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SUMMATION OPERATORS AND 'EXPLICIT FORMULAS'

DAVID JOYNER*

Abstract. A general class of summation operators on $L^2(\mathbf{A})$ is analyzed using the spectral theorem. As an example of number-theoretical interest, Weil's 'explicit formula' is used to reformulate the Riemann hypothesis in terms of tempered distributions.

1 – Introduction

In this paper, a simple theory is given for the operators of the form

$$\phi(x) \mapsto \sum_{\gamma \in L} a(\gamma)\, \phi(x+\gamma)\,, \quad x \in \mathbf{R} \times \Pi \mathbf{Z}_p \subseteq \mathbf{A}\,,$$

where the $a(\gamma)$ are complex numbers satisfying $a(\gamma) \ll (1+|\gamma|_{\mathbf{A}})^C$, C a positive constant, $L \subseteq \mathbf{A}^\times/\mathbf{Q}^\times$ is usually a semigroup contained in the ideles of $\mathbf{Q}$, and the sum, using a suitable summability method, will usually be defined initially as a distribution-valued function then extended as an unbounded operator to a dense subspace of $L^2(\mathbf{R} \times \Pi \mathbf{Z}_p) \hookrightarrow L^2(\mathbf{A})$. These operators form a fairly broad class of unbounded normal operators, well-suited for explicit computations, and provide a useful source of examples.

A short summary of results follows. In Section 2, a classification is given for the sequence $\{a(\gamma)\}_{\gamma \in L}$ having the property that the above operator is unitarily equivalent to a certain multiplication operator. For such $a(\gamma)$, the invariant subspaces are easy to classify, its adjoint

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*NSF Fellow

operator can be explicitly determined, and its (usually continuous) spectrum can be given. Finally, in Section 3 a number-theoretic example is given concerning the Weil distribution.

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2 – Basic Theory

Some notation must be introduced. Let

$$h(x) = h_\infty(x_\infty)\, h_2(x_2) \ldots h_p(x_p) \ldots, \quad x \in \mathbf{A}\,, \tag{1}$$

where $h_\infty(t) = (2\pi)^{-1/2} \exp(-t^2/2)$, and

$$h_p(x_p) = \begin{cases} 1 & |x_p|_p \leq 1 \\ 0 & \text{otherwise}\,. \end{cases}$$

Let $\langle\ ,\ \rangle$ be the natural pairing of $\mathbf{A}$, as a self-dual locally compact topological group, let dx be a Haar measure on $\mathbf{A}$ normalized so that the Fourier transform

$$\hat{f}(y) = \int f(y)\, \langle x, y\rangle\, dx$$

satisfies

$$f(x) = \int \hat{f}(y)\, \overline{\langle x, y\rangle}\, dy\,.$$

In this case, recall

$$\hat{h}(x) = h(x)\,. \tag{2}$$

Let ω be a quasi-character (in the sense of Weil) of $\mathbf{A}^\times/\mathbf{Q}^\times$, and let

$$Z(\omega) = \sum_{\gamma \in L} a(\gamma)\, \omega(\gamma)$$

denote the formal series on which the involution $*$ acts by

$$Z^*(\omega) = \sum_{\gamma \in L} \overline{a(\gamma)}\, \omega(-\gamma)\,.$$

It will always be assumed that $Z(\omega)$ converges absolutely somewhere. Recall any quasi-character of $\mathbf{A}^\times/\mathbf{Q}^\times$ can be written in the form

$$\omega(x) = |x|_{\mathbf{A}}^s \, \psi(x)\,, \quad s \in \mathbf{C}, \quad x \in \mathbf{A}^\times \,, \tag{3}$$

where $x \mapsto |x|_{\mathbf{A}}^s$ is a principal quasi-character trivial on $\mathbf{Q}^\times \times \Pi \mathbf{Z}_p^\times$ and ψ is a character of finite order, trivial on $\mathbf{Q}^\times \times \mathbf{R}_+^\times$. In this way we may regard $Z(\omega)$, in its region of absolute convergence, as an analytic function of s.

The Dirichlet series $Z(\omega) = Z(s, \psi)$ is said to satisfy the *summability condition* (briefly, *SC*) if

a) Z is analytically continuable to $\{\operatorname{Re} s \geq 0\}$,

and

b) For all s, $\operatorname{Re} s \geq 0$, we have

$$Z(s,\psi) \ll e^{(\frac{\pi}{2}-\epsilon)|\operatorname{Im} s|}$$

According to Hardy [3], if Z satisfies the *SC* then for all s with $\operatorname{Re} s \geq 0$, and ω given by (3),

$$Z(\omega) = \lim_{r \to 1^-} \sum_{\gamma \in L} a(\gamma)\, r^{|\gamma|}\, \omega(\gamma) \tag{4}$$

(the limit being uniform on compacta), and conversely. Observe that any Hecke L-series corresponding to a nontrivial ψ satisfies the *SC*.

Let $\chi_0(x) = \exp 2\pi i\, \sigma(x)$, where

$$\sigma(x) = -x_\infty + x_2 + \ldots + x_p + \ldots \pmod 1\,,$$

so that $\langle x, y\rangle = \chi_0(xy)$. Let $\chi_y : x \mapsto \chi_0(xy)$. The condition which determines whether or not the operator

$$\phi(x) \mapsto \lim_{r \to 1^-} \sum_{\gamma \in L} a(\gamma)\, r^{|\gamma|}\, \phi(x+\gamma) \tag{5}$$

is unitarily equivalent to multiplication by $Z(\chi_y)$ is precisely that (4) hold with $\omega(y) = \chi_y$ (assuming a weak finiteness condition). However, the stronger *SC* is much more convenient in applications.

Denote the operator (5) by $P = P(Z)$, and let $L = L(Z)$ denote

$$\phi(x) \mapsto \int Z(\chi_y)\, \overline{\langle x, y\rangle}\, \hat{\phi}(x)\, dx\,.$$

Let $S = \mathbf{R} \times \Pi Z_p$, let $L^2(S) \hookrightarrow L^2(\mathbf{A})$ be functions restricted to S, and let H be the subspace of Fourier transforms of functions in $L^2(S)$. Note that H can also be realized as the space of functions whose Fourier transforms belong to $L^2(S)$. Since L is unitarily equivalent to multiplication by $Z(\chi_y)$ it is easy to check that $L^2(S) \cong H$ is an invariant subspace of L.

As usual, an unbounded operator T on a Hilbert space is called *normal* in case T is densely defined (so T^* exists), closed, and $TT^* = T^*T$. From this it follows $D(T) = D(T^*)$, where D denotes domain (see Rudin [5, chapter 13] for example).

The main result is the following

Theorem. *Assume $h(y)\, Z(\chi_y) \in L^2(\mathbf{A})$.*

a) *Z satisfies (4) with $\omega = \chi_y$ if and only if*

(i) *$L(Z)$ and $P(Z)$ are normal in H,*

(ii) *$P = L$ on $D(L) = D(P) \subseteq H$,*

(iii) *for all $f \in D(L)$,*

$$\|P(Z)\, f\| = \|P(Z^*)\, f\| = \|L(Z)\, f\| = \|L(\overline{Z})\, f\| < \infty .$$

Assume Z satisfies (4) with $\omega = \chi_y$, $y \in \mathbf{A}$, and is continuous in y.

b) *On $D(L) = D(P) \subseteq H$,*

$$L(Z)^* = L(\overline{Z}), \quad P(Z)^* = P(Z^*) .$$

c) $\sigma(P) = \sigma(L) = \{Z(\chi_y) \mid y \in S\}$, $Z \neq \text{constant} \Rightarrow \sigma_{\text{point}}(P) = \emptyset$.

d) *Let $E \subseteq D(P)$ be a subspace stable under $\phi(x) \mapsto \phi(-x)$. E is an invariant subspace of P if and only if $\widehat{E}$, the image of E under Fourier transform, is of the form $L^2(M) \hookrightarrow L^2(S)$, where $M \subseteq S$ is a measurable, not locally null subset, and functions in $L^2(M)$ are extended by 0 outside of M.*

e) *If $Z(\chi_y)$ has a zero then $P(Z) = L(Z)$ has dense range on $L^2(S)$.*

f) *P commutes with the unitary representation of H induced by*

$$R_y \mapsto \phi(x) \mapsto \phi(x - y), \quad y \in S ,$$

on $L^2(S)$.

Remark. The assumptions made on Z can be weakened in certain cases. Consider, for example, the distribution

$$P(x) : \phi \mapsto \sum_{\alpha \in \mathbf{Q}} \phi(\alpha x) , \quad x \in \mathbf{A} ,$$

where ϕ is a Schwartz-Bruhat function. With respect to the $L^2(\mathbf{A})$ inner product, the adjoint P^* is given by

$$P^*(x)\,\phi = \sum_{\alpha \in \mathbf{Q}} |\alpha|^{-1}\,\phi(x/\alpha) ,$$

where the $\alpha = 0$ term is defined distributionally as the limit $\alpha \to 0^+$. Let

$$\phi^*(x) = |x|_{\mathbf{A}}^{-1}\,\phi(1/x) .$$

Evidently, this is an involution and the Poisson summation formula can be rewritten as $P(x)\,\phi = P^*(x)\,(\widehat{\phi})^*$. This, of course, leads to the functional equation for all the Hecke L-series (provided one first replaces $\mathbf{Q}$ by an arbitrary number field k).

Since the proof of the theorem is based on well-known ideas it shall only be sketched in the following series of lemmas.

Lemma 1. *The SC implies $h(y)\,Z(\chi_y) \in L^2(\mathbf{A})$ and that Z satisfies (4) with $\omega = \chi_y$. Moreover, $h(y)\,Z(\chi_y) \in L^2(\mathbf{A})$ implies that $L(Z(\chi_y))$ is densely defined on H.*

Proof: The first claim is clear. The second claim follows from

$$\int |L\,\phi(x)|^2\,dx = \int |Z(\chi_y)|^2\,|\phi(y)|^2\,dy$$

(by Plancheral's formula), $h(x) = \widehat{h}(x)$, and Wiener's L^2-Tauberian Theorem. ∎

Lemma 2. *If $L(Z)$ is densely defined then $L(Z)$ is normal and $L(Z)^* = L(\overline{Z})$.*

Proof: $L(Z)^* = L(\overline{Z})$ follows from the formula

$$\int \phi(x)\,\overline{\psi(x)}\,dx = \int \widehat{\phi}(y)\,\overline{\widehat{\psi}(y)}\,dy .$$

To see L is normal it suffices to prove that the graph

$$G = \left\{ (\hat{\phi}, (L\phi)^\wedge), \quad \phi \in D(L) \right\}$$

is closed in $L^2(S) \times L^2(S)$. This follows from a standard density argument. ∎

Lemma 3. *Assume $L(Z)$ is densely defined. $Z(\chi_y)$ satisfies (4), uniformly on compact α in y, if and only if $P(Z) = L(Z)$.*

Proof: Let

$$T_R\,\phi(x) = \int_{|y|_{\mathbf{A}} \le R} \hat{\phi}(y)\,\overline{\langle x, y\rangle}\,dy\,,$$

so T_R forms an approximate identity. Let

$$P_{N,r}\,\phi(x) = \sum_{|\gamma|_{\mathbf{A}} \le N} \alpha(\gamma)\,\phi(x+\gamma)\,r^{|\gamma|_{\mathbf{A}}}\,.$$

For convenience, write $|\ | = |\ |_{\mathbf{A}}$. It is easy to compute

$$P_{N,r}\,T_R\,\phi(x) = \int_{|\gamma| \le R} \left(\sum_{|\gamma| \le N} \alpha(\gamma)\,r^{|\gamma|}\,\overline{\langle \gamma, y\rangle} \right) \hat{\phi}(y)\,\overline{\langle x, y\rangle}\,dy$$

and

$$(P_{N,r}\,\phi)^\wedge(y) = \sum_{|\gamma| \le N} \alpha(\gamma)\,r^{|\gamma|}\,\overline{\langle \gamma, y\rangle}\,\hat{\phi}(y)\,.$$

Therefore, T_R commutes with $P_{N,r}$. The lemma now follows from a limiting argument. ∎

Proof of the theorem: Parts a), b), and f) now follow from Lemmas 1,2,3 and an easy computation of $P(Z)^*$, $L(Z)^*$. Parts c) and e) follow from well-known facts about normal operators (see Rudin [5, Theorem 13.27]). Part d) follows from a well-known result on multiplication operators (see Mozak [4, Lemma 7.2 (c)]). ∎

3 – An Example

There are many distributions of the form

$$\phi \mapsto \sum_{n=1}^{\infty} a_n \, \phi(\lambda_n) \,,$$

and under certain conditions on a_n, λ_n these give rise to summation operators (replacing ϕ by $R_x \, \phi$ where R is the regular representation). In this context, such distributions coming from the Weil explicit formula [7] provide interesting examples.

First, recall Weil's formula. Let $L(s,\chi)$ be the Hecke L-function corresponding to an idele class character χ of a number field k. Let

$$W(F) = \sum_{\rho} \widehat{F}(\rho) \,, \tag{6}$$

where the sum runs over all non-trivial zeros of $L(s,\chi)$,

$$\widehat{F}(s) = \int_{-\infty}^{\infty} F(x) \, e^{(s-1/2)x} \, dx \,,$$

and F (to begin with) belongs to $C_c^\infty(\mathbf{R})$. In the notation of Benedetto [1], let

$$\begin{aligned} W_2(F) &= \int_{-\infty}^{\infty} F(x) \, (e^{x/2} + e^{-x/2}) \, dx \\ &= \int_0^{\infty} \Big(F(\log y) + F(-\log y)\Big) \frac{dy}{\sqrt{y}} \,, \end{aligned}$$

$$W_3(F) = \sum_{\underline{a} \neq 0} \frac{\Lambda(\underline{a})}{\sqrt{N\underline{a}}} \Big[\chi(\underline{a}) \, F(\log N\underline{a}) + \chi(\underline{a})^{-1} \, F(-\log N\underline{a})\Big] \,,$$

where $\Lambda(\underline{a})$ is the von Mangoldt function on the ring of integral ideals of k and $N\underline{a}$ is the norm of an integral ideal $\underline{a}$. Let

$$W_0(F) = 2 \, F(0) \log A \,,$$

where $A = A(k,\chi) > 0$ is a number theoretic constant (see [7]), and let

$$W_1(F) = \sum_{m=1}^{r_1+r_2} \text{P.V.} \int_{-\infty}^{\infty} F(x) \, e^{-i\,\phi_m\,x} \, K_{\eta_k, f_m}(x) \, dx \,,$$

where $\eta_k \in \{1, 2\}$, ϕ_m, f_m, r_1 and r_2 are as in Weil [7], and

$$K_{1,f}(x) = \frac{e^{(\frac{1}{2}-f)|x|}}{|e^x - e^{-x}|}, \quad K_{2,f}(x) = \frac{e^{-f|x|/2}}{|e^{x/2} - e^{-x/2}|}.$$

As in Benedetto [1], define $V = V_\chi$ by

$$\delta(\chi)\, W_2 + W_3 = V_\chi + \check{V}_{\overline{\chi}}, \quad \check{V}(F)(x) = V(F(-x)), \quad \delta(\chi) = \left\{ \begin{matrix} 1 & \chi \neq \chi_0 \\ 0 & \text{otherwise} \end{matrix} \right\},$$

and put

$$V = T + U,$$

where

$$T(F) = \sum_{\underline{a}} \int_{N\underline{a}-1}^{N\underline{a}} \left(\frac{\delta(\chi)}{\sqrt{x}} - \frac{\Lambda(\underline{a})\,\chi(\underline{a})}{\sqrt{N\underline{a}}} \right) dx \cdot F(\log N\underline{a}),$$

$$U(F) = \sum_{\underline{a}} \int_{N\underline{a}-1}^{N\underline{a}} \frac{F(\log x) - \delta(\chi)\, F(\log N\underline{a})}{\sqrt{x}}\, dx.$$

Weil's *explicit formula* states

$$W = W_0 + W_1 + (\delta(\chi)\, W_2 + W_3) \tag{7}$$

as distributions on $C_c^\infty(\mathbf{R})$. For these distributions, the following result is known.

Theorem (Benedetto [1]). **(a)** *W_0, W_1 and U are all tempered distributions. In particular, W is tempered if and only if T is tempered.*

(b) *If W is tempered then the Riemann hypothesis for $L(s,\chi)$ holds.* ∎

Remark. The converse to (b) was stated without proof.

Let $S = S(\mathbf{R})$ be the Schwartz space of rapidly decreasing functions. The statement $W \in S'$ means that the (bracketed) Weil explicit formula is valid for all rapidly decreasing functions. For comparison, recall that Weil [7] assumed all his functions satisfied

A) $F'(x)$ exists everywhere except possibly at a finite number of points α_j where

$$F'(\alpha_j) = \frac{1}{2}\left(F'(\alpha_j + 0) + F'(\alpha_j - 0)\right),$$

B) $F(x)$ and $F'(x)$ are both

$$\ll e^{-(\frac{1}{2}+\varepsilon)|x|} .$$

Clearly, the growth condition for S is much weaker than B).

In view of this, it may be of interest to examine the operator

$$T(y)(F) = \sum_{\underline{a}} \int_{N\underline{a}-1}^{N\underline{a}} \left(\frac{\delta(\chi)}{\sqrt{x}} - \frac{\Lambda(\underline{a})\,\chi(\underline{a})}{\sqrt{N\underline{a}}} \right) dx\, F(y + \log N\underline{a}) .$$

As a distribution, $T(y)$ has the following remarkable property.

Theorem. a) *The Riemann hypothesis holds for $L(s,\chi)$ if and only if, for some real y, $T(y) \in S'$.*

b) *Assume the Riemann hypothesis and let $C(y) \geq 1$ be a monotonely increasing function.*

$$\sum_{N\underline{a} \leq x} \chi(\underline{a})\,\Lambda(\underline{a}) = \delta(\chi)\,x + 0\left(\sqrt{x}\; C(\log x)\right)$$

if and only if

$$\sum_{|\rho| \leq x} \frac{\sin(\gamma \log x)}{\gamma} \ll C(\log x) ,$$

where the sum runs over zeros $\rho = \frac{1}{2} + i\gamma$ of $L(s,\chi)$.

Remarks. a) The proof of a) implicity uses a classical special case (8) of the Weil explicit formula, mentioned below in the proof of b). The proof of a) also uses the following fact (implied by a theorem of L. Schwartz) concerning the second primitive $T_{(2)}$ of T : $T_{(2)} \in S' \Rightarrow T \in S'$. This unfortunately forces the functions to be C^{∞}.

b) Part b) is essentially well-known although this version has not been stated in the literature. It is implicit for example in Gallagher [2] and in one sense it improves Benedetto [1, Proposition 4.1]. It is well-known from Koch that the Riemann hypothesis implies $C(y) \ll y^2$ (the converse is obvious).

Proof of the theorem: a) This is a direct consequence of the calculations in Benedetto [1, pp.161-162], using a structure theorem of L. Schwartz for tempered distributions [6, Théorème VI, p.239] and Koch's result mentioned above.

b) This is immediate from

$$\sum_{N\underline{a}\le x} \chi(a)\,\Lambda(\underline{a}) = \delta(\chi)\,x - \sum_{|\gamma|\le x} \frac{x^\rho}{\rho} + 0\,(\log^2 x) \tag{8}$$

since

$$\sum_{|\rho|\le x} \frac{x^\rho}{\rho} = 2\sqrt{x} \sum_{|\gamma|\le x} \frac{\sin(\gamma \log x)}{\gamma} + 0\,(\sqrt{x})\,. \blacksquare$$

Now consider the distribution-valued function

$$T_\sigma(x)\colon\ \phi \mapsto \lim_{r\to 1^-} \sum_{\underline{a}\ne 0} N\underline{a}^{-\sigma}\, C(\underline{a})\, r^{N\underline{a}}\, \phi(x + \log N\underline{a})\ , \tag{9}$$

$\phi \in C_c^\infty(\mathbf{R})$, where

$$\begin{aligned} C(\underline{a}) &= \int_{N\underline{a}-1}^{N\underline{a}} \frac{\delta(\chi)}{\sqrt{x}} - \frac{\Lambda(\underline{a})\,\chi(\underline{a})}{\sqrt{N\underline{a}}}\, dx \\ &= (\delta(\chi) - \chi(\underline{a})\, N\underline{a})/\sqrt{N\underline{a}} + 0\,(N\underline{a}^{-3/2})\,. \end{aligned}$$

Clearly $T_0(0)$ is identical with Benedetto's distribution T.

The singularity at $s = 1/2$ of

$$Z(s) = \sum_{\underline{a}\ne 0} C(\underline{a}) N\underline{a}^{-s}$$

is removable and $Z(s)$ analytically continues as a meromorphic function of the entire plane having simple poles at $\rho - 1/2$, ρ a zero of $L(s,\chi)$.

Let

$$Z_\sigma(s) = \sum_{\underline{a}\ne 0} C(\underline{a})\, N\underline{a}^{-\sigma}\, N\underline{a}^{-s}\ ,$$

so Z_σ satisfies the *SC* if and only if $\sigma > \theta$, where

$$\theta = \sup\Big\{\beta \mid L(\beta + i\gamma, \chi) = 0\Big\} - 1/2\,.$$

Considering only the real (i.e., infinite) component of the operator $L(Z_\sigma)$, the Plancheral formula implies, for $f \in D(L) \subseteq L^2(\mathbf{R})$, $\sigma > \theta$,

$$\|T_\sigma(x)\,\phi\|^2 = \|L(Z_\sigma)\,f\|^2 = \int_{-\infty}^{\infty} |Z_\sigma(it)|^2\, |\hat{f}(t)|^2\, dt =$$

$$= \int_{-\infty}^{\infty} |\varsigma(\sigma + \tfrac{1}{2} + it, k) - \tfrac{L'}{L}(\sigma + \tfrac{1}{2} + it, X)|^2 \, |\hat{f}(t)|^2 \, dt + 0(\|f\|^2) ,$$

where $\varsigma(s,k)$ is the Dedekind zeta function. This shows, that, for $\sigma = 0$, $L(Z_0)$ cannot be thought of as a densely defined operator on $L^2(\mathbf{R})$. However, there is a rather natural subspace, given below, on which $L(Z_\sigma)$ is defined, $0 \le \sigma \le \theta$.

Let

$$S_0 = \left\{ \phi \in S \mid \sum_{n=1}^{\infty} \frac{|\phi^{(n)}(x)|}{n!} \in S \right\} ,$$

so for example $e^{-x^2/2} \in S_0$. On this space, which is dense in $L^2(\mathbf{R})$ by Wiener's L^2-Tauberian theorem, one can compute the inverse Fourier transform K_σ of $L(\sigma + \frac{1}{2} + it, \chi)$, regarded as a tempered distribution. (The exact expression for K_σ is a rather messy convolution and not of particular importance.) When K_σ acts on S_0 by convolution, K_σ: $\phi \in S_0 \to K_\sigma * \phi \in S$, and, of course,

$$(K_\sigma * \phi)^\wedge(it) = L(\sigma + \tfrac{1}{2} + it, \chi)\, \hat{\phi}(t) .$$

In particular, $L(Z_\sigma)$ is defined on $K_\sigma * S_0$, even if $0 \le \sigma \le \theta$. Moreover, functions in $K_0 * S_0$ satisfying Weil's conditions A) and B) above must belong to the kernel of the Weil distribution W.

Although these remarks may illuminate the structure of W and T, unfortunately I do not think they can be used to attack the Riemann hypothesis.

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David Joyner
Dept. of Mathematics, Institute for Advanced Study,
Princeton, NJ 08540

and

Present adress: David Joyner
Dept. of Mathematics, U.S. Naval Academy,
Annapolis, MD 21402