Decomposing Lie Algebra Representations Using Crystal Graphs

David Joyner and Roland Martin

March 25, 2008

Abstract

We use the theory of crystal graphs to give a simple graph-theoretical algorithm for determining the branching rule for decomposing a representation of a simple Lie algebra when restricted to a simple subalgebra. We also describe a computer package for determining such decompositions graphically.

1 Introduction

When modeling elementary particle interactions and symmetry breaking in physics it becomes important to understand how tensor products of representations of simple Lie algebras decompose into irreducible subrepresentations and how irreducible representations decompose when restricted to simple subalgebras. The classification of irreducible finite dimensional representations of simple Lie algebras is well-understood ([2]). In principal, this classification yields straightforward, if cumbersome, algorithms for decomposing

1) tensor products of such representations and

2) representations when restricted to simple subalgebras.

The purpose of this paper is to present simpler algorithms for solving these problems using crystal graphs ([5]). We, further, describe an implementation of these algorithms in a Maple package ([3],[4]).

2 Crystal Graphs

Let $\mathcal{G}$ be a simple Lie algebra with highest weight λ. Let π denote a finite dimensional representation of $\mathcal{G}$.

DEFINITION 2.1: The *crystal graph* of the representation $\pi(\mathcal{G})$ is a colored digraph $\Gamma = (V, E)$, where

1) the vertex set V is given by the partially ordered set of all weights of $\mathcal{G}$ oriented from left to right beginning with λ and

2) the edge set E is defined by the condition that $v \in V$ is connected to $v^{'} \in V$ if and only if there is a simple root vector X of $\mathcal{G}$ such that $\pi(X)v = v^{'}$. In this case, we label or "color" this edge by X (or by the simple root associated to X).

Note that Γ is unique, $\mathrm{card}(V) = \dim(\mathcal{G})$ and that, if Γ is connected, π is an irreducible representation of $\mathcal{G}$ ([5]). For example, in the case $\mathcal{G}$=$\mathcal{SL}_2$ and π equal the identity representation, Γ is given by

 .

Let $\pi^{'}$ denote a second finite dimensional representation of $\mathcal{G}$ with highest weight $\lambda^{'}$ and $\Gamma^{'}$ its associated crystal graph.

DEFINITION 2.2: The *crystal graph product*, $\Gamma \times \Gamma^{'}$, is formed by the following rules.

1) Form a rectangular grid of the vertices $V \times V^{'}$, indexing the grid vertically from bottom to top with the vertices of V and horizontally left to right with the vertices of $V^{'}$.

2) Set $i = 1$.

3) If $i \leq \mathrm{card}(V^{'})$, then goto step 4, else goto step 13.

4) Set $j = 1$.

5) If $j \leq \mathrm{card}(V)$, then goto step 6, else goto step 8.

6) For k from j to $\mathrm{card}(V)$ connect $(v_j, v_i^{'})$ and $(v_k, v_i^{'})$ with a labeled edge if v_j is connected to v_k in Γ with an edge of that label and there does not already exist an edge of that label in $\Gamma \times \Gamma^{'}$ with head $(v_j, v_i^{'})$.

7) Increment j and goto step 5.

8) Set $\ell = i + 1$.

9) If $\ell \leq \mathrm{card}(V^{'})$ then goto step 10, else step 12.

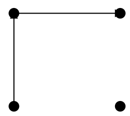
10) For j from 1 to $\mathrm{card}(V)$ connect $(v_j, v_i^{'})$ and $(v_j, v_\ell^{'})$ with a labeled edge if $v_i^{'}$ is connected to $v_\ell^{'}$ in $\Gamma^{'}$ with an edge of that label and there does not already exist an edge of that label in $\Gamma \times \Gamma^{'}$ with tail $(v_j, v_i^{'})$.

11) Increment ℓ and goto step 9.

12) Increment i and goto step 3.

13) End.

It is a theorem of Kashiwara ([5]) that the crystal graph product $\Gamma \times \Gamma^{'}$ is the crystal graph associated to the tensor product representation $\pi \otimes \pi^{'}(\mathcal{G})$. The irreducible representations of $\mathcal{G}$ associated to the connected components of $\Gamma \times \Gamma^{'}$ give the decomposition of $\pi \otimes \pi^{'}(\mathcal{G})$. For example, in the case where Γ is the crystal graph associated to the identity representation of $\mathcal{SL}_2$, $\Gamma \times \Gamma$ decomposes as


,

implying $\pi \otimes \pi(\mathcal{G})$ decomposes into a direct sum of a trivial one dimensional representation and a three dimensional representation. Note that the three node connected component of $\Gamma \times \Gamma$ contains the vertex (λ, λ). One concludes that this three node connected component is the crystal graph of the Cartan product $\pi * \pi(\mathcal{G})$ by canonically identifying (λ, λ) with the weight 2λ and by using the uniqueness property of crystal graphs. This result agrees exactly with the classical Clebsch-Gordon formula ([2]).

We remark that these definitions are essentially algorithms and are implemented in the Maple package **crystal** ([3],[4]).

3 The Branching Algorithm

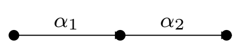
We continue with the notation of §2.

DEFINITION 3.1: A subgraph $\gamma \subset \Gamma$ is called a *branching graph* if γ is obtained from Γ by deleting all edges with labels in a subset of the set of simple roots.

Let $\mathcal{H} \subset \mathcal{G}$ be a subalgebra. The decomposition of $\pi(\mathcal{G})$ when restricted to $\mathcal{H}$ is called a branching rule. Recall that any simple Lie algebra has a Dynkin diagram labeled by its simple roots ([2]).

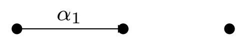
THEOREM 3.1: The *branching rule* for $\pi(\mathcal{G})$ with respect to $\mathcal{H}$ is determined by the branching graph γ obtained from Γ by deleting all edges labeled with simple roots not contained in the Dynkin diagram of $\mathcal{H}$.

As an example, consider the identity representation of $\mathcal{SL}_3$ and the subalgebra $\mathcal{SL}_2 \subset \mathcal{SL}_3$. Denote by α_1, α_2 the simple roots of $\mathcal{SL}_3$. The associated crystal graph Γ of $\mathcal{SL}_3$ is, then,



.

An examination of the Dynkin diagrams for $\mathcal{SL}_2$ and $\mathcal{SL}_3$ implies that the branching rule for $\mathcal{SL}_3$ with respect to $\mathcal{SL}_2$ is determined by the branching graph



obtained from Γ by deleting the edge labeled by α_2. In particular, $\mathcal{SL}_3$ decomposes into a direct sum of $\mathcal{SL}_2$ and a trivial, one dimensional representation when restricted to $\mathcal{SL}_2$.

To check our result, recall that every representation π of $\mathcal{SL}_n$ is obtained from n fundamental representations by means of the Cartan product ([2]). It follows that a representation π of $\mathcal{SL}_3$ is uniquely determined by a triplet of non-negative integers (a, b, c). The identity representation of $\mathcal{SL}_3$ is determined by $(1, 0, 0)$. The classical formula for the branching rule of $\mathcal{SL}_3$ with respect to $\mathcal{SL}_2$ is

$$\mathrm{Res}^{\mathcal{SL}_3}_{\mathcal{SL}_2}((1,0,0)) = \oplus(a', b')$$

where (a', b') determines a unique representation of $\mathcal{SL}_2$ with $1 \geq a' \geq 0 \geq b' \geq 0$ ([2]). Since the identity representation of $\mathcal{SL}_2$ is determined by $(1, 0)$ our result checks immediately with the classical formula.

We again remark that this theorem is essentially an algorithm and has been implemented in the Maple package **crystal** ([3],[4]). The validity of this algorithm will be proven in a future paper.

4 The crystal Package

The **crystal** package ([4]) contains programs to compute the crystal graph of a multiplicity free, irreducible representation associated to the highest weight of a fundamental representation, compute the crystal graph product of two crystal graphs, compute branching graphs and display crystal graphs. The **crystal** package is, by design, compatible with the **coxeter** and **weyl** packages of John Stembridge ([6]), and implicitly uses both of those packages.

For example, the **weyl** and **crystal** commands

```
weyl[weights](A2);
L1:=crystal[weight_system](-(1/3*e2-2/3*e1+1/3*e3),A2);
crystal[graphrep](L1,A2,G1);
crystal[showgraph](G1,1,3);
crystal[graphprodrep](G1,G1,G2);
crystal[showgraphprod](G2,1,3,1,3);
crystal[linsubgraphrep](2*e1,A2,G2,G3);
crystal[graphprodrep](G1,G3,G4);
crystal[showgraphprod](G4,1,3,1,6);
crystal[linsubgraphrep](v1X2,A2,G2,G5);
crystal[graphprodrep](G1,G5,G6);
```

crystal[showgraphprod](G6,1,3,1,3);

describe how the triple tensor product of the identity representation of $\mathcal{SU}_3$ with itself decomposes into a direct sum of a 10 dimensional representation, two eight dimensional representations and a trivial, one dimensional representation as is indicated by the crystal graphs G4 and G6. We remark that this example has important ramifications in particle physics because it provides supporting evidence for the existence of quarks ([1]). The interested reader may view G4 and G6, and other interesting crystal graphs at the world wide web address http://web.usna.navy.mil/~wdj/crystal2.htm .

The utility of this package is further enhanced by existing Maple commands. For example, $\mathcal{SO}_{10} \otimes \mathcal{SO}_{10}$ can be shown to decompose into a direct sum of a 54 dimensional representation, a 45 dimensional representation and a trivial representation by computing the crystal graph of $\mathcal{SO}_{10}$, computing the crystal graph product of $\mathcal{SO}_{10}$ with itself, and using the existing **networks**[components] and **networks**[vertices] commands.

The **crystal** package also provides the means of quickly deriving branching rules such as

$$\mathrm{Res}^{\mathcal{SL}_{n+m}}_{\mathcal{SL}_n}((1,0,\dots,0)) = \oplus(a_1^{'},\dots,a_n^{'})$$

where $1 \geq a_1^{'} \geq 0 \geq a_2^{'} \geq \dots \geq 0$.

5 References

[1] H. Georgi, Lie Algebras in Particle Physics, Benjamin/Cummings Publishing, London, 1982.

[2] J. E. Humphreys, Introduction to Lie Algebras and Representation Theory, Springer Verlag, New York, 1972.

[3] D. Joyner and R. Martin, *A Constructive Algorithm for Determining Branching Rules of Lie Group Representations*, preprint 1997.

[4] D. Joyner and R. Martin, **crystal**, available at the world wide web address http://web.usna.navy.mil/~wdj/papers.html

[5] Masaki Kashiwara, *Crystalizing the q-analog of universal enveloping algebras*, Comm. Math. Physics, **133** (1990) pp. 249-260.

[6] John Stembridge, *A Maple package for root systems and finite Coxeter groups*, preprint, 1992.