. The cardioid $\rho = a(1 \cos\theta)$.

    $R = \frac{2}{3}\sqrt{2a\rho}$.

11. The lemniscate $\rho^2 = a^2 \cos\, 2\theta$.

    $R = \frac{a^2}{3\rho}$.

12. The parabola $\rho = a \sec^2 \frac{\theta}{2}$.

    Ans. $R = 2a \sec^3 \frac{\theta}{2}$.

13. The curve $\rho = a sin^3 \frac{\theta}{3}$.

14. The trisectrix $\rho = 2a \cos\theta - a$.

    Ans. $R = \frac{a(5-4\cos\theta)^{\frac{3}{2}}}{9-6\cos\theta}$.

15. The equilateral hyperbola $\rho^2 \cos\, 2\theta = a^2$.

    Ans. $R = \frac{\rho^3}{a^2}$.

16. The conic $\rho = \frac{a(1-e^2)}{1-e\cos\theta}$.

    Ans. $R = \frac{a(1-e^2)(1-2e\cos\theta+e^2)^{\frac{3}{2}}}{(1-e\cos\theta)^3}$.

17. The curve

$$\begin{cases} x = 3t^2, \\ y = 3t - t^3, \end{cases}$$

$t = 1$.

Ans. $R = 6$.

In Sage :

Sage

```
sage: t = var('t')
sage: x = 3*t^2
sage: y = 3*t-t^3
sage: Rnum = (x.diff(t)^2+y.diff(t)^2)^(3/2)
sage: Rdenom = x.diff(t)*y.diff(t,2)-y.diff(t)*x.diff(t,2)
sage: R = Rnum/Rdenom
sage: R(1)
-6
```

18. The hypocycloid

$$\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t, \end{cases}$$

$t = t_1$.

Ans. $R = 3a \sin\, t_1 \cos\, t_1$.

In Sage :

Sage

```
sage: t = var('t')
sage: x = cos(t)^3
sage: y = sin(t)^3
sage: Rnum = (x.diff(t)^2+y.diff(t)^2)^(3/2)
sage: Rdenom = x.diff(t)*y.diff(t,2)-y.diff(t)*x.diff(t,2)
sage: R = Rnum/Rdenom
sage: R.simplify_trig()
-sqrt(9*cos(t)^2 - 9*cos(t)^4)
```

You can simplify this last result even further if you want.

19. The curve

$$\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t), \end{cases}$$

$t = \frac{\pi}{2}$.

Ans. $R = \frac{\pi a}{2}$.

20. The curve

$$\begin{cases} x = a(m\cos t + \cos mt), \\ y = a(m\sin t - \sin mt), \end{cases}$$

$t = t_0$.

Ans. $R = \frac{4ma}{m-1} \sin\left(\frac{m+1}{2}\right) t_0$.

21. Find the radius of curvature for each of the following curves at the point indicated; draw the curve and the corresponding circle of curvature:

(a) $x = t^2$, $2y = t$; $t = 1$.
(b) $x = t^2$, $y = t^3$; $t = 1$.
(c) $x = \sin\, t$, $y = \cos\, 2t$; $t = \frac{\pi}{6}$.
(d) $x = 1 - t$, $y = t^3$; $t = 3$.
(e) $x = t$, $y = 6t - 1$; $t = 2$.
(f) $x = 2e^t$, $y = e^{-t}$; $t = 0$.
(g) $x = \sin\, t$, $y = 2\cos t$; $t = \frac{\pi}{4}$.
(h) $x = t^3$, $y = t^2 + 2t$; $t = 1$.

22. An automobile race track has the form of the ellipse $x^2 + 16y^2 = 16$, the unit being one mile. At what rate is a car on this track changing its direction

(a) when passing through one end of the major axis?

(b) when passing through one end of the minor axis?

(c) when two miles from the minor axis?

(d) when equidistant from the minor and major axes?

Ans. (a) 4 radians per mile; (b) $\frac{1}{16}$ radian per mile.

23. On leaving her dock a steamship moves on an arc of the semi cubical parabola $4y^2 = x^3$. If the shore line coincides with the axis of y, and the unit of length is one mile, how fast is the ship changing its direction when one mile from the shore?

Ans. $\frac{24}{125}$ radians per mile.

24. A battleship 400 ft. long has changed its direction 30^o while moving through a distance equal to its own length. What is the radius of the circle in which it is moving?

Ans. 764 ft.

25. At what rate is a bicycle rider on a circular track of half a mile diameter changing his direction?

Ans. 4 rad. per mile = $43'$ per rod.

26. The origin being directly above the starting point, an aeroplane follows approximately the spiral $\rho = \theta$, the unit of length being one mile. How rapidly is the aeroplane turning at the instant it has circled the starting point once?

27. A railway track has curves of approximately the form of arcs from the following curves. At what rate will an engine change its direction when passing through the points indicated (1 mi. = unit of length):

(a) $y = x^3$, $(2, 8)$?	(d) $y = e^x$, $x = 0$?
(b) $y = x^2$, $(3, 9)$?	(e) $y = \cos x$, $x = \frac{\pi}{4}$?
(c) $x^2 - y^2 = 8$, $(3, 1)$?	(f) $\rho\theta = 4$, $\theta = 1$?

11.8 Circle of curvature

The circle of curvature is sometimes called the *osculating circle*. It was defined from another point of view in §11.6.

Definition 11.8.1. (Second definition) If a circle be drawn through three points $P_0 = (x_0, y_0)$, $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ on a plane curve, and if P_1 and P_2 be made to approach P_0 along the curve as a limiting position, then the circle will in general approach in magnitude and position a limiting circle called the circle of curvature of the curve at the point P_0. The center of this circle is called the *center of curvature*.

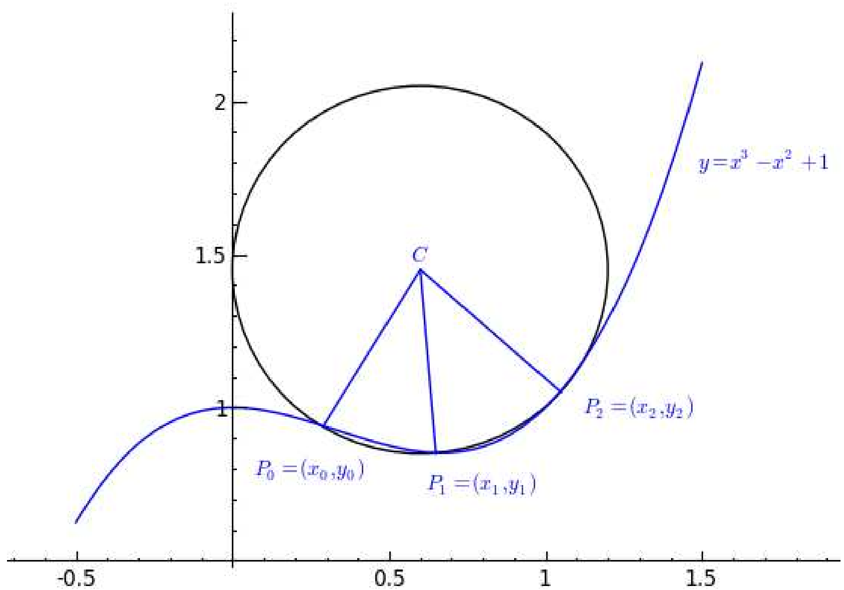

Figure 11.5: Geometric visualization of the circle of curvature.

Let the equation of the curve be

$$y = f(x); \tag{11.7}$$

and let x_0, x_1, x_2 be the abscissas of the points P_0, P_1, P_2 respectively, $C = (\alpha', \beta')$ the coordinates of the center, and R' the radius of the circle passing through the three points. Then the equation of the circle is

$$(x - \alpha')^2 + (y - \beta')^2 = (R')^2;$$

and since the coordinates of the points P_0, P_1, P_2 must satisfy this equation, we have

$$\begin{cases} (x_0 - \alpha')^2 + (y_0 - \beta')^2 - (R')^2 = 0, \\ (x_1 - \alpha')^2 + (y_1 - \beta')^2 - (R')^2 = 0, \\ (x_2 - \alpha')^2 + (y_2 - \beta')^2 - (R')^2 = 0. \end{cases} \tag{11.8}$$

Now consider the function of x defined by

$$F(x) = (x - \alpha')^2 + (y - \beta')^2 - (R')^2,$$

in which $y = f(x)$ using (11.7).

Then from equations (11.8) we get

$$F(x_0) = 0, \quad F(x_1) = 0, \quad F(x_2) = 0.$$

Hence, by Rolle's Theorem (§10.1), $F'(x)$ must vanish for at least two values of x, one lying between x_0 and x_1, say x', and the other lying between x_1 and x_2 say x''; that is,

$$F'(x') = 0, \quad F'(x'') = 0.$$

Again, for the same reason, $F''(x)$ must vanish for some value of x between x' and x'', say x_3; hence

$$F''(x_3) = 0.$$

Therefore the elements α', β', R' of the circle passing through the points P_0, P_1, P_2 must satisfy the three equations

$$F(x_0) = 0, \quad F'(x') = 0, \quad F''(x_3) = 0.$$

Now let the points P_1 and P_2 approach P_0 as a limiting position; then x_1, x_2, x', x'', x_3 will all approach x_0 as a limit, and the elements α, β, R of the osculating circle are therefore determined by the three equations

$$F(x_0) = 0, \qquad F'(x_0) = 0, \qquad F''(x_0) = 0;$$

or, dropping the subscripts, which is the same thing,

$$(x-\alpha)^2 + (y-\beta)^2 = R^2 \tag{11.9}$$

$$(x-\alpha) + (y-\beta)\frac{dy}{dx} = 0, \tag{11.10}$$

differentiating (11.9).

$$1 + \left(\frac{dy}{dx}\right)^2 + (y-\beta)\frac{d^2y}{dx^2} = 0, \tag{11.11}$$

differentiating (11.10). Solving (11.10) and (11.11) for $x-\alpha$ and $y-\beta$, we get $\left(\frac{d^2y}{dx^2} \neq 0\right)$,

$$\begin{cases} x-\alpha = \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \\ y-\beta = -\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}; \end{cases} \tag{11.12}$$

hence the coordinates of the center of curvature are

$$\begin{aligned} \alpha &= x - \frac{\frac{dy}{dx}\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \\ \beta &= y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}, \end{aligned} \tag{11.13}$$

assuming $\frac{d^2y}{dx^2} \neq 0$.

Substituting the values of $x-\alpha$ and $y-\beta$ from (11.12) in (11.9), and solving for R, we get

$$R = \pm\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

which is identical with (11.5), [§11.5]. This is summarized in the following statement.

Theorem 11.8.1. The radius of the circle of curvature equals the radius of curvature.

11.9 Second method for finding center of curvature

Here we shall make use of the definition of circle of curvature given in §11.6. Draw a figure showing the tangent line, circle of curvature, radius of curvature, and center of curvature $C = (\alpha, \beta)$ corresponding to the point $P = (x, y)$ on the curve. For example, in Figure 11.5, replace P_2 by P, replace (α', β') by (α, β), and imagine the tangent line to the curve drawn at P. Call the origin in the plane O, the projection of P to the x-axis D, the projection of C to the x-axis A, and call B the projection of P onto the segment CA. This is depicted in Figure 11.6.

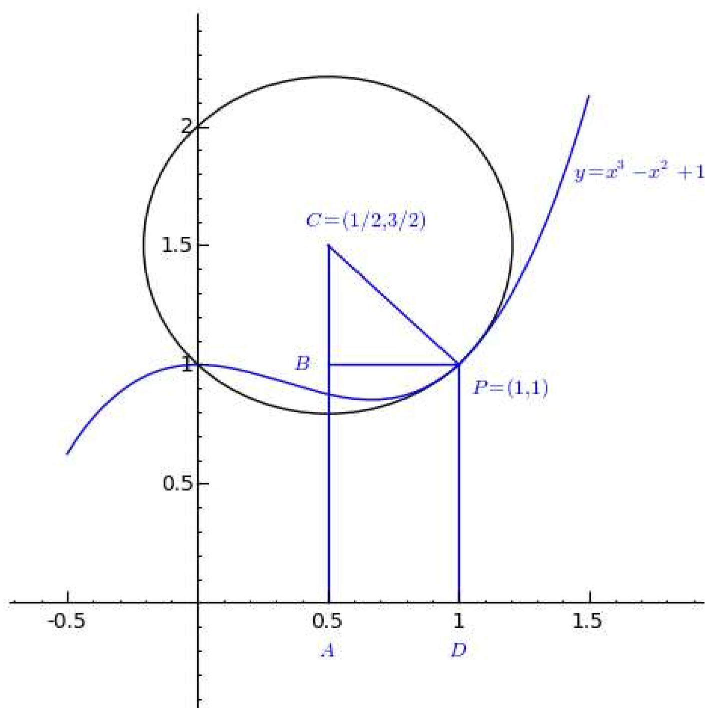

Figure 11.6: Circle of curvature.

Then

$$\alpha = OA = OD - AD = OD - BP = x - BP,$$
$$\beta = AC = AB + BC = DP + BC = y + BC.$$

But $BP = R\sin\tau$, $BC = R\cos\tau$. Hence

$$\alpha = x - R\sin\tau, \quad \beta = y + R\cos\tau. \tag{11.14}$$

Example 11.9.1. We shall solve for the radius of curvature of $y = x^3 - x^2 + 1$ at $x = 1$ using Sage .

Sage

```
sage: y = x^3-x^2+1
sage: Dy = diff(y,x)
sage: D2y = diff(y,x,x)
sage: R = (1+Dy^2)^(3/2)/D2y
sage: R(1)
1/sqrt(2)
sage: alpha = x - Dy*(1+Dy^2)/D2y
sage: beta = y + (1+Dy^2)/D2y
sage: alpha(1)
1/2
sage: beta(1)
3/2
```

From (8.8) [§8.3], and (11.5) [11.5],

$$\sin\tau = \frac{\frac{dy}{dx}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}},$$
$$\cos\tau = \frac{1}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}},$$
$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

Substituting these back in (11.14), we get

$$\alpha = x - \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}; \quad \beta = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}. \tag{11.15}$$

From Lemma 7.8.1 [§7.8], we know that at a point of inflection

$$\frac{d^2y}{dx^2} = 0.$$

Therefore, by (11.3) [§11.4], the curvature $K = 0$. From (11.5) [§11.5], and (11.15) [§11.9], we see that in general α, β, R increase without limit as $\frac{d^2y}{dx^2} \to 0$.

Example 11.9.2. Find the coordinates of the center of curvature of the parabola $y^2 = 4px$ corresponding (a) to any point on the curve; (b) to the vertex.

Solution. $\frac{dy}{dx} = \frac{2p}{y}$; $\frac{d^2y}{dx^2} = -\frac{4p^2}{y^3}$.

(a) Substituting in (11.13) [§11.8],

$$\alpha = x + \frac{y^2 + 4p^2}{y^2} \cdot \frac{2p}{y} \cdot \frac{y^3}{4p^2} = 3x + 2p.$$

$$\beta = y - \frac{y^2 + 4p^2}{y^2} \cdot \frac{y^3}{4p^2} = -\frac{y^3}{4p^2}.$$

Therefore $\left(3x + 2p, -\frac{y^3}{4p^2}\right)$ is the center of curvature corresponding to any point on the curve.

(b) $(2p, 0)$ is the center of curvature corresponding to the vertex $(0, 0)$.

11.10 Center of curvature

In this section, we discuss how the center of curvature can be thought of geometricaly as the limiting position of the intersection of normals at neighboring points. Let the equation of a curve be

$$y = f(x). \tag{11.16}$$

The equations of the normals to the curve at two neighboring points P_0 and P_1 are (using (5.2) [§5.3]),

$$(x_0 - x) + (y_0 - y)\frac{dy_0}{dx_0} = 0, \quad (x_1 - x) + (y_1 - y)\frac{dy_1}{dx_1} = 0.$$

If the normals intersect at $C' = (\alpha', \beta')$, the coordinates of this point must satisfy both equations, giving

$$\begin{cases} (x_0 - \alpha') + (y_0 - \beta)\frac{dy_0}{dx_0} = 0, \\ (x_1 - \alpha') + (y_1 - \beta')\frac{dy_1}{dx_1} = 0. \end{cases} \tag{11.17}$$

Now consider the function of x defined by

$$\phi(x) = (x - \alpha') + (y - \beta')\frac{dy}{dx},$$

in which $y = f(x)$ using (11.16). Then equations (11.17) show that

$$\phi(x_0) = 0, \quad \phi(x_1) = 0.$$

But then, by Rolle's Theorem (§10.1), $\phi'(x)$ must vanish for some value of x between x_0 and x_1 say x'. Therefore α' and β' are determined by the two equations

$$\phi(x_0) = 0, \quad \phi'(x') = 0.$$

If now P_1 approaches P_0 as a limiting position, then x' approaches x_0, giving

$$\phi(x_0) = 0, \quad \phi'(x_0) = 0,$$

and $C'(\alpha', \beta')$ will approach as a limiting position the center of curvature $C(\alpha, \beta)$ corresponding to P_0 on the curve. For if we drop the subscripts and write the last two equations in the form

$$(x - \alpha') + (y - \beta')\frac{dy}{dx} = 0, \quad 1 + \left(\frac{dy}{dx}\right)^2 + (y - \beta')\frac{d^2y}{dx^2} = 0,$$

it is evident that solving for α' and β' will give the same results as solving (11.10) and ((11.11) for α and β. Hence we have the following result.

Theorem 11.10.1. The center of curvature C corresponding to a point P on a curve is the limiting position of the intersection of the normal to the curve at P with a neighboring normal.

11.11 Evolutes

The locus of the centers of curvature of a given curve is called the *evolute* of that curve. Consider the circle of curvature corresponding to a point P on a curve. If P moves along the given curve, we may suppose the corresponding circle of curvature to roll along the curve with it, its radius varying so as to be always equal to the radius of curvature of the curve at the point P. The curve described by the center of the circles is the evolute.

It is instructive to make an approximate construction of the evolute of a curve by estimating (from the shape of the curve) the lengths of the radii of curvature at different points on the curve and then drawing them in and drawing the locus of the centers of curvature.

Formula (11.13) gives the coordinates of any point (α, β) on the evolute expressed in terms of the coordinates of the corresponding point (x, y) of the given curve. But y is a function of x; therefore

$$\alpha = x - \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}, \qquad \beta = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

give us at once the parametric equations of the evolute in terms of the parameter x.

To find the ordinary rectangular equation of the evolute we eliminate x between the two expressions. No general process of elimination can be given that will apply in all cases, the method to be adopted depending on the form of the given equation. In a large number of cases, however, the student can find the rectangular equation of the evolute by taking the following steps:

General directions for finding the equation of the evolute in rectangular coordinates.

- FIRST STEP. Find α, β from (11.15).
- SECOND STEP. Solve the two resulting equations for x and y in terms of α and β.
- THIRD STEP. Substitute these values of x and y in the given equation. This gives a relation between the variables α and β which is the equation of the evolute.

Example 11.11.1. Find the equation of the evolute of the parabola $y^2 = 4px$.

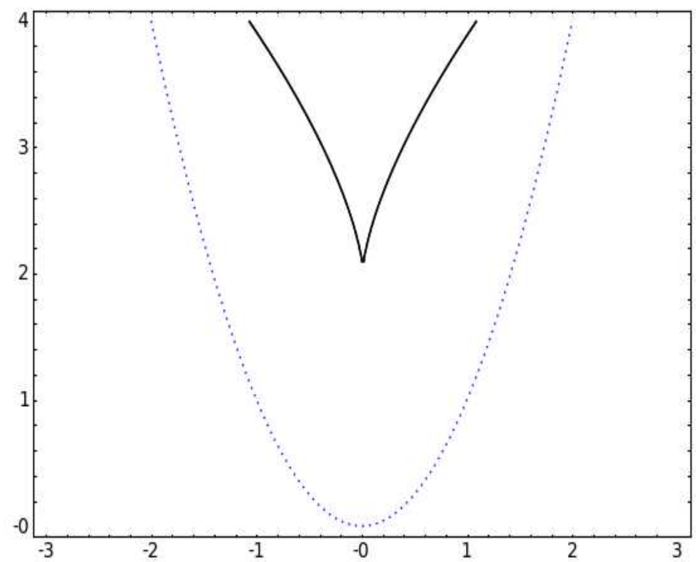

Figure 11.7: Evolute of a parabola.

Solution. $\frac{dy}{dx} = \frac{2p}{y}, \frac{d^2y}{dx^2} = -\frac{4p^2}{y^3}$.
First step. $\alpha = 3x + 2p$, $\beta = -\frac{y^3}{4p^2}$.
Second step. $x = \frac{\alpha - 2p}{3}$, $y = -(4p^2\beta)^{\frac{1}{3}}$.
Third step $(4p^2\beta)^{\frac{2}{3}} = 4p\left(\frac{\alpha-2p}{3}\right)$ or $p\beta^2 = \frac{4}{27}(\alpha - 2p)^3$.

Remembering that α denotes the "x-coordinate" and β the "y-coordinate" of a rectangular system of coordinates, we see that the evolute of the parabola $y = 4x^2$ is the "cusp" $y^2 = 4(x-2)^3/27$. The curve (dotted) and its evolute (solid) are plotted in Figure 11.7.

Example 11.11.2. Find the equation of the evolute of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.
Solution. $\frac{dy}{dx} = -\frac{b^2x}{a^2y}, \frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.
First step. $\alpha = \frac{(a^2-b^2)x^3}{a^4}$, $\beta = -\frac{(a^2-b^2)y^3}{b^4}$.
Second step. $x = \left(\frac{a^4\alpha}{a^2-b^2}\right)^{\frac{1}{3}}$, $y = -\left(\frac{b^4\beta}{a^2-b^2}\right)^{\frac{1}{3}}$.
Third step. $(a\alpha)^{\frac{2}{3}} + (b\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$, the equation of the evolute of the ellipse.

When the equations of the curve are given in parametric form, we proceed to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, as in §11.5, from

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3} \tag{11.18}$$

and then substitute the results in formulas (11.15). This gives the parametric equations of the evolute in terms of the same parameter that occurs in the given equations.

Example 11.11.3. The parametric equations of a curve are

$$x = \frac{t^2+1}{4}, \qquad y = \frac{t^3}{6}. \tag{11.19}$$

Find the equation of the evolute in parametric form, plot the curve and the evolute, find the radius of curvature at the point where $t = 1$, and draw the corresponding circle of curvature.

Solution. $\frac{dx}{dt} = \frac{t}{2}$, $\frac{d^2x}{dt^2} = \frac{1}{2}$, $\frac{dy}{dt} = \frac{t^2}{2}$, $\frac{d^2y}{dt^2} = t$. Substituting in above formulas (11.18) and then in (11.15), gives

$$\alpha = \frac{1 - t^2 - 2t^4}{4}, \qquad \beta = \frac{4t^3 + 3t}{6}, \tag{11.20}$$

the parametric equations of the evolute. Assuming values of the parameter t, we calculate x, y; α, β from (11.19) and (11.20). The curve (solid) and its evolute (dotted) are plotted in Figure 11.9.

The point $(\frac{1}{4}, 0)$ is common to the given curve and its evolute. The given curve (a "cusp") lies entirely to the right and the evolute entirely to the left of $x = \frac{1}{4}$.

Example 11.11.4. Find the parametric equations of the evolute of the cycloid,

$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t). \end{cases} \tag{11.21}$$

Solution. As in Example 11.5.2, we get

$$\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}, \qquad \frac{d^2y}{dx^2} = -\frac{1}{\alpha(1 - \cos t)^2}.$$

Substituting these results in formulas (11.15), we get the answer:

$$\begin{cases} \alpha = a(t + \sin t), \\ \beta = -a(1 - \cos t). \end{cases} \tag{11.22}$$

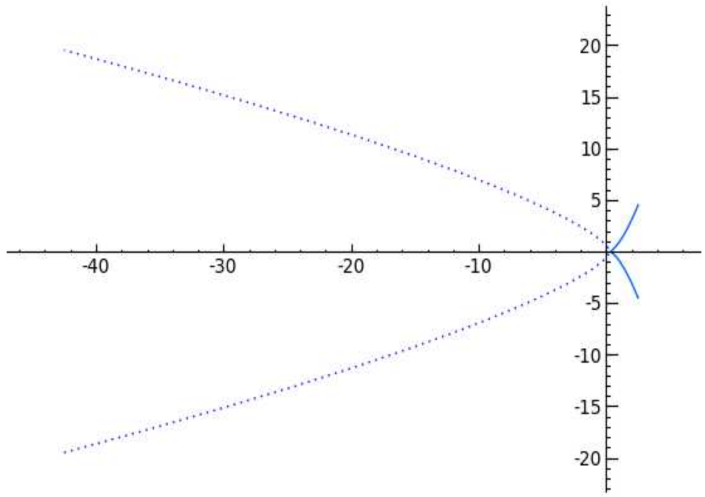

Figure 11.8: Evolute of an parametric curve.

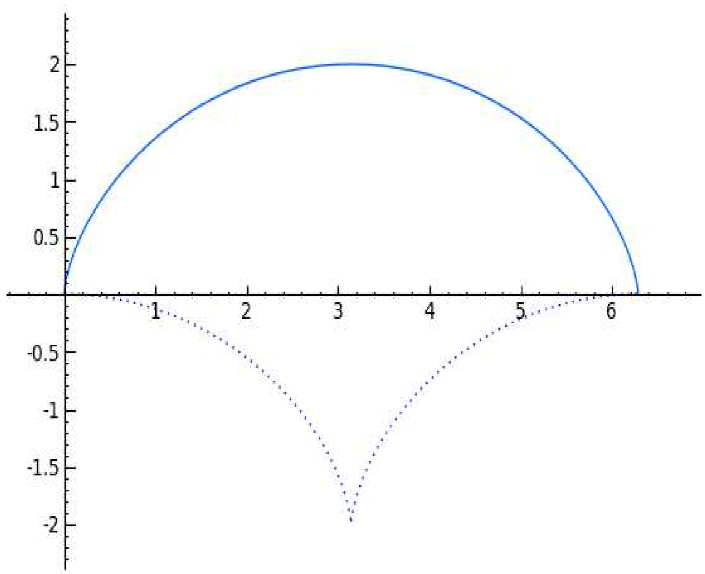

Figure 11.9: Evolute of a cycloid.

The curve (solid) and its evolute (dotted) are plotted in Figure 11.9.

11.12 Properties of the evolute

From (11.14),

$$\alpha = x - R\sin\,\tau, \qquad \beta = y + R\cos\,\tau. \tag{11.23}$$

Let us choose as independent variable the lengths of the arc on the given curve; then x, y, R, T, α, β are functions of s. Differentiating (11.23) with respect to s gives

$$\frac{d\alpha}{ds} = \frac{dx}{ds} - R\cos\tau\frac{d\tau}{ds} - \sin\tau\frac{dR}{ds}, \tag{11.24}$$

$$\frac{d\beta}{ds} = \frac{dy}{ds} - R\sin\tau\frac{d\tau}{ds} + \cos\tau\frac{dR}{ds}. \tag{11.25}$$

But $\frac{dx}{ds} = \cos\tau$, $\frac{dy}{ds} = \sin\tau$, from (8.5); and $\frac{d\tau}{ds} = \frac{1}{R}$, from (11.1) and (11.2).
Substituting in (11.24) and (11.25), we obtain

$$\frac{d\alpha}{ds} = \cos\tau - R\cos\tau\cdot\frac{1}{R} - \sin\tau\frac{dR}{ds} = -\sin\tau\frac{dR}{ds}, \tag{11.26}$$

and

$$\frac{d\beta}{ds} = \sin\tau - R\sin\tau\cdot\frac{1}{R} + \cos\tau\frac{dR}{ds} = \cos\tau\frac{dR}{ds}. \tag{11.27}$$

Dividing (11.27) by (11.26) gives

$$\frac{d\beta}{d\alpha} = -\cot\tau = -\frac{1}{\tan\tau} = -\frac{1}{\frac{dy}{dx}}. \tag{11.28}$$

But $\frac{d\beta}{d\alpha} = \tan\tau$ = slope of tangent to the evolute at C, and $\frac{dy}{dx} = \tan\tau$ = slope of tangent to the given curve at the corresponding point $P = (x, y)$.
Substituting the last two results in (11.28), we get

$$\tan\tau' = -\frac{1}{\tan\tau}.$$

Since the slope of one tangent is the negative reciprocal of the slope of the other, they are perpendicular. But a line perpendicular to the tangent at P is a normal to the curve. Hence

A normal to the given curve is a tangent to its evolute.

Again, squaring equations (11.26) and (11.27) and adding, we get

$$\left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2 = \left(\frac{dR}{ds}\right)^2. \tag{11.29}$$

But if s' = length of arc of the evolute, the left-hand member of (11.29) is precisely the square of $\frac{ds'}{ds}$ (from (8.12), where $t = s$, $s = s'$, $x = \alpha$, $y = \beta$). Hence (11.29) asserts that

$$\left(\frac{ds'}{ds}\right)^2 = \left(\frac{dR}{ds}\right)^2, \quad \text{or} \quad \frac{ds'}{ds} = \pm\frac{dR}{ds}.$$

That is, the radius of curvature of the given curve increases or decreases as fast as the arc of the evolute increases. In our figure this means that

$$P_1C_1 - PC = \text{arc } CC_1.$$

The length of an arc of the evolute is equal to the difference between the radii of curvature of the given curve which are tangent to this arc at its extremities.

Thus in Example 11.11.4, we observe that if we fold Q_vP_v ($= 4a$) over to the left on the evolute, P_v will reach to O', and we have:

The length of one arc of the cycloid (as $OO'Q_v$) is eight times the length of the radius of the generating circle.

11.13 Exercises

Find the coordinates of the center of curvature and the equation of the evolute of each of the following curves. Draw the curve and its evolute, and draw at least one circle of curvature.

1. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^1} = 1$.

   Ans. $\alpha = \frac{(a^2+b^2)x^3}{a^4}$, $\beta = -\frac{(a^2+b^2)y^3}{b^4}$; evolute $(a\alpha)^{\frac{2}{3}} - (b\beta)^{\frac{2}{3}} = (a^2+b^2)^{\frac{2}{3}}$.

2. The hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

   $\alpha = x + 3x^{\frac{1}{3}}y^{\frac{2}{3}}$, $\beta = y + 3x^{\frac{2}{3}}y^{\frac{1}{3}}$; evolute $(\alpha+\beta)^{\frac{2}{3}} + (\alpha-\beta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

3. Find the coordinates of the center of curvature of the cubical parabola $y^3 = a^2x$.

   Ans. $\alpha = \frac{a^4+15y^4}{6a^2y}$, $\beta = \frac{a^4y-9y^5}{2a^4}$.

4. Show that in the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ we have the relation $\alpha + \beta = 3(x+y)$.

5. Given the equation of the equilateral hyperbola $2xy = a^2$ show that

$$\alpha + \beta = \frac{(y+x)^3}{a^2}, \alpha - \beta = \frac{(y-x)^3}{a^2}.$$

From this derive the equation of the evolute $(\alpha+\beta)^{\frac{2}{3}} - (\alpha-\beta)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

Find the parametric equations of the evolutes of the following curves in terms of the parameter t. Draw the curve and its evolute, and draw at least one circle of curvature.

6. The hypocycloid $\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t. \end{cases}$

Ans. $\begin{cases} \alpha = a\cos^3 t + 3a\cos t\sin^2 t, \\ \beta = 3a\cos^2 t\sin t + a\sin^3 t. \end{cases}$.

7. The curve $\begin{cases} x = 3t^2, \\ y = 3t - t^3. \end{cases}$

Ans. $\begin{cases} \alpha = \frac{3}{2}(1 + 2t^2 - t^4), \\ \beta = -4t^3. \end{cases}$

8. The curve $\begin{cases} x = a(\cos t + t\sin t), \\ y = a(\sin t - t\cos t). \end{cases}$.

Ans. $\begin{cases} \alpha = a\cos t, \\ \beta = a\sin t. \end{cases}$.

9. The curve $\begin{cases} x = 3t, \\ y = t^2 - 6. \end{cases}$.

Ans. $\begin{cases} \alpha = -\frac{4}{3}t^3, \\ \beta = 3t^2 - \frac{3}{2}. \end{cases}$.

10. The curve $\begin{cases} x = 6 - t^2 \\ y = 2t. \end{cases}$.

Ans. $\begin{cases} \alpha = 4 - 3t^2, \\ \beta = -2t^3. \end{cases}$.

11. The curve $\begin{cases} x = 2t, \\ y = t^2 - 2. \end{cases}$.

Ans. $\begin{cases} \alpha = -2t^3, \\ \beta = 3t^2. \end{cases}$.

12. The curve $\begin{cases} x = 4t, \\ y = 3 + t^2. \end{cases}$.

Ans. $\begin{cases} \alpha = -t^3, \\ \beta = 11 + 3t^2. \end{cases}$.

13. The curve $\begin{cases} x = 9 - t^2, \\ y = 2t. \end{cases}$.

Ans. $\begin{cases} \alpha = 7 - 3t^2, \\ \beta = -2t^3. \end{cases}$.

14. The curve $\begin{cases} x = 2t, \\ y = \frac{1}{3}t^3. \end{cases}$.

Ans. $\begin{cases} \alpha = \frac{4t - t^5}{4}. \\ \beta = \frac{12 + 5t^4}{6t}. \end{cases}$.

15. The curve $\begin{cases} x = \frac{1}{3}t^3, \\ y = t^2. \end{cases}$.

Ans. $\begin{cases} \alpha = \frac{4t^3 + 12t}{3} \\ \beta = -\frac{2t^2 + t^4}{2}. \end{cases}$.

16. The curve $\begin{cases} x = 2t, \\ y = \frac{3}{t}. \end{cases}$.

Ans $\begin{cases} \alpha = \frac{12t^4 + 9}{4t^3} \\ \beta = \frac{27 + 4t^4}{6t}. \end{cases}$.

17. $x = 4 - t^2$, $y = 2t$.
18. $x = 2t$, $y = 16 - t^2$.
19. $x = t$, $y = \sin\ t$.
20. $x = \frac{4}{t}$, $y = 3t$.
21. $x = t^2$, $y = \frac{1}{6}t^3$.
22. $x = t$, $y = t^3$.
23. $x = \sin t$, $y = 3\cos t$.

24. $x = 1 - \cos t$, $y = t - \sin t$.
25. $x = \cos^4 t$, $y = \sin^4 t$.
26. $x = a \sec t$, $y = b \tan t$.

CHAPTER

TWELVE

Appendix: Collection of formulas

12.1 Formulas for reference

For the convenience of the reader we give the following list of elementary formulas from Algebra, Geometry, Trigonometry, and Analytic Geometry.

1. Binomial Theorem (n being a positive integer):

$$\begin{aligned}(a+b)^n = a^n + na^{n-1}b \quad &+ \quad \tfrac{n(n-1)}{2!}a^{n-2}b^2 + \tfrac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots \\ &+ \quad \tfrac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1} + \cdots\end{aligned}$$

2. $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-1)n$.

3. In the quadratic equation $ax^2 + bx + c = 0$,

   when $b^2 - 4ac > 0$, the roots are real and distinct;

   when $b^2 - 4ac = 0$, the roots are real and equal;

   when $b^2 - 4ac < 0$, the roots are complex.

4. When a quadratic equation is reduced to the form $x^2 + px + q = 0$,

   p = sum of roots with sign changed, and

   q = product of roots.

5. In an arithmetical series, a, $a+d$, $a+2d$, ...,

$$s = \sum_{i=0}^{n-1} a + id = \frac{n}{2}[2a + (n-1)d].$$

6. In a geometrical series, a, ar, ar^2, ...,

$$s = \sum_{i=0}^{n-1} ar^i = \frac{a(r^n - 1)}{r - 1}.$$

7. $\log ab = \log a + \log b$.

8. $\log \frac{a}{b} = \log a - \log b$.

9. $\log a^n = n \log a$.

10. $\log \sqrt[n]{a} = \frac{1}{n} \log a$.

11. $\log 1 = 0$.

12. $\log e = 1$.

13. $\log \frac{1}{a} = -\log a$.

14. [1] Circumference of circle = $2\pi\, r$.

15. Area of circle = $\pi\, r^2$.

16. Volume of prism = Ba.

17. Volume of pyramid = $\frac{1}{3}Ba$.

18. Volume of right circular cylinder = $\pi\, r^2 a$.

19. Lateral surface of right circular cylinder = $2\pi\, ra$.

20. Total surface of right circular cylinder = $2\pi\, r(r+a)$.

21. Volume of right circular cone = $2\pi\, r(r+a)$.

[1] In formulas 14-25, r denotes radius, a altitude, B area of base, and s slant height.

22. Lateral surface of right circular cone = $\pi\, rs$.

23. Total surface of right circular cone = $\pi\, r(r+s)$.

24. Volume of sphere = $\frac{4}{3}\pi\, r^3$.

25. Surface of sphere = $4\pi\, r^2$.

26. $\sin x = \frac{1}{\csc x}$;
 $\cos x = \frac{1}{\sec x}$;
 $\tan x = \frac{1}{\cot x}$.

27. $\tan x = \frac{\sin x}{\cos x}$;
 $\cot x = \frac{\cos x}{\sin x}$.

28. $\sin^2 x + \cos^2 x = 1$;
 $1 + \tan^2 x = \sec^2 x$;
 $1 + \cot^2 x = \csc^2 x$.

29. $\sin x = \cos\left(\frac{\pi}{2} - x\right)$;
 $\cos x = \sin\left(\frac{\pi}{2} - x\right)$;
 $\tan x = \cot\left(\frac{\pi}{2} - x\right)$.

30. $\sin(\pi\ - x) = \sin x$;
 $\cos(\pi\ - x) = -\cos x$;
 $\tan(\pi\ - x) = -\tan x$.

31. $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

32. $\sin(x-y) = \sin x \cos y - \cos x \sin y$.

33. $\cos(x \pm y) = \cos x \cos y + \mp \sin x \sin y$.

34. $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.

35. $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$.

36. $\sin 2x = 2 \sin x \cos x$; $\cos 2x = \cos^2 x - \sin^2 x$; $\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$.

37. $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$; $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$; $\tan x = \frac{2\tan\frac{1}{2}x}{1-\tan^2\frac{1}{2}x}$.

38. $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$; $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.

39. $1 + \cos x = 2\cos^2\frac{x}{2}$; $1 - \cos x = 2\sin^2\frac{x}{2}$.

40. $\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$; $\cos x/2 = \pm\sqrt{\frac{1+\cos x}{2}}$; $\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$.

41. $\sin x + \sin y = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.

42. $\sin x - \sin y = 2\cos\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$.

43. $\cos x + \cos y = -2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$.

44. $\cos x - \cos y = -2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$.

45. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$; Law of Sines.

46. $a^2 = b^2 + c^2 2bc\cos A$; Law of Cosines.

47. $d = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$; distance between points (x_1, y_1) and (x_2, y_2).

48. $d = \frac{Ax_1+By_1+C}{\pm\sqrt{A^2+B^2}}$; distance from line $Ax + By + C = 0$ to (x_1, y_1).

49. $x = \frac{x_1+x_2}{2}$, $y = \frac{y_1+y_2}{2}$; coordinates of middle point.

50. $x = x_0 + x'$, $y = y_0 + y'$; transforming to new origin (x_0, y_0).

51. $x = x'\cos\theta - y'\sin\theta$, $y = x'\sin\theta + y'\cos\theta$; transforming to new axes making the angle theta with old.

52. $x = \rho\ \cos\theta$, $y = \rho\ \sin\theta$; transforming from rectangular to polar coordinates.

53. $\rho = \sqrt{x^2+y^2}$, $\theta = \arctan\frac{y}{x}$; transforming from polar to rectangular coordinates.

54. Different forms of equation of a straight line:

    (a) $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$, two-point form (or point-point form);
    (b) $\frac{x}{a} + \frac{y}{b} = 1$, intercept form;

(c) $y - y_1 = m(x - x_1)$, slope-point form;

(d) $y = mx + b$, slope-intercept form;

(e) $x\cos\alpha + y\sin\alpha = p$, normal form ($\alpha$ is the angle the normal line crosses the x-axis and p is the length of the shortest segment between the line in question and the origin);

(f) $Ax + By + C = 0$, general form.

55. $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$, angle between two lines whose slopes are m_1 and m_2.

 $m_1 = m_2$ when lines are parallel, and

 $m_1 = -\frac{1}{m_2}$ when lines are perpendicular.

56. $(x - \alpha)^2 + (y - \beta)^2 = r^2$, equation of circle with center (α, β) and radius r.

Many of these facts are already known to Sage :

Sage

```
sage: a,b = var("a,b")
sage: log(sqrt(a))
log(a)/2
sage: log(a/b).simplify_log()
log(a) - log(b)
sage: sin(a+b).simplify_trig()
cos(a)*sin(b) + sin(a)*cos(b)
sage: cos(a+b).simplify_trig()
cos(a)*cos(b) - sin(a)*sin(b)
sage: (a+b)^5
(b + a)^5
sage: expand((a+b)^5)
b^5 + 5*a*b^4 + 10*a^2*b^3 + 10*a^3*b^2 + 5*a^4*b + a^5
```

"Under the hood" Sage used Maxima to do this simplification.

12.2 Greek alphabet

letters	**names**	**letters**	**names**
A, α	alpha	N, ν	nu
B, β	beta	Ξ, ξ	xi
Γ, γ	gamma	O, o	omicron
Δ, δ	delta	Π, π	pi
E, ϵ	epsilon	P, ρ	rho
Z, ζ	zeta	Σ, σ	sigma
H, η	eta	T, τ	tau
Θ, θ	theta	Y, υ	upsilon
I, ι	iota	Φ, ϕ	phi
K, κ	kappa	X, χ	chi
Λ, λ	lambda	Ψ, ψ	psi
M, μ	mu	Ω, ω	omega

12.3 Rules for signs of the trigonometric functions

Quadrant	Sin	Cos	Tan	Cot	Sec	Csc
First	+	+	+	+	+	+
Second	+	-	-	-	-	+
Third	-	-	+	+	-	-
Fourth	-	+	-	-	+	-

12.4 Natural values of the trigonometric functions

Angle in Radians	Angle in Degrees	Sin	Cos	Tan	Cot	Sec	Csc
0	0	0	1	0	∞	1	∞
$\frac{\pi}{6}$	30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4}$	45	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
$\frac{\pi}{2}$	90	1	0	∞	0	∞	1
π	180	0	-1	0	∞	-1	∞
$\frac{3\pi}{2}$	270	-1	0	∞	0	∞	-1
2π	360	0	1	0	∞	1	∞

You can create a table of trig values (for $0 < \theta < \pi/4$ in radians) like this in Sage :

Sage

```
sage: RR15 = RealField(15)
sage: rads1 = [n*0.0175 for n in range(1,6)]
sage: rads2 = [0.0875+n*0.0875 for n in range(1,9)]
sage: rads = rads1+rads2
sage: trigs = ['radian', "sin", "cos", "tan", "cot"]
sage: tbl = [[RR(y)]+[RR15(eval(x+"(%s)"%y)) for x in trigs[1:]] for y in rads]
sage: print trigs; print Matrix(tbl)
[0.01750  0.9998 0.01750   57.14]
[0.03499  0.9994 0.03502   28.56]
[0.05247  0.9986 0.05255   19.03]
[0.06994  0.9976 0.07011   14.26]
[0.08739  0.9962 0.08772   11.40]
[ 0.1741  0.9847  0.1768   5.656]
[ 0.2595  0.9658  0.2687   3.722]
[ 0.3429  0.9394  0.3650   2.740]
[ 0.4237  0.9058  0.4677   2.138]
[ 0.5012  0.8653  0.5792   1.726]
[ 0.5749  0.8182  0.7026   1.423]
[ 0.6442  0.7648  0.8423   1.187]
[ 0.7086  0.7056   1.004  0.9958]
```

The first column are the values of $\sin(x)$ at $x \in \{0.01750, 0.03500, ...0.7875\}$ (measured in radians). The second, third and fourth rows are the corresponding values for $\cos$, $\tan$ and $\cot$, respectively.

CHAPTER

THIRTEEN

Appendix: A mini-Sage tutorial

The goal for this chapter[1] is to introduce to a Sage –newcomer some ways of using Sage in calculus, emphasizing examples over detailed explanations or programming background. We hope that you will consult the more detailed documentation, such as the Sage Tutorial [T], available (free) on the Sage website if you want to learn more.

What is Sage ?

First, if you are a newcomer to Sage then welcome!

Sage is a free, open-source mathematics software that supports research and teaching in algebra, geometry, number theory, cryptography, numerical computation, and related areas. Both the Sage development model and the technology in Sage itself are distinguished by an extremely strong emphasis on openness, community, cooperation, and collaboration: we are building the car, not reinventing the wheel.

For an undergraduate student needing mathematical software, Sage does basically the same type of computations you would use `Maple` or `Mathematica` for[2], but it is free. Even the heavily discounted student price of these programs

[1]Much of the material in this chapter appears in the Tutorial on the Sage website. The author of the Tutorial is The Sage Group and the tutorial is licensed under the Creative Commons attribution license, `http://creativecommons.org/licenses/by/3.0/us/`.

[2]Since these mathematical software programs are different, some problems can be solved more

can be weeks (or more) of a students' salary[3]? . Sage is easy to use (that is, for a beginner it is at least as easy to use as the commercial "competition") and costs a lot less!

A main goal for Sage is to create the best available software for (among many other mathematical topics)

- number theory ("What's the 10 millionth prime?"),
- algebra ("How many legal positions does the Rubik's cube have?"),
- geometry ("What is an algebraic equation describing the intersection of a sphere and a cone?"),
- probability/statistics ("What is the probability of a royal straight flush in 5-card stud poker game?"), and
- numerical computation ("What is the 10 millionth digit of π"?),

using the best possible GPL-compatible (open source) software. Currently, Sage includes are: `Maxima` (for calculus and other symbolic computation), `Singular` (for algebra), `R` (for statistics), `Pari` (for number theory), `GAP` (for more algebra!), `SciPy` (for numerical computation), and over 60 more. Sage is headed by the mathematician William Stein, who is at the University of Washington, in Seattle. Sage is free and open source and will *always remain so*.

Though much of Sage is implemented using Python, no Python background is needed to read this chapter nor to follow the examples in this book. However, to become expert in Sage you will want to learn Python (a great language, used at places such as Google and Industrial Light and Magic) at some point.

13.1 Ways to Use Sage

You can use Sage in several ways.

- **Notebook graphical interface,**

easily or faster in Sage that the others, and conversely. However, for *most* of what you will need to do, the functionality and speed is about the same.

[3]Though costs are an important and practical matter, we will be leaving the cost aside, and arguing for Sage purely on the basis of quality, openness and customizability. However, it is worth noting that the latest license for `Mathematica` is reported by `Wikipedia` to be at least 2500 US dollars. Sage is free.

- **Interactive command line,**
- **Programs:** By writing interpreted and compiled programs in Sage /Python, and
- **Scripts:** by writing stand-alone Python scripts that use the Sage library.

The first two mentioned will be discussed below. For the latter two ways of using Sage , please see the Sage Tutorial [T] as they are more advanced than what is needed here.

Here is a brief Sage example, to illustrate the ease-of-use and some capabilites. (More examples will be given later, but for a more complete tutorial, please see [T].) To find the area under the curve $y = x^2$ from $x = 0$ to $x = 1$, your can type in the following commands to see that Sage tells you the answer is $1/3 = 0.333\ldots$.

Sage

```
sage: x = var('x')
sage: integral(x^2,x,0,1)
1/3
sage: n(integral(x^2,x,0,1))
0.333333333333333
```

If you use the Sage Notebook (described later) then you can use Sage to create an interactive application which allows you to approximate this area using mid-point based rectangles. This is illustrated in Figure 13.1.

Use your mouse to move the slider to vary the level of the approximation. The Sage code can be found at [W].

13.2 Longterm Goals for Sage

- **Useful**: Sage 's intended audience is mathematics students (from high school to graduate school), teachers, and research mathematicians. The aim is to provide software that can be used to explore and experiment with mathematical constructions in algebra, geometry, number theory, calculus,

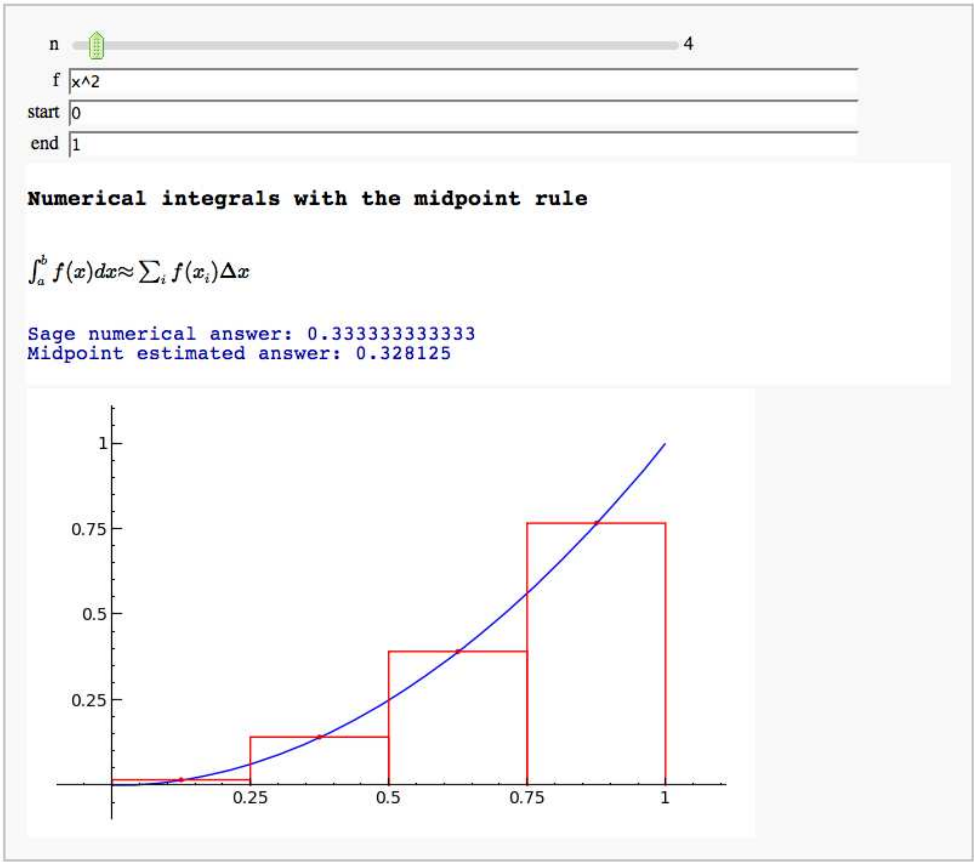

Figure 13.1: Approximate area under $y = x^2$

numerical computation, etc. Sage helps make it easier to interactively experiment with mathematical objects.

- **Efficient:** Be fast. Sage uses highly-optimized mature software like GMP, PARI, GAP, and NTL, and so is very fast at certain operations.

- **Free and open source:** The source code must be freely available and readable, so users can understand what the system is really doing and more easily extend it. Just as mathematicians gain a deeper understanding of a theorem by carefully reading or at least skimming the proof, people who do computations should be able to understand how the calculations work by reading documented source code. If you use Sage to do computations in a paper you publish, you can rest assured that your readers will always have free access to Sage and all its source code, and you are even allowed to archive and re-distribute the version of Sage you used.

- **Easy to compile:** Sage should be easy to compile from source for Linux, OS X and Windows users. This provides more flexibility for users to modify the system.
- **Cooperation:** Provide robust interfaces to most other computer algebra systems, including PARI, GAP, Singular, Maxima, KASH, Magma, Maple, and Mathematica. Sage is meant to unify and extend existing math software.
- **Well documented:** Tutorial, programming guide, reference manual, and how-to, with numerous examples and discussion of background mathematics.
- **Extensible:** Be able to define new data types or derive from built-in types, and use code written in a range of languages.
- **User friendly**: It should be easy to understand what functionality is provided for a given object and to view documentation and source code. Also attain a high level of user support.

13.3 The Sage command line

The session below shows and example of "tab-completion": start typing the beginning of a command and hit the TAB key. Sage will then return a list of possible completions. *Very* handy!

When you start Sage you will get a small Sage banner and then the Sage command-line prompt `sage:`. If you want to use the graphical user interface (GUI), type `notebook()` at the prompt and hit return. If you are happy to work at the command line, here is an example of what a short Sage session could look like:

Sage

```
sage: 2^3
8
sage: t = var("t")
sage: integrate(t*sin(t^2),t)
-cos(t^2)/2
```

```
sage: plot[TAB]
plot                    plot_slope_field     plotkin_bound_asymp
plot3d                  plot_vector_field    plotkin_upper_bound
```

13.4 The Sage notebook

The Sage Notebook can be tried out for free by anyone with an internet connection and a good browser at `http://www.sagenb.org` (this also works with the `iPhone` but not all cell-phones are configured for this).

The following screenshot illustrates a Notebook worksheet. Worksheets can be saved (as text or as an `sws` file in Sage worksheet format), downloaded, emailed (for use by someone else), shared (with "collaborators"), or published (if created on a public Sage server).

- Connect to Sage running locally *or elsewhere* (via ethernet).
- Create embedded graphics (in 2- and 3-d).
- Typeset mathematical expressions using LaTeX.
- Add and delete input, re-executing entire block of commands at once.
- Start and interrupt multiple calculations at once.
- The notebook also works with `Maxima`, `Python`, `R`, `Singular`, LaTeX, html, etc.!

Here are the commands used to create the output in the Notebook session in the

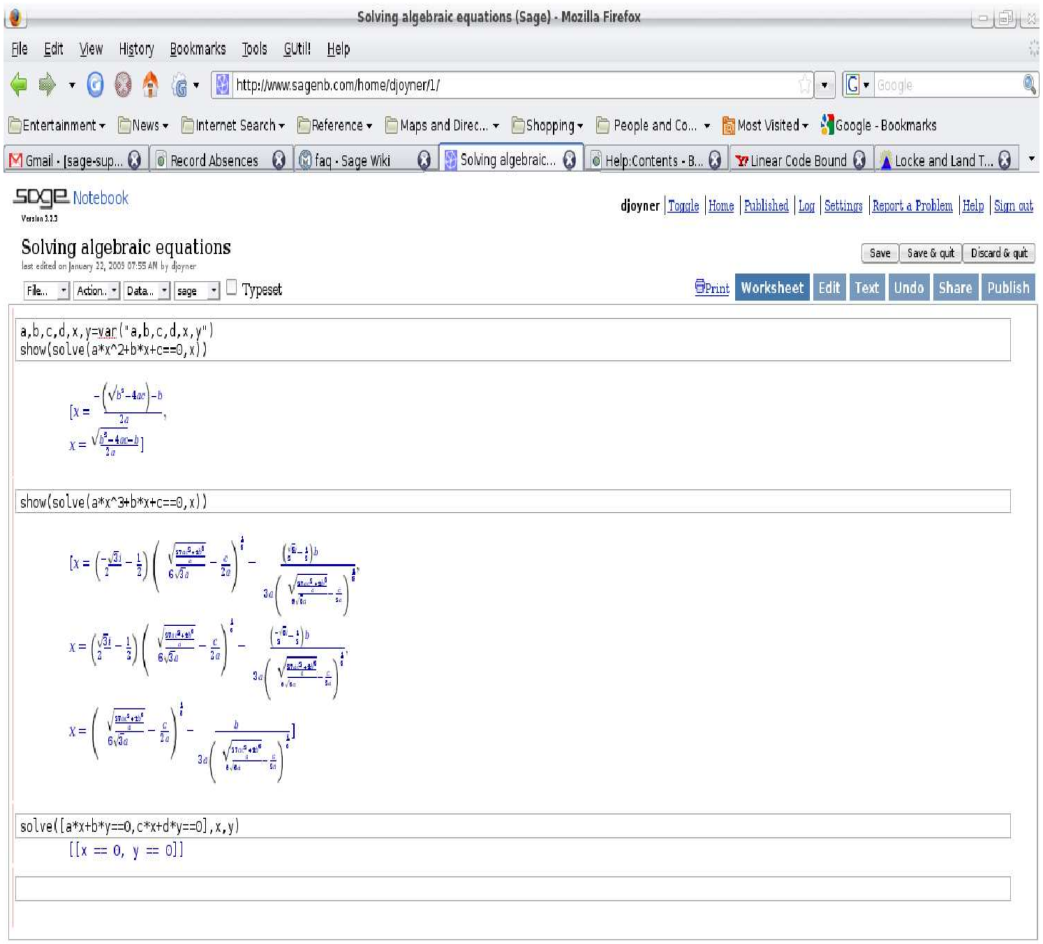

Figure 13.2: Sage screenshot

above screenshot:

Sage Notebook

```
a,b,c,d,x,y=var('a,b,c,d,x,y')
show(solve(a*x^2+b*x+c==0,x))
show(solve(a*x^3+b*x+c==0,x))
solve(a*x+b*y==0,c*x+d*y==0,x,y)
```

Sage notebook screenshot (an uploaded *.sws file)

• If you enjoy playing with the Rubik's cube, there are several programs for solving the Rubik's cube in Sage :

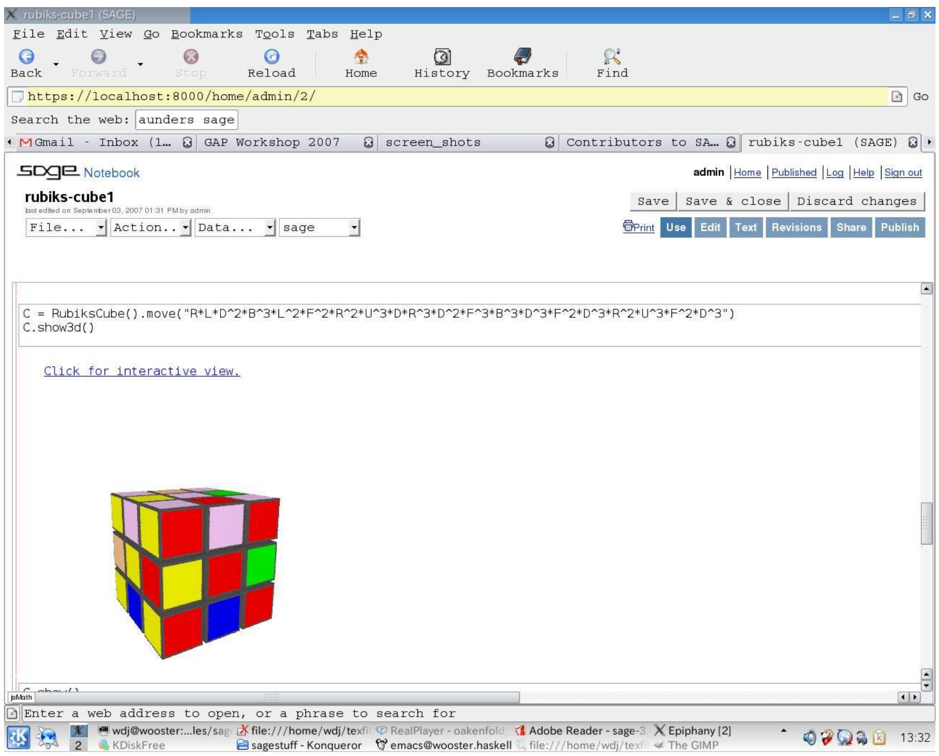

Figure 13.3: Sage notebook screenshot.

You can rotate the Rubik's cube interactively with your mouse.

• If you are interested in experimenting with calculus, Sage has excellent symbolic computation and graphics capabilities. This is a Notebook session (to be typed in a notebook "cell", then executed):

Sage Notebook

```
var('x')
RR = RealField(15)
f   = sin(x)*e^(-x)
p   = plot(f,-1,5, thickness=2)
pt_list = (RR(0),RR(0.25),RR(0.5),RR(0.75),RR(1),RR(1.25),RR(1.5),RR(1.75),\
           RR(2),RR(2.25),RR(2.5),RR(2.75),RR(3),RR(3.25),RR(3.5))
@interact
def _(pt=pt_list):
  dot = point((pt,f(pt)),pointsize=80,rgbcolor=(1,0,0))
  fp = f.diff()
  slope = fp(pt)
  sp = plot(f(pt)+slope*(x-pt),(x,-1, 5), color='green', thickness=2)
  html('<font color=red>Tangent to y = exp(-x)sin(x) at x = %s</font>'%RR(pt))
  show(dot + p + sp, ymin = -.5, ymax = 1)
```

When these are all typed in a single cell and executed, using javascript Sage displays an interactive graphic (see Figure 13.4) with a slider bar which allows you to vary the point at which the tangent line is drawn to the graph of $f(x) = e^{-x}\sin(x)$ using your mouse.

Such interactive commands are easy to write in Sage !

13.5 A guided tour

This section is a guided tour of some of what is available in Sage . For many more examples, see the Sage Reference Manual [R], which has thousands more examples. Also note that you can interactively work through this tour in the Sage notebook by clicking the `Help` link.

(If you are viewing the tutorial in the Sage notebook, press `shift-enter` to evaluate any input cell. You can even edit the input before pressing shift-enter. On some Macs you might have to press `shift-return` rather than `shift-enter`.)

13.5.1 Assignment, Equality, and Arithmetic

With some minor exceptions, Sage uses the Python programming language, so most introductory books on Python will help you to learn Sage .

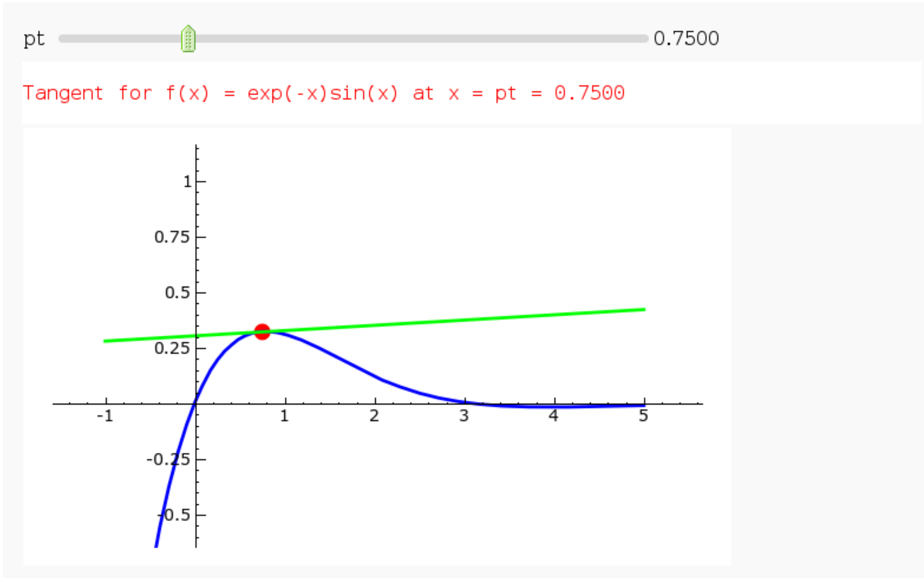

Figure 13.4: Interact example

Sage uses = for assignment. It uses ==, <=, >=, < and > for comparison:

Sage

```
sage: a = 5
sage: a
5
sage: 2 == 2
True
sage: 2 == 3
False
sage: 2 < 3
True
sage: a == 5
True
```

Sage provides all of the basic mathematical operations:

Sage

```
sage: 2**3 # ** means exponent
8
sage: 2^3 # ^ is a synonym for ** (unlike in Python)
8
sage: 10 % 3 # for integer arguments, % means mod, i.e., remainder 1
sage: 10/4 5/2
sage: 10//4 # for integer arguments, // returns the integer quotient 2
sage: 4 * (10 // 4) + 10 % 4 == 10 True
```

```
sage: 3ˆ2*4 + 2%5
38
```

The computation of an expression like `3ˆ2*4 + 2%5` depends on the order in which the operations are applied; this is specified in the "operator precedence table" in *Arithmetical binary operator precedence*.

Sage also provides many familiar mathematical functions; here are just a few examples:

Sage

```
sage: sqrt(3.4)
1.84390889145858
sage: sin(5.135)
-0.912021158525540
sage: sin(pi/3)
sqrt(3)/2
```

As the last example shows, some mathematical expressions return 'exact' values, rather than numerical approximations. To get a numerical approximation, use either the function `n` or the method `n` (and both of these have a longer name, `numerical_approx`, and the function `N` is the same as `n`)). These take optional arguments `prec`, which is the requested number of bits of precision, and `digits`, which is the requested number of decimal digits of precision; the default is 53 bits of precision.

Sage

```
sage: exp(2)
eˆ2
sage: n(exp(2))
7.38905609893065
sage: sqrt(pi).numerical approx()
1.77245385090552
sage: sin(10).n(digits=5)
-0.54402
sage: N(sin(10),digits=10)
-0.5440211109
sage: numerical approx(pi, prec=200)
3.1415926535897932384626433832795028841971693993751058209749
```

13.5.2 Getting Help

Sage has extensive built-in documentation, accessible by typing the name of a function or a constant (for example), followed by a question mark:

Sage

```
sage: tan?
Type:     <class sage.calculus.calculus.Function tan>
Definition:  tan( [noargspec] )
Docstring:

    The tangent function

    EXAMPLES:
        sage: tan(pi)
        0
        sage: tan(3.1415)
        -0.0000926535900581913
        sage: tan(3.1415/4)
        0.999953674278156
        sage: tan(pi/4)
        1
        sage: tan(1/2)
        tan(1/2)
        sage: RR(tan(1/2))
        0.546302489843790
        sage: sudoku?
    File:        sage/local/lib/python2.5/site-packages/sage/games/sudoku.py
    Type:        <type function>
    Definition:  sudoku(A)
    Docstring:

    Solve the 9x9 Sudoku puzzle defined by the matrix A.

    EXAMPLE:
        sage: A = matrix(ZZ,9, [5,0,0, 0,8,0, 0,4,9, 0,0,0, 5,0,0,
        0,3,0, 0,6,7, 3,0,0, 0,0,1, 1,5,0, 0,0,0, 0,0,0, 0,0,0, 2,0,8,
        0,0,0, 0,0,0, 0,0,0, 0,1,8, 7,0,0, 0,0,4, 1,5,0, 0,3,0, 0,0,2,
        0,0,0, 4,9,0, 0,5,0, 0,0,3])
        sage: A
        [5 0 0 0 8 0 0 4 9]
        [0 0 0 5 0 0 0 3 0]
        [0 6 7 3 0 0 0 0 1]
        [1 5 0 0 0 0 0 0 0]
        [0 0 0 2 0 8 0 0 0]
        [0 0 0 0 0 0 0 1 8]
        [7 0 0 0 0 4 1 5 0]
        [0 3 0 0 0 2 0 0 0]
        [4 9 0 0 5 0 0 0 3]
        sage: sudoku(A)
        [5 1 3 6 8 7 2 4 9]
        [8 4 9 5 2 1 6 3 7]
        [2 6 7 3 4 9 5 8 1]
        [1 5 8 4 6 3 9 7 2]
        [9 7 4 2 1 8 3 6 5]
        [3 2 6 7 9 5 4 1 8]
        [7 8 2 9 3 4 1 5 6]
        [6 3 5 1 7 2 8 9 4]
        [4 9 1 8 5 6 7 2 3]
```

13.5.3 Basic Algebra and Calculus

Sage can perform various computations related to basic algebra and calculus: for example, finding solutions to equations, differentiation, integration, and plotting. See the "Sage reference manual" for more examples.

Solving Equations

Solving Equations Exactly

The `solve` function solves equations. To use it, first specify some variables; then the arguments to `solve` are an equation (or a system of equations), together with the variables for which to solve:

Sage
```
sage: x = var(x)
sage: solve(x2 + 3*x + 2, x)
[x == -2, x == -1]
```

You can solve equations for one variable in terms of others:

Sage
```
sage: x, b, c = var(x b c)
sage: solve([x2 + b*x + c == 0],x)
[x == (-sqrt(b2 - 4*c) - b)/2, x == (sqrt(b2 - 4*c) - b)/2]
```

You can also solve for several variables:

Sage
```
sage: x, y = var(x, y)
sage: solve([x+y==6, x-y==4], x, y)
[[x == 5, y == 1]]
```

In the following example, first we solve the system symbolically:

Sage
```
sage: var(x y p q)
(x, y, p, q)
sage: eq1 = p+q==9
sage: eq2 = q*y+p*x==-6
sage: eq3 = q*y2+p*x2==24
```

```
sage: solve([eq1,eq2,eq3,p==1],p,q,x,y)
[[p == 1, q == 8, x == (-4*sqrt(10) - 2)/3,
y == (sqrt(2)*sqrt(5) - 4)/6],
[p == 1, q == 8, x == (4*sqrt(10) - 2)/3,
y == (-sqrt(2)*sqrt(5) - 4)/6]]
```

For numerical approximations of the solutions, you can instead use:

Sage

```
sage: solns = solve([eq1,eq2,eq3,p==1],p,q,x,y, solution dict=True)
sage: [[s[p].n(30), s[q].n(30), s[x].n(30), s[y].n(30)] for s in solns]
[[1.0000000, 8.0000000, -4.8830369, -0.13962039],
[1.0000000, 8.0000000, 3.5497035, -1.1937129]]
```

(The function `n` prints a numerical approximation, and the argument is the number of bits of precision.)

Solving Equations Numerically

Often times, `solve` will not be able to find an exact solution to the equation or equations specified. When it fails, you can use `find_root` to find a numerical solution. For example, solve does not return anything intersting for the following equation:

Sage

```
sage: theta = var(theta)
sage: solve(cos(theta)==sin(theta))
[sin(theta) == cos(theta)]
```

On the other hand, we can use `find_root` to find a solution to the above equation in range $0 < \theta < \pi/2$:

Sage

```
sage: find root(cos(theta)==sin(theta),0,pi/2)
0.78539816339744839
```

Differentiation, Integration, etc.

Sage knows how to differentiate and integrate many functions. For example, to differentiate $\sin(u)$ with respect to u, do the following:

Sage

```
sage: u = var(u)
sage: diff(sin(u), u)
cos(u)
```

To compute the fourth derivative of $\sin(x^2)$:

Sage

```
sage: diff(sin(x2), x, 4)
16*x4*sin(x2) - 12*sin(x2) - 48*x2*cos(x2)
```

To compute the partial derivatives of $x^2 + 17y^2$ with respect to *x* and *y*, respectively:

Sage

```
sage:
sage:
sage:
2*x
sage:
34*y
x, y = var(x,y)
f = x2 + 17*y2
f.diff(x)
f.diff(y)
```

We move on to integrals, both indefinite and definite. To compute $\int x \sin(x^2)\, dx$ and $\int_0^1 \frac{x}{x^2+1}\, dx$

Sage

```
sage: integral(x*sin(x2), x)  # Sage always omits the ’’+C’’
-cos(x2)/2
sage: integral(x/(x2+1), x, 0, 1)
log(2)/2
```

To compute the partial fraction decomposition of $\frac{1}{x^2-1}$:

Sage

```
sage: f = 1/((1+x)*(x-1))
sage: f.partial fraction(x)
1/(2*(x - 1)) - 1/(2*(x + 1))
sage: print f.partial fraction(x)
    1           1
--------- - ---------
2 (x - 1)   2 (x + 1)
```

13.5.4 Plotting

Sage can produce two-dimensional and three-dimensional plots.

Two-dimensional Plots

In two dimensions, Sage can draw circles, lines, and polygons; plots of functions in rectangular coordinates; and also polar plots, contour plots and vector field plots. We present examples of some of these here. For more examples of plotting with Sage, see also the Sage Reference Manual [R].

This command produces a yellow circle of radius 1, centered at the origin:

Sage
```
sage: circle((0,0), 1, rgbcolor=(1,1,0))
```

You can also produce a filled circle:

Sage
```
sage: circle((0,0), 1, rgbcolor=(1,1,0), fill=True)
```

You can also create a circle by assigning it to a variable; this does not plot it:

Sage
```
sage: c = circle((0,0), 1, rgbcolor=(1,1,0))
```

To plot it, use `c.show()` or `show(c)`, as follows:

Sage
```
sage: c.show()
```

Alternatively, evaluating `c.save('filename.png')` will save the plot to the given file.

Now, these 'circles' look more like ellipses because the axes are scaled differently. You can fix this:

Sage
```
sage: c.show(aspect ratio=1)
```

The command `show(c, aspect_ratio=1)` accomplishes the same thing, or you can save the picture using `c.save('filename.png', aspect_ratio=1)`.

It's easy to plot basic functions:

Sage
```
sage: plot(cos, (-5,5))
```

Once you specify a variable name, you can create parametric plots also:

Sage
```
sage: x = var(x)
sage: parametric plot((cos(x),sin(x)3),0,2*pi,rgbcolor=hue(0.6))
```

You can combine several plots by adding them:

Sage
```
sage:
sage:
sage:
sage:
sage:
x = var(x)
p1 = parametric plot((cos(x),sin(x)),0,2*pi,rgbcolor=hue(0.2))
p2 = parametric plot((cos(x),sin(x)2),0,2*pi,rgbcolor=hue(0.4))
p3 = parametric plot((cos(x),sin(x)3),0,2*pi,rgbcolor=hue(0.6))
show(p1+p2+p3, axes=false)
```

A good way to produce filled-in shapes is to produce a list of points (`L` in the example below) and then use the `polygon` command to plot the shape with boundary formed by those points. For example, here is a green deltoid:

Sage
```
sage: L = [[-1+cos(pi*i/100)*(1+cos(pi*i/100)),\
... 2*sin(pi*i/100)*(1-cos(pi*i/100))] for i in range(200)]
sage: polygon(L, rgbcolor=(1/8,3/4,1/2))
```

(You don't type the "..." above - they are filled in automatically by Sage when you type `\<shift-enter>`, which is a way you can create a newline without executing the Sage command.) Type `show(p, axes=false)` to see this without any axes.

You can add text to a plot:

Sage

```
sage: L = [[6*cos(pi*i/100)+5*cos((6/2)*pi*i/100),\
...    6*sin(pi*i/100)-5*sin((6/2)*pi*i/100)] for i in range(200)]
sage: p = polygon(L, rgbcolor=(1/8,1/4,1/2))
sage: t = text("hypotrochoid", (5,4), rgbcolor=(1,0,0))
sage: show(p+t)
```

Calculus teachers draw the following plot frequently on the board: not just one branch of arcsin but rather several of them: i.e., the plot of $y = \sin(x)$ for x between -2π and 2π, flipped about the 45 degree line. The following Sage commands construct this:

Sage

```
sage: v = [(sin(x),x) for x in srange(-2*float(pi),2*float(pi),0.1)]
sage: line(v)
```

Since the tangent function has a larger range than sine, if you use the same trick to plot the inverse tangent, you should change the minimum and maximum coordinates for the *x*-axis:

Sage

```
sage: v = [(tan(x),x) for x in srange(-2*float(pi),2*float(pi),0.01)]
sage: show(line(v), xmin=-20, xmax=20)
```

Sage also computes polar plots, contour plots and vector field plots (for special types of functions). Here is an example of a contour plot:

Sage

```
sage: f = lambda x,y: cos(x*y)
sage: contour plot(f, (-4, 4), (-4, 4))
```

Three-Dimensional Plots

Sage produces three-dimensional plots using an open source package called [Jmol]. Here are a few examples:

Yellow Whitney's umbrella http://en.wikipedia.org/wiki/Whitney_umbrella:

Sage

```
sage: u, v = var(u,v)
sage: fx = u*v
sage: fy = u
sage: fz = v2
sage: parametric plot3d([fx, fy, fz], (u, -1, 1), (v, -1, 1),\
...   frame=False, color="yellow")
```

Once you have evaluated `parametric_plot3d`, so that the plot is visible, you can click and drag on it to rotate the figure. Type `parametric_plot3d?` for more examples.

13.5.5 Some common issues with functions

Some aspects of defining functions (e.g., for differentiation or plotting) can be confusing. In this section we try to address some of the relevant issues.

Here are several ways to define things which might deserve to be called "functions":

1. Define a Python function (as described for example in the Sage Tutorial `http://www.sagemath.org/doc/tutorial/tour_functions.html` or the Python docs at `http://docs.python.org/tutorial/`). These functions can be plotted, but not always differentiated or integrated.

Sage

```
sage: def f(z): return z2
sage: type(f)
<type function>
sage: f(3)
9
sage: plot(f, 0, 2)
```

In the last line, note the syntax. Using `plot(f(z), 0, 2)` instead will give an error, because `z` is a dummy variable in the definition of `f` and is not defined outside of that definition. Indeed, just `f(z)` returns an error. The following will work in this case, although in general there are issues and so it should probably be avoided (see item 4 below).

Sage

```
sage: z = var("z")
sage: derivative(f(z),z)
2*z
sage: plot(f(z), 0, 2)
```

At this point, `f(z)` is a symbolic expression, the next item in our list.

2. Define a “callable symbolic expression”. These can be plotted, differentiated, and integrated.

Sage

```
sage: g(x) = x2
sage: g
# g sends x to x2
x |--> x2
sage: g(3)
9
sage: Dg = g.derivative(); Dg
x |--> 2*x
sage: Dg(3)
6
sage: type(g)
<class sage.calculus.calculus.CallableSymbolicExpression>
sage: plot(g, 0, 2)
```

Note that while `g` is a callable symbolic expression, `g(x)` is a related, but different sort of object, which can also be plotted, differentated, etc., albeit with some issues: see item 5 below for an illustration.

Sage

```
sage: type(g(x))
<class sage.calculus.calculus.SymbolicArithmetic>
sage: g(x).derivative()
2*x
sage: plot(g(x), 0, 2)
```

3. Use a pre-defined Sage ‘calculus function’. These can be plotted, and with a little help, differentiated, and integrated.

Sage

```
sage: type(sin)
<class sage.calculus.calculus.Function sin>
sage: plot(sin, 0, 2)
sage: type(sin(x))
<class sage.calculus.calculus.SymbolicComposition>
sage: plot(sin(x), 0, 2)
```

By itself, `sin` cannot be differentiated, at least not to produce `cos`.

Sage

```
sage: f = sin
sage: f.derivative()
0
```

Using `f = sin(x)` instead of `sin` works, but it is probably even better to use `f(x) = sin(x)` to define a callable symbolic expression.

Sage

```
sage: S(x) = sin(x)
sage: S.derivative()
x |--> cos(x)
```

Here are some common problems, with explanations:

4. Accidental evaluation.

Sage

```
sage: def h(x):
... if x<2:
...     return 0
... else:
...     return x-2
```

The issue: `plot(h(x), 0, 4)` plots the line $y = x - 2$, not the multi-line function defined by `h`. The reason? In the command `plot(h(x), 0, 4)`, first `h(x)` is evaluated: this means plugging `x` into the function `h`, which means that `x`$<$`2` is evaluated.

Sage

```
sage: type(x<2)
<class sage.calculus.equations.SymbolicEquation>
```

When a symbolic equation is evaluated, as in the definition of `h`, if it is not obviously true, then it returns False. Thus `h(x)` evaluates to `x-2`, and this is the function that gets plotted.

The solution: don't use `plot(h(x), 0, 4)`; instead, use

Sage

```
sage: plot(h, 0, 4)
```

5. Accidentally producing a constant instead of a function.

Sage

```
sage: f = x
sage: g = f.derivative()
sage: g
1
```

The problem: `g(3)`, for example, returns an error, saying “ValueError: the number of arguments must be less than or equal to 0.”

Sage

```
sage: type(f)
<class sage.calculus.calculus.SymbolicVariable>
sage: type(g)
<class sage.calculus.calculus.SymbolicConstant>
```

`g` is not a function, it’s a constant, so it has no variables associated to it, and you can’t plug anything into it.

The solution: there are several options.

- Define `f` initially to be a symbolic expression.

Sage

```
sage: f(x) = x
# instead of f = x
sage: g = f.derivative()
sage: g
x |--> 1
sage: g(3)
1
sage: type(g)
<class sage.calculus.calculus.CallableSymbolicExpression>
```

- Or with `f` as defined originally, define `g` to be a symbolic expression.

Sage

```
sage: f = x
sage: g(x) = f.derivative() # instead of g = f.derivative()
sage: g
x |--> 1
sage: g(3)
1
sage: type(g)
<class sage.calculus.calculus.CallableSymbolicExpression>
```

- Or with `f` and `g` as defined originally, specify the variable for which you are substituting.

Sage

```
sage: f = x
sage: g = f.derivative()
sage: g
1
sage: g(x=3) # instead of g(3)
1
```

Please see the Sage Tutorial for more details and examples.

13.6 Try it!

Sage users are a *community*. Please join us!

`http://www.sagemath.org/`

`http://www.sagenb.com/`

CHAPTER

FOURTEEN

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