On the Montgomery-Dyson Hypothesis

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The Montgomery-Dyson hypothesis, first mentioned in [**Mon1**], is a statistical statement that the imaginary parts of the zeros of the Riemann zeta function behave like eigenvalues of a random Hermitian matrix of unitary type. In for example [**Odl**] numerical evidence is brought out in support of this hypothesis. In an apparently unrelated work, Montgomery [**Mon2**] conjectured that the maximum value of $S(t) := \frac{1}{\pi} arg\ \zeta(\frac{1}{2}+it)$,

$$\max_{|t|\leq T} S(t),$$

is of the same order of magnitude as $(\log T/\log\log T)^{1/2}$. There is some heuristic evidence in support of this hypothesis as well. In this note we present evidence, stated in the form of the Lemma below, that these two hypotheses appear to be inconsistent.

Pick an ordering of the complex zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\gamma > 0$ in such a way that $\rho' = \beta' + i\gamma'$ occurs after $\rho = \beta + i\gamma$ if $\gamma' > \gamma$. Let γ_n denote the imaginary part of the n^{th} non-trivial zero of $\zeta(s)$ lying above the real axis. Let

$$\delta_n := (\gamma_{n+1} - \gamma_n)\frac{\log \gamma_n}{2\pi},$$

so that the average of the δ_n equals 1. Let $F_r(t)$ denote the density function of the normalized eigenvalues of a large $r \times r$ random Hermitian matrix of unitary type (see [**dCM**]). Let $\alpha(x) = \alpha_r(x)$ denote a function such that

$$\int_{\alpha(x)}^{\infty} F_r(t)dt \sim x^{-1},$$

as $x \to \infty$ and let $\beta(x) = \beta_r(x)$ denote a function such that

$$\int_0^{\beta(x)} F_r(t)dt \sim x^{-1},$$

as $x \to 0$. We call the statement

$$\lim_{N\to\infty} \frac{1}{N}\#\{n \leq N \mid \delta_n \in (a,b)\} = \lim_{r\to\infty} \int_a^b F_r(t)dt,$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

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the weak Montgomery-Dyson hypothesis, and the statement

$$\max_{n\leq N} \delta_n \approx \alpha(N),$$

as $N \to \infty$, as the strong Montgomery-Dyson hypothesis. Incidentally, the same principle leads to the lower bound

$$\min_{n\leq N} \delta_n \approx \beta(N),$$

as $N \to \infty$. By $\approx$ we mean both side have the same order of magnitude (so $c_1\alpha(N) \leq \max_{n\leq N} \delta_n \leq c_2\alpha(N)$ for some constants c_i). A. Odlyzko has pointed out that, for r large enough,

$$F_r(t) \approx t^{-1/4}\exp(-\frac{\pi}{8}t^2),$$

as $t \to \infty$ (see [**dCM**]), and

$$F_r(t) = \frac{\pi^2}{3}t^2 + O(t^4),$$

as $t \to 0$ (see [**Meh**, Arr. A12, p. 202]). These suggest that we should have

$$\alpha(N) \sim \frac{2\sqrt{2}}{\pi}(\log N)^{1/2}$$

in the strong Montgomery-Dyson hypothesis and

$$\beta(N) \sim (\frac{9}{\pi^2 N})^{1/3}$$

(see also [**Odl**, (2.5.1)]). This is a much more precise version of a conjecture made in [**Mon3**]. The numerical evidence at hand appears to conform to these expectations [**Odl**].

In this paper we present the simple Lemma below as evidence that the statement

$$\alpha(N) \approx (\log N)^{1/2}$$

appears to be inconsistent with the statement $\max_{|t|\leq T} S(t) << (\log T/\log\log T)^{1/2}$ (however, it is not inconsistent with $\max_{|t|\leq T} S(t) >> (\log T)^{1/2}$). A little notation is needed first: for constants $c_1 > 0$ and $c_2 > 0$, let $f(t)$ denote a function satisfying

$$c_1(\log t/\log\log t)^{1/2} \leq f(t) \leq c_2 \log t/\log\log t,$$

for $t > 10$, and $f(t + O(1)) < 2f(t)$, for t sufficiently large. For each fixed c, let

$$g(t) = g_c(t) = cf(t)/\log t.$$

LEMMA. *Suppose that for each constant $c > 0$*
(A) for $t > t_0(c)$ and

$$\frac{1}{2} + \frac{1}{100} g_c(t) \leq \sigma \leq \frac{1}{2} + \frac{1}{2} g_c(t),$$

we have $|Re\ \log \zeta(\sigma + it)| \leq f(t)$,
(B) for all $\sigma \geq \frac{1}{2}$, $t > t_1(c)$, we have

$$|\zeta(\sigma + it)| \leq \exp(f(t)).$$

Then there exists a $c = c(c_1, c_2) \geq 1$ with the property that every disk $|s - \frac{1}{2} - it| \leq g_c(t)$, $t > t_3(c)$ contains a zero (the constant $t_3(c)$ is not effectively computable).

REMARK: If Montgomery's conjecture concerning $S(t)$ were true then it might be reasonable to expect that one may take $f(t) \approx (\log t / \log\log t)^{1/2}$ in (A)-(B). The Lemma then implies that every pair $\rho = \beta + i\gamma$, $\rho' = \beta' + i\gamma'$ of consecutive zeros satisfies

$$|\gamma' - \gamma| << (\log \gamma \log\log \gamma)^{-1/2}.$$

For the sake of comparison, it might be pointed out that the strong Montgomery-Dyson hypothesis suggests that there exist infinitely many consecutive zeros satisfying

$$|\gamma' - \gamma| >> (\log \gamma)^{-1/2}.$$

PROOF OF THE LEMMA: Let $g(t) = g_c(t)$ for brevity. Recall the functional equation

$$\zeta(1 - s) = 2^{1-s}\pi^{-s} \cos(\frac{\pi s}{2})\Gamma(s)\zeta(s),$$

and Stirling's formula

$$\Gamma(s) = (\frac{2\pi}{s})^{1/2}(\frac{s}{e})^s(1 + O(|s|^{-1})),$$

uniformly in $|arg\ s| < \pi - \epsilon$, as $s \to \infty$.

Let $r_1 = r_1(t) = \frac{1}{5}g(t)$, $r_2 = r_2(t) = \frac{1}{3}g(t)$, $r_3 = r_3(t) = \frac{1}{2}g(t)$, and let $C_t := \{s \mid |s - \frac{1}{2} - it| \leq g(t)\}$. The r_i will be the radii for three concentric circles defined as follows. Assume that the Lemma is false, so there exists an infinite sequence $t_n \to \infty$ such that $\zeta(s) \neq 0$ in C_{t_n} and $C_{t_m} \cap C_{t_n} = \emptyset$ for $m \neq n$. By (A), for each n there is an $s_0 = s_0(n)$ such that $|s_0 - \frac{1}{2} - \frac{1}{4}g(t_n) - it_n| \leq \frac{1}{50}g(t_n)$ and $|\log \zeta(s_0)| \leq c_3 f(t_n)$, where $\log \zeta(s)$ is the branch on C_{t_n} defined so that $|Im\ \log \zeta(s)| < \pi$ for all $s \in C_{t_n}$. To make explicit the fact that $\log\zeta(s)$ depends on n via this choice of the branch, let us denote $F_n(s) = \log\zeta(s)$, for $s \in C_{t_n}$.

Let $C_i = C_i(n) = \{s \mid |s - s_0(n)| = r_i(t_n)\}$. By Caratheodory's lemma and our assumptions, we have

$$\max_{s\in C_1} |F_n(s)| \le c_4 f(t_n),$$

for some absolute constant c_4. On C_3 the functional equation and Caratheodory's lemma imply

$$\max_{s\in C_3} |F_n(s)| << \max_{s\in C_3} |\log\frac{\Gamma(1/2 - s/2)}{\Gamma(s/2)}| + f(t_n) \le c_5 c f(t_n),$$

for some absolute constant c_5. Let $\theta_1 = \frac{\log(r_3/r_2)}{\log(r_3/r_1)}$, $\theta_3 = \frac{\log(r_2/r_1)}{\log(r_3/r_1)}$, so by Hadamard's three circles theorem

$$\max_{s\in C_2} |F_n(s)| \le c_6 c^{\theta_3} f(t_n),$$

for some absolute constant c_6. Note $\theta_3 < 1$ is independent of n. Our assumptions imply that

$$Re\log\frac{1}{\zeta(\frac{1}{2} + \frac{1}{20}g(t_n) \pm it_n)} \le f(t_n).$$

By the functional equation, $Re\log\frac{1}{\zeta(s_n)} = Re\log\frac{1}{\zeta(1-s_n)} + Re\log\chi(1-s_n)$, where $s_n = \frac{1}{2} - \frac{1}{20}g(t_n) + it_n$, and $\chi(s) = \pi^{s-1/2}\frac{\Gamma(1/2-s/2)}{\Gamma(s/2)} \approx |t|^{1/2-\sigma}$ for $\sigma - \frac{1}{2} << (\log\log t)^{-1}$. It follows that

$$\begin{aligned} Re\log\frac{1}{\zeta(s_n)} &\le f(t_n) - \frac{1}{20}g(t_n)\log t_n + O(1) \\ &= (1 - \frac{c}{20})f(t_n) + O(1), \end{aligned}$$

for any branch of $\log\zeta(s)$. For $c > 1$ chosen sufficiently large, this bound contradicts our earlier estimate for $\max_{s\in C_2} |F_n(s)|$.

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