A Constructive Algorithm for Determining Branching Rules of Lie Group Representations

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Abstract

We prove a branching rule for crystal graphs.

1 Introduction

Let $\mathfrak{g}_2$ be a complex Lie algebra and $\mathfrak{g}_1$ a subalgebra. Assume that $\mathfrak{g}_i$, $i = 1, 2$, are semi-simple and that there are Cartan subalgebras $\mathfrak{h}_i \subset \mathfrak{g}_i$, $i = 1, 2$, such that $\mathfrak{h}_1 \subset \mathfrak{h}_2$. Assume further that the half-spaces in the dual spaces $\mathfrak{h}_i^*$, $i = 1, 2$, determining the positive roots are compatible.

Let Λ_i denote the set of dominant integral weights of $\mathfrak{g}_i$ and let W_i denote the Weyl group of $\mathfrak{h}_i \subset \mathfrak{g}_i$, $i = 1, 2$. For a dominant weight $\lambda_i \in \Lambda_i$, let π_{λ_i} denote the corresponding irreducible representation of $\mathfrak{g}_i$, $i = 1, 2$. Let $m_\lambda(\pi_{\lambda'})$ denote the multiplicity with which the weight λ occurs in the restriction of $\pi_{\lambda'}$ to the Cartan subalgebra.

One method of obtaining information about the restriction of a representation of $\mathfrak{g}_2$ to $\mathfrak{g}_1$ is to use the following rule.

Klimyk's formula (See [FH, page 428], for example.): Let $n(\lambda_1, \lambda_2)$ denote the multiplicity of π_{λ_1} occurs in the restriction $\mathrm{res}_{\mathfrak{g}_1}(\pi_{\lambda_2})$ of π_{λ_2} to $\mathfrak{g}_1$, so

$$\mathrm{res}_{\mathfrak{g}_1}(\pi_{\lambda_2}) = \oplus_{\lambda_1 \in \Lambda_1} n(\lambda_1, \lambda_2)\pi_{\lambda_2}.$$

Then

$$n(\lambda_1, \lambda_2) = \sum_{w_1 \in W_1} sgn(w_1) \sum_{\substack{\lambda_2' \in \Lambda_2 \\ \mathrm{res}_{\mathfrak{g}_1}(\lambda_2') = \lambda_1 + \rho_1 - w_1(\rho_1)}} m_{\lambda_2'}(\pi_{\lambda_2}).$$

We shall give an alternative approach.

2 Crystal graphs

Let π denote a finite dimensional representation of $\mathfrak{g}$. For each simple root α of $\mathfrak{g}$, let X_α denote a corresponding root vector.

DEFINITION 2.1 (See [K1].): A *crystal basis* of π is a basis B of the $\mathfrak{h}$-module $\pi|_{\mathfrak{h}}$ (in particular, each element of B is a weight vector) with the following property: if $v \in B$ then either

(a) $\pi(\mathfrak{g})(v) = 0$ or

(b) there is a unique simple root α of $\mathfrak{g}$, a unique $v' \in B$, and a $c \neq 0$ such that $\pi(X_\alpha)(v) = cv'$.

DEFINITION 2.2 (See [K1].): The *crystal graph* of the representation π is a colored digraph $\Gamma = (V, E)$, where

1) the vertex set V is indexed by a crystal basis of π oriented from left to right beginning (at the left) with the basis vector of highest weight λ and arranging all isotypic basis vectors (i.e., basis vectors with the same weight) arbitrarily, and

2) the edge set E is defined by the condition that $v \in V$ is connected to $v^{'} \in V$ if and only if there is a simple root vector X of $\mathfrak{g}$ such that $\pi(X)v \in \mathbb{C} \cdot v^{'}$. In this case, we label or "color" this edge by X (or by the simple root associated to X).

Note that Γ is unique, $\mathrm{card}(V) = \dim(\mathfrak{g})$ and that, if Γ is connected, π is an irreducible representation of $\mathfrak{g}$.

THEOREM 2.3: If $\mathfrak{g}$ is semisimple then each finite dimensional representation of $\mathfrak{g}$ has a crystal basis.

For the proof of this, see for example [K2].

3 The Branching Algorithm

DEFINITION 3.1: A subgraph $\gamma \subset \Gamma$ is called a *branching graph* if γ is obtained from Γ by deleting all edges with labels in a subset of the set of simple roots.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. The decomposition of π when restricted to $\mathfrak{h}$ is called a *branching rule*.

Recall that any simple Lie algebra has a Dynkin diagram labeled by its simple roots ([H]).

THEOREM 3.1: The *branching rule* for $\pi(\mathfrak{g})$ with respect to $\mathfrak{h}$ is determined by the branching graph γ obtained from Γ by deleting all edges labeled with simple roots not contained in the Dynkin diagram of $\mathfrak{h}$.

proof: Consider the crystal graph of π_{λ_2}. The vertices of this graph Γ_2 are labeled by the weights of π_{λ_2}. Let v, v' denote two distinct vertices. There is a directed edge from v to v' if and only if there is a simple root α of $\mathfrak{g}_2$ and a root vector X_α of $\mathfrak{g}_2$ such that

$$\pi_{\lambda_2}(X_\alpha)(v) \in \mathbb{C} \cdot v'. \qquad (*)$$

Consider now the crystal graph of $\mathrm{res}_{\mathfrak{g}_1}(\pi_{\lambda_2})$. The vertices of this graph Γ_1 may be identified with the vertices of Γ_2. Since the roots of $\mathfrak{g}_1$ are, by assumption, compatible with those of $\mathfrak{g}_2$, the edges of Γ_1 may be regarded as a subset of the edges of Γ_2. An edge $v \stackrel{\alpha}{\to} v'$ of Γ_2 "survives" this restriction process to Γ_1 if and only if

(a) the root α corresponds to a simple root α_1 of $\mathfrak{g}_1$ by restriction from the Cartan algebra $\mathfrak{h}_2$ to $\mathfrak{h}_1$,

(b) (*) holds for this α_1, v, v'.

This completes the proof of the theorem. $\square$

For an implementation of these ideas using the computer algebra system MAPLE, see [JM1], [JM2].

4 References

[FH] Fulton and Harris, Representation Theory, Springer Verlag, New York, 1991.

[H] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer Verlag, New York, 1972.

[JM1] D. Joyner and R. Martin, *Branching rules for Lie algebra representations using crystal graphs*, Symbolic and Algebraic Computation Newsletter, no. 2, 1997.

[JM2] D. Joyner and R. Martin, the Maple package **crystal** available at http://www.nadn.navy.mil/MathDept/~wdj/papers.html .

[K1] Masaki Kashiwara, *Crystalizing the q-analog of universal enveloping algebras*, Comm. Math. Physics, **133** (1990) pp. 249-260.

[K2] Masaki Kashiwara, *On crystal bases of the q-analog of universal enveloping algebras*, Duke Math. J., **63** (1991) pp. 465-516.