first draft

# Zeta Functions and their Associated Operators

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Associated Operators

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Abstract

The zeros of $\zeta(s)$ are characterized by eigenvalues of the associated operators. Machinery is build up for which, hopefully, the end product will be the solution of an important conjecture of Montgomery. Our method yields, as a by-product, a theorem reducing the sufficiency of Turoin's result $\pi(x) - \mathrm{Li}x \ll_\varepsilon x^{1/2+\varepsilon} \Rightarrow$ Riemann hypothesis.

Zeta Functions and their Associated Operators.

A well-known result due to Turán is the equivalence of the Riemann hypothesis with the bound, valid for every $\varepsilon > 0$ and $x > 1$, $|\pi(x)-\mathrm{Li}(x)| \ll_{\varepsilon} x^{1/2+\varepsilon}$. A related result, given in section 2, states that we can substantially reduce the sufficiency of Turán's theorem by merely placing an appropriate bound on the number of integral values of x which violate $|\pi(x)-\mathrm{Li}(x)| \ll_{\varepsilon} x^{1/2+\varepsilon}$. Moreover, the proof of this result can be said to be of methodological interest as it utilizes the Grosswald-Schnitzer method [1] concerning the analytic continuation of Euler products, a new method which helps us prove a more necessary result, Theorem 3.

Aside from this we need to only use the statement, proven in section 1, that there is a complete probability measure on the set of zeta functions of certain discrete real semigroups (see Lemma 1 below) such that the set of functions for which the extended Riemann hypothesis is false is of measure zero. We refer the interested reader to related results (using a different probability measure) in Wintner [3], and Heyde [7].

In section 3 we turn to the question of how to derive the nontrival zeros from the eigenvalues of certain linear operators over associated Hilbert spaces. In a weak sense this settles a well-known conjecture (Montgomery [5]) that the zeros of $\zeta(s)$ are characterized by the eigenvalues of a linear operator on a Hilbert space. (Our result actually only answers this question affirmatively if the quasi-Riemann hypothesis is true.) Utilizing the results of section 2 we then describe and compare similar operators.

In the conclusion, all the results are unified in a conjecture (and the discussion preceeding it) which is suggested naturally from the theorems to follow.

## Section 1.

We make no title to novelty for the results in this section since most of the arguments are well-known, in fact we will prove and clarify certain unverified claims made in Kosambi [2].

In a probability space $\{(0,2),P\}$ consider an infinite sequence of independent random variables $\{x_n(t)\}_{n=1}^{\infty}$, $t \in (0,1)$, such that each $x_n$ is uniformly distributed in $(0,2)$ with mean 1. Since the variance of each $x_n$ satisfies

$$V(x_n) = \int_0^2 \frac{(x-1)^2}{2}\, dx = 1/3 ,$$

the variance of the sum $s_n = x_1+x_2+\cdots+x_n$ satisfies

$$V(s_n) = n/3 . \tag{1}$$

Furthermore, if we let the sequence $\{g_n\}_{n=1}^{\infty}$ defined by

$$g_n := Li^{-1}(s_n)$$

(where for $\lambda > 1$, $Li(\lambda) := P.V. \int_0^\lambda \frac{dt}{\log t}$ ) generate the random multiplicative semigroup $G \cup \{1\}$, then we have the following

Lemma 1: The generators satisfy

(a) $1 < g_1 < g_2 < \cdots$.

(b) $P\{|g_n - Li^{-1}(n)| > 2n^{1/2}\log^{3/2}n\} < n^{-3/2}$, for every $n \geq N_0 \geq 2$.

(c) with probability 1, G is uniquely factorable into products containing only the elements $g_1, g_2, \cdots$ and their powers.

(d) If $\pi_G(x)$ "counts" the generators less than or equal to $x$, then $P\{|\pi_G(x)-Li(x)| > 2x^{1/2}\log x\} < 2x^{-3/2}\log^{3/2}x$, for all $x \geq x_0 > 2$.

Proof: (a): This is immediate when we note that $0 < s_1 < s_2 < \cdots$, that $Li^{-1}(x)$ is strictly increasing for $x > 1$, and that $Li(\alpha) = 0$ holds (by continuity) for some $1 < \alpha < 2$.

(b) The well-known Bernstein inequality states that the probability that $|(x_1-1)+(x_2-1)+\cdots+(x_n-1)| > t\sqrt{2V(s_n-n)}$ occurs is less than $e^{-t^2}$. For $n \geq 2$, let $t = \sqrt{3/2 \log n}$ to arrive at

$$P\{|s_n-n| > n^{1/2}\log^{1/2}n\} < n^{-3/2} \tag{2}$$

in virtue of $V(s_n-n) = V(s_n)$ and equation (1).

Now, $Li(x) \sim \frac{x}{\log x}$, so $Li^{-1}(x) \sim x \log x$. We may thus choose $N_0$ so large that (2), combined with the mean value theorem, implies that, with probability less than $n^{-3/2}$, either

(i) $Li^{-1}(s_n) > Li^{-1}(n-n^{1/2}\log^{1/2}n) > Li^{-1}(n) - (1+\varepsilon)n^{1/2}\log^{3/2}n$

occurs, or

(ii) $Li^{-1}(s_n) < Li^{-1}(n+n^{1/2}\log^{1/2}n) < Li^{-1}(n) + (1+\varepsilon)n^{1/2}\log^{3/2}n$

does, for all $n \geq N_0(\varepsilon)$. Taking $\varepsilon = 1$ gives (b).

(c) We determine the probability $P_n$ that the n-th generator $g_n$ can be factored into the product of any previously occuring generators or their powers: $g_1^{e_1}g_2^{e_2}\cdots g_{n-1}^{e_{n-1}}$, where the $e_i \geq 0$ are integers.

Since $1 < g_1 < g_2 < \cdots < g_{n-1}$ we need only consider a finite set of possible $e_i$, as for some $e \geq 1$, $g_1^e > g_n$, $g_2^e > g_n$, $\cdots$, $g_{n-1}^e > g_n$. So suppose without loss of generality $e_i \leq e < \infty$, for all $1 \leq i \leq n-1$. The number of element in this set is evidently $e(n-1)$, which is finite. On the other hand, the sample space for $g_n$ has cardinality $|\mathbb{R}|$, since it is a non-empty interval. Thus $P_n = 0$. To conclude the proof we sum over all the $P_n$, and the claim follows.

(d) It follows from (b) that there won't occur more than x generators in $(n,n+x)$, with probability 1. Thus, since $\pi_G$ is increasing we have, with $P \leq n^{-3/2}$, by inequality (b) - (i),

(i) $\pi_G(g_n) = n > \pi_G(Li^{-1}(n) - (1+\varepsilon)n^{1/2}\log^{3/2}n)$

$$= \pi_G(Li^{-1}(n)) - [\pi_G(Li^{-1}(n)) - \pi_G(Li^{-1}(n-(1+\varepsilon)n^{1/2}\log^{3/2}n)]$$

$$> \pi_G(Li^{-1}(n)) - (1+2\varepsilon)n^{1/2}\log^{3/2}n, \quad n \geq N_0(\varepsilon).$$

Similarly, with $P \le n^{-3/2}$, $n \ge N_0(\varepsilon)$

(ii) $\pi_G(g_n) = n < \pi_G(\mathrm{Li}^{-1}(n)) + (1+2\varepsilon)n^{1/2}\log^{3/2}n$.

Thus we have the result, $n \ge N_0$

$$P\{|\pi_G(\mathrm{Li}^{-1}(n))-n| > (1+2\varepsilon)n^{1/2}\log^{3/2}n\} < n^{-3/2} .$$

Since $\mathrm{Li}^{-1}(n+1) - \mathrm{Li}^{-1}(n) \ll \log n$, this implies that for all $\mathrm{Li}^{-1}(n) \le x \le \mathrm{Li}^{-1}(n+1)$, $n \le y \le n+1$, $n \ge N_0(\varepsilon)$

$$P\{|\pi_G(x)-y| > (1+3\varepsilon)n^{1/2}\log^{3/2}n\} < n^{-3/2} .$$

But with a change of variables, this is just

$$P\{|\pi_G(x)-\mathrm{Li}(x)| > (1+4\varepsilon)x^{1/2}\log x\} < 2x^{-3/2}\log^{3/2}x \tag{3}$$

since, for $x \ge X_0(\varepsilon)$,

$$(1+3\varepsilon)(\mathrm{Li}(x))^{1/2}\log^{3/2}(\mathrm{Li}(x)) < (1+4\varepsilon)x^{1/2}\log x.$$

Let $\varepsilon = 1/4$, then (3) gives the result stated. <u>QED</u>.

Since G is uniquely factorable with probability 1, it follows that we may factor almost every zeta function $\zeta_G(s)$ over G, i.e., in formal terms

$$\zeta_G(s) \overset{\text{def.}}{=} \sum_{a\in G} a^{-s} = \prod_{n=1}^{\infty} (1-g_n^{-s})^{-1} . \tag{4}$$

In fact, by the Borel-Cantelli Lemmas, Lemma 1(b) implies that almost every $\zeta_G(s)$ is analytic Res > 1. Moreover, for every fixed $\delta > 0$, $\delta' > 0$, $\zeta_G(s)$ is uniformly bounded in Res $\ge 1 + \delta$, with probability at least $1 - \delta'$.

<u>Theorem 1</u>: With probability 1, $\zeta_G(s)$ is analytic and zero-free in the half-plane Res > 1/2, except for a simple pole at s = 1.

Proof: Note that if we define the random function of x, $\theta_G(x)$, to be

$$\theta_G(x) = \sum_{g \le x} \log g ,$$

where g runs over the generators of G, then we have

$$\theta_G(x) - x = \int_e^x \log t \, d[\pi_G(t) - Li(t)] + O(1) . \tag{5}$$

To avoid questions of convergence we have set the lower limit of integration equal to e = 2.718 ··· .

Fix an arbitrary $0 < \varepsilon < 1$. Choose $N = N(\varepsilon)$ so that (5) and Lemma 1(d) imply that

$$\begin{aligned} \theta_G(x) - x &= \int_N^x \log t \cdot d\{\pi_G(t) - Li(t)\} + O_N(1) \\ &= \log x \cdot \{\pi_G(x) - Li(x)\} - \int_N^x \frac{\pi_G(t) - Li(t)}{t} dt + O_N(1) \\ &\ll_N x^{1/2} \log^2 x , \end{aligned} \tag{6}$$

with probability at least $1 - \varepsilon/2$.

Define $\psi_G$ formally by

$$\psi_G(x) = \theta_G(x) + \theta_G(x^{1/2}) + \theta_G(x^{1/3}) + \cdots .$$

In virtue of Lemma 1(a) $g_1 \geq Li^{-1}(0) = 1 + \text{constant} = 1 + c > 1$, so it follows that $\theta_G(x^{1/n}) = 0$ occurs for all $n > \frac{1}{c}\log x$. Hence we always have

$$\theta_G(x) \leq \psi_G(x) \leq \theta_G(x) + \theta_G(x^{1/2})\,\frac{1}{c}\log x .$$

By (6) we have $\theta_G(x) \sim x$ with probability 1, hence we conclude from (6) and the above equation that

$$(7) \qquad \psi_G(x) - x \ll_\varepsilon x^{1/2}\log^2 x \ , \ (x \geq N) .$$

This equation is applied to equation (10) (which we derive below) to conclude the theorem.

Formal logarithmic derivation of the Euler product (4) yields with probability 1,

$$(8) \qquad -\frac{\zeta_G'}{\zeta_G}(s) = \sum_{a\in G}\frac{\Lambda(a)}{a^s} = \int_1^\infty x^{-s}d\psi_G(x) = s\int_1^\infty x^{-s-1}\psi_G(x)\,dx ,$$

where Res > 1 and

$$\Lambda(a) = \begin{cases} \dfrac{\log g}{n} & \text{, if } a = g^n \text{ is a power of a generator,} \\ 0 & \text{, otherwise.} \end{cases}$$

Furthermore, integration by parts yields

$$(9) \qquad -\frac{1}{1-s} = \int_1^\infty x^{-s}dx = -1 + s\int_1^\infty x^{-s-1}x\,dx\,.$$

Now, $\frac{\zeta_G'}{\zeta_G}(s)$ and $\frac{-1}{s-1}$ are logarithmic derivatives of the functions $\log \zeta_G(s)$ and $\log(1-s)$, respectively, hence (8) and (9) yield for $\operatorname{Re} s > 1$

$$(10) \qquad -\frac{d}{ds}\log\{(s-1)\zeta_G(s)\} = 1 + s\int_1^\infty x^{-s-1}\{\psi_G(x)-x\}dx$$

with probability 1.

The theorem follows immediately from (7) and (10) by analytic continuation. QED.

Remark: Note that $g_{n+1} - g_n \ll \log g_n$, hence a well-known result of Rankin [8] implies that we cannot find $x_n$'s in $(0,2)$ so that $Li^{-1}(x_1+\cdots x_n) = p_n$, for all $n \geq 1$. However, one may suggest that the exceptions to $p_{k+1} - p_k \ll \log p_k$ are of (relative) density zero, so the implication of theorem 1 in the next section is not unusual.

## Section 2.

We are now ready to prove

**Theorem 2:** Let $\theta = \sup_{\beta} \{\beta: \zeta(\beta+i\gamma) = 0,\ \beta,\gamma \in \mathbb{R}\}$ (so $\frac{1}{2} \le \theta \le 1$), $\sigma_0 \ge \frac{1}{2}$ be fixed, and define the counting function $p_\varepsilon$ as

$$(11) \qquad p_\varepsilon(N) := \sum_{\substack{p_n \le N \\ |p_n - Li^{-1}(n)| \ge n^{1/2+\varepsilon}}} 1 ,$$

where $Li^{-1}$ is the inverse function of $Li(x) = \text{P.V.}\int_0^x \frac{dt}{\log t}$, and $p_n$ is the $n^{\text{th}}$ prime. Suppose that for every sufficiently small $\varepsilon = \varepsilon(\sigma_0) > 0$, $p_\varepsilon(N) \ll N^{1+\sigma_0-\theta+\varepsilon}$. Then $\zeta(s)$ is zero-free for all $\text{Re}\, s \geq \sigma_0$, i.e., $\theta \le \sigma_0$.

Proof: Let $u_n := Li^{-1}(n)$ and $s := \sigma + it$, $\sigma, t \in \mathbb{R}$. We claim that the zeta function $\zeta_L(s)$ defined for $\text{Re}\, s > 1$ by

$$(12) \qquad \zeta_L(s) := \prod_{n=1}^{\infty} (1-u_n^{-s})^{-1}$$

has a simple pole at $s = 1$ and is zero-free for all $\text{Re}\, s > \frac{1}{2}$. For clarity we note that this result, proved using Theorem 1 and the Grosswald-Schnitzer method of analytic continuation, will be combined with (11) and another application of the continuation method to show that $\zeta(s)$ is zero-free for $\text{Re}\, s > \sigma_0 \ge \frac{1}{2}$.

To proceed with the proof of (12) let the set

$$\{1 < g_1 < g_2 < \cdots : (\forall n \ge N_0 > 2)\, |g_n - u_n| < n^{\frac{1}{2}} \log^3 n\}$$

(where $N_0$ is given by Lemma 1(b)) generate a semigroup H with unique factorization subject to $\zeta_H(s) \neq 0$, for all Res $> \frac{1}{2}$. Such an H exists by Theorem 1 and, moreover, we can choose $\zeta_H$ to be analytic in Res $> \frac{1}{2}$, except for a simple pole at s = 1. To apply the continuation method we therefore consider the quotient, defined in Res > 1,

$$(13) \qquad \phi_1(s) := \frac{\zeta_L(s)}{\zeta_H(s)} = \prod_{n=1}^{\infty} \left( \frac{1-g_n^{-s}}{1-u_n^{-s}} \right).$$

We will reduce (12) to the analyticity of (13) in Res $> \frac{1}{2}$.

Consider the formal expansion

$$(14) \qquad \log \phi_1(s) = \sum_{n=1}^{\infty} \log \left( \frac{1-g_n^{-s}}{1-u_n^{-s}} \right)$$

$$= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{\infty} \frac{1}{m} u_n^{-ms} - \sum_{m=1}^{\infty} \frac{1}{m} g_n^{-ms} \right)$$

$$= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{m_0} + \sum_{m=m_0+1}^{\infty} \right) \frac{1}{m} (u_n^{-ms} - g_n^{-ms})$$

where $m_0$ is a fixed integer greater than both $\sigma^{-1}$ and $N_0$. We have the bound on the second inner sum

$$\left| \sum_{m=m_0+1}^{\infty} \frac{1}{m} (u_n^{-ms} - g_n^{-ms}) \right| \le \sum_{m=m_0+1}^{\infty} \frac{1}{m} (u_n^{-m\sigma} + g_n^{-m\sigma})$$

$$< 3 \sum_{m=m_0+1}^{\infty} \frac{1}{m} u_n^{-m\sigma} .$$

Since $m\sigma > 1$ and $u_n \sim n \log n$ it follows that

$$\sum_{n=1}^{\infty} \sum_{m=m_0+1}^{\infty} \frac{1}{m} (u_n^{-ms} - g_n^{-ms}) \tag{15}$$

is absolutely convergent. It remains to establish the convergence of

$$\sum_{n=1}^{\infty} \sum_{m=1}^{m_0} \frac{1}{m} (u_n^{-ms} - g_n^{-ms}) .$$

To see this let us note that $|g_n - u_n| < n^{\frac{1}{2}} \log^3 n$ and $u_n \sim n \log n$ yield

$$\left|\frac{g_n}{u_n} - 1\right| << n^{-\frac{1}{2}} \log^2 n .$$

Expanding certain terms into their Taylor series and approximating them to the first order, we thus have for sufficiently large $n > N_1 = N_1(m,s) > 2$.

$$(16) \qquad \left|1 - \left(\frac{g_n}{u_n}\right)^{ms}\right| = |1 - [1 - 0(n^{-\frac{1}{2}} \log^2 n)]^{ms}|$$

$$= |1 - \exp\{ms \log(1 - 0(n^{-\frac{1}{2}} \log^2 n))\}|$$

$$= |1 - \exp\{ms(-0(n^{-\frac{1}{2}} \log^2 n))\}|$$

$$= |1 - (1 - ms\, 0(n^{-\frac{1}{2}} \log^2 n))|$$

$$= 0(m|s|n^{-\frac{1}{2}} \log^2 n) \ .$$

The claim made in (12) now follows from (15) and

$$(17) \qquad \left|\sum_{n=1}^{\infty} \sum_{m=1}^{m_0} \frac{1}{m} (u_n^{-ms} - g_n^{-ms})\right| \le \sum_{n=1}^{\infty} \sum_{m=1}^{m_0} \frac{1}{m} |u_n^{-ms} - g_n^{-ms}|$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{m_0} \frac{1}{m} g_n^{-m\sigma} \left|1 - \left(\frac{g_n}{u_n}\right)^{ms}\right|$$

$$= \sum_{m=1}^{m_0} \frac{1}{m} \sum_{n=1}^{\infty} g_n^{-m\sigma} 0(m|s|n^{-\frac{1}{2}} \log^2)$$

$$= 0\left(\sum_{n=1}^{\infty} |s| m_0 \frac{\log^2 n}{g_n^{\sigma} n^{1/2}}\right)$$

$$= 0\left(m_0 |s| \sum_{n=1}^{\infty} \frac{\log^{2-\sigma} n}{n^{1/2+\sigma}}\right)$$

$$= 0\left(m_0 |s| \frac{1}{\frac{1}{2} + \sigma - 1}\right) ,$$

where the third step utilized (16).

We need only combine (15) and (17) into the representation (14) to deduce the convergence of the logarithmic expansion of $\log \phi_1(s)$. Therefore, since $\phi_1(s)$ is bounded, $\zeta_L$ and $\zeta_H$ have the same zeros and poles from which (12) now follows.

To proceed with the proof of the theorem note that, due to (12), it suffices to show that (11) implies the convergence of

$$\phi_2(s) := \frac{\zeta_L(s)}{\zeta(s)} = \prod_{n=1}^{\infty} \left( \frac{1-p_n^{-s}}{1-u_n^{-s}} \right) \tag{18}$$

for all $\mathrm{Re}\, s > \sigma_0$, where $p_n$ is the $n^{\underline{th}}$ rational prime. Again we consider

$$\begin{aligned}
\log \phi_2(s) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} (u_n^{-ms} - p_n^{-ms}) \\
&= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{m_0} + \sum_{m=m_0+1}^{\infty} \right) \frac{1}{m} (u_n^{-ms} - p_n^{-ms}) \\
&= \sum\nolimits_1 + \sum\nolimits_2 ,
\end{aligned}$$

where $m_0$ is an integer greater than $\sigma^{-1}$.

As the convergence of $\sum_2$ follows in exactly the same manner as (15) we need only consider, for every sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
\textstyle\sum_1^N &:= \sum_{p_n \le N} \sum_{m=1}^{m_0} \frac{1}{m} (u_n^{-ms} - p_n^{-ms}) \\
&= \sum_{p_n \le N} \sum_{m=1}^{m_0} \frac{1}{m} u_n^{-ms} \left( 1 - \left( \frac{u_n}{p_n} \right)^{ms} \right) \\
&= \left( \sum_{\substack{p_n \le N \\ |p_n - u_n| \ge n^{1/2+\varepsilon}}} + \sum_{\substack{p_n \le N \\ |p_n - u_n| < n^{1/2+\varepsilon}}} \right) \sum_{m=1}^{m_0} \frac{1}{m} u_n^{-ms} \left( 1 - \left( \frac{u_n}{p_n} \right)^{ms} \right) \\
&= \textstyle\sum_{11}^N + \sum_{12}^N
\end{aligned} \tag{19}$$

as $N \to \infty$. The second sum $\sum_{12}^{N}$ is handled in exactly the same manner as in the derivation of (16) and (17)

$$\sum_{12}^{N} = 0\left(\sum_{p_n \le N} |s| m_0 u_n^{-\sigma-\frac{1}{2}+\varepsilon}\right).$$

We now fix an arbitrary $0 < \varepsilon < \dfrac{\sigma + \frac{1}{2} - 1}{2}$. Then a simple application of the prime number theorem to the limit of summation certainly yields

$$(20)\quad \sum_{12}^{N} = 0\left(|s| m_0 \sum_{n \le \frac{2N}{\log n}} n^{-\sigma-\frac{1}{2}+\varepsilon}\right) = 0\left(|s| m_0 \frac{1}{\sigma + \frac{1}{2} - 1}\right).$$

Evidently, $\sum_{12}^{N}$ is bounded as $N \to \infty$.

As for $\sum_{11}^{N}$, it is a well-known fact that $\theta = \sup_{\beta}\{\beta: \zeta(\beta+i\gamma) = 0\}$ implies $|\pi(x)-\mathrm{Li}(x)| \ll_{\varepsilon} x^{\theta+\varepsilon}$. From this an argument similar to that given in section 1 for the proof of Lemma 1(b) will yield $|p_n - u_n| \ll_{\varepsilon} p_n^{\theta+\varepsilon}$ (however, if $\theta = 1$ we must bound $|p_n - u_n|$ by the sharp form of the prime number theorem and consider the following sums more carefully). With this in mind we deal with $\sum_{11}^{N}$ in the following manner

$$(21)\qquad \sum_{11}^{N} = \sum_{m=1}^{m_0} \frac{1}{m} \sum_{\substack{p_n \le N \\ |p_n - u_n| \ge n^{1/2+\varepsilon}}} u_n^{-ms}\left(1 - \left(\frac{p_n + 0(p_n^{\theta+\varepsilon})}{p_n}\right)^{ms}\right)$$

$$= \sum_{m=1}^{m_0} \frac{1}{m} \sum_{\substack{p_n \le N \\ |p_n - u_n| \ge n^{1/2+\varepsilon}}} u_n^{-ms}(1 - (1 + 0(p_n^{\theta-1+\varepsilon}))^{ms})$$

$$<< \sum_{m=1}^{m_0} \frac{1}{m} \sum_{\substack{p_n \le N \\ |p_n - u_n| \ge n^{1/2+\varepsilon}}} u_n^{-m\sigma} |ms\ 0(p_n^{\theta-1+\varepsilon})|$$

$$<< \sum_{m=1}^{m_0} |s| \sum_{\substack{p_n \le N \\ |p_n - u_n| \ge n^{1/2+\varepsilon}}} u_n^{-\sigma-1+\theta+\varepsilon}$$

$$<< m_0|s| \sum_{\substack{p_n \le N \\ |p_n - u_n| \ge n^{1/2+\varepsilon}}} n^{-\sigma-1+\theta+\varepsilon}$$

$$= m_0|s| \int_1^N x^{-\sigma-1+\theta+\varepsilon}\, dp_\varepsilon(x)\ ,$$

where $dp_\varepsilon$ is the counting measure with jump discontinuities equal to 1 associated with (11). Formal integration by parts yields, for $0 < \varepsilon < \frac{1}{4}$,

$$\int_1^N x^{-\sigma-1+\theta+\varepsilon} dp_\varepsilon(x) = p_\varepsilon(N) N^{-\sigma-1+\theta+\varepsilon} -$$

$$- (\theta+\varepsilon-\sigma-1)\int_1^N p_\varepsilon(x) x^{-\sigma-2+\theta+\varepsilon} dx$$

$$<< \ p_\varepsilon(N) N^{-\sigma-1+\theta+\varepsilon}\ .$$

In virtue of the hypothesis, (11), this last quantity is bounded, for sufficiently small $\varepsilon > 0$, as $N \to \infty$, hence (21) becomes

(22) $$\textstyle\sum_{11}^{N} \ll m_0|s|p_\varepsilon(N)N^{-\sigma-1+\theta+\varepsilon}$$
$$\ll_\varepsilon m_0|s| .$$

We conclude the proof noting that by substituting (20) and (22) into (19) we've established the convergence of

$$\sum_{n=1}^{\infty} \sum_{m=1}^{m_0} \frac{1}{m} (u_n^{-ms} - p_n^{-ms}) \qquad (\mathrm{Re}\, s > \sigma_0) ,$$

and hence also that of the logarithmic expansion of $\log \phi_2(s)$. It follows $\zeta$ has the same zeros as $\zeta_L$ in $\mathrm{Re}\, s > \sigma_0$; $\zeta(s)$ is zero-free in the region $\mathrm{Re}\, s > \sigma_0$ and $\theta \le \sigma_0$. QED.

In conclusion we note that the same method yields

Theorem 3: Suppose that $1 = a_1 < a_2 \le a_3 \le \cdots$ generate a (fixed) multiplicative semigroup A with unique factorization, where for each $\varepsilon > 0$

(23i) $$|p_n - a_n| \ll_\varepsilon n^{\theta+\varepsilon} \quad (0 \le \theta < 1,\ p_n \text{ the } n^{\text{th}} \text{ prime})$$

and for all sufficiently small $\varepsilon = \varepsilon(\sigma_0) > 0$ and all $\theta$ satisfying (23i),

$$\text{(23ii)} \quad A_\varepsilon(N) := \sum_{\substack{|p_n - a_n| \geq n^{\sigma_0+\varepsilon} \\ p_n \leq N}} 1 \ll_\varepsilon N^{1+\sigma_0-\theta+\varepsilon} . \quad (\sigma_0 > 0)$$

Then the function defined by $\zeta_A(s) := \prod_{n=1}^{\infty} (1-a_n^{-s})^{-1}$ in $\mathrm{Re}\, s > 1$ can be analytically continued to $\mathrm{Re}\, s > \sigma_0 > 0$, where it will have the same zeros and poles as $\zeta(s)$.

The form we shall require in the next section is expressed as

Corollary 1: If $k \geq 1$ is an integer and if $\theta = \frac{1}{2k}$, then $\zeta_A(s)$ and $\zeta(s)$ have the same zeros in $\mathrm{Re}\, s > \frac{1}{2k}$.

## Section 3.

The main result in this section is to obtain a class of linear operators, each acting on a dense subspace of a Hilbert space, from which we can derive the non-trival zeros of $\zeta(s)$, provided we are given the eigenvalues. The class of such operators is of cardinality one if and only if the Riemann hypothesis is true. Moreover, we not only see that Theorem 3 yields a number of related operators also characterizing these zeros, but we exhibit some operator-classes which do not yield eigenvalues corresponding to zeros of a zeta function with real part greater than 1/2. The question of whether this can be established for the operator(s) related to $\zeta(s)$ is, of course, equivalent to the Riemann hypothesis.

We define $S_k$ to be the vector space consisting of all complex-valued functions $f \in L^2(0,1)$ such that for each integral j, $0 \le j \le N$ with $N \ge 0$ fixed, the distributional derivatives of f satisfy

$$(24) \qquad \int_0^1 |D^j f(y)|^2 y^{2j-1-(1/k)} dy < \infty .$$

Equivalently, $S_k$ is the completion of $C^N(0,1)$ with respect to the norm

$$(25) \qquad \{\sum_{j=0}^{N} \int_0^1 |D^j f(y)|^2 y^{2j-1-1/k} dy\}^{1/2} , \quad f \in C^N(0,1) .$$

Indeed, we have

Lemma 2: The vector space $S_k$ is complete with respect to the norm $||\cdot||_k$ given by

$$(26) \qquad ||f||_k : = \{\sum_{j=0}^{N} \int_0^1 |D^j f(y)|^2 y^{2j-1-1/k} dy\}^{1/2} < \infty .$$

Proof: The verification that $||\cdot||_k$ is a norm is straightforward.

To see that this induces a complete space, let us first notice that (25) differs from the norm of the one-dimensional Sobolev space $W^{N,2}(0,1)$ (see Yosida [4] p. 55) only from the weight $y^{2j-1-1/k}$. Since the Sobolev space $W^{N,2}$ is a Hilbert space we leave the standard proof that $R_k$ is a Hilbert subspace of $W^{N,2}$ to the reader. QED.

We now proceed to slightly restructure this weighted Sobolev space into a more suitable form as follows. Let $f \in S_k$. Since $||f||_k < \infty$ we may write

$$(27) \qquad \frac{1}{k}\int_0^1 |(1y)^j D^j f(1y)|^2 y^{-1-1/k} dy = \int_1^\infty \left|\left(\frac{1}{t^k}\right)^j D^j f\left(\frac{1}{t^k}\right)\right|^2 dt < \infty$$

where we have substituted $t^k = y^{-1}$. Since for $C^\infty$ functions the distributional derivative coincides with the classical derivative, in virtue of (27) it is now easy to verify that for $\sigma > 1/2k$, $u_{\sigma,t}$, $v_{\sigma,t} \in S_k$, where $\sigma, t \in \mathbb{R}$,

$$(28) \qquad u_{\sigma,t}(y) := y^\sigma \cos(t \log y) ,$$

and

$$(29) \qquad v_{\sigma,t}(y) := y^\sigma \sin(t \log y) .$$

We introduce the equivalent inner product

$$(30) \qquad (f,g)_k := \sum_{j=0}^{N} \int_1^\infty t^{-2jk} (D^j f)(t^{-k}) \overline{(D^j g)(t^{-k})} dt$$

$f,g \in S_k$. In virtue of (27) and Lemma 2, $S_k$ is a Hilbert space with this inner product.

Define the mapping $T: \mathcal{D}_T \subseteq S_k \to S_k$ with the corresponding domain $\mathcal{D}_T$ of $T$ as follows. If $0 \le y \le 1$ and $f \in S_k$ write

(31) $$(Tf)(y) := \sum_{n=1}^{\infty} (-1)^n f\left(\frac{y}{n}\right)$$

provided the sum "converges" (in some well-defined manner) to an element in $S_k$, $Tf \in S_k$. The collection of such $f \in S_k$ we denote as $\mathcal{D}_T$.

Define the mapping M and its domain $\mathcal{D}_M \subseteq S_k$ as follows. For each $0 < y < 1$, $f \in S_k$, let

(32) $$(Mf)(y) := -f(y)\log y$$

provided $Mf \in S_k$. Again, the collection of $f \in S_k$ satisfying this property is denoted $\mathcal{D}_M$.

Lemma 3: $T: \mathcal{D}_T \to S_k$, $M: \mathcal{D}_M \to S_k$ are densely defined linear operators, $\overline{\mathcal{D}}_T = \overline{\mathcal{D}}_M = S_k$ (the closure being taken with respect to the norm topology included by (30)).

Proof: Consider the subspace of $C^N(0,1)$, normed by

(33) $$\{\sum_{j=0}^{N} \int_0^1 |y^j D^j f(y)|^2 y^{-1-1/k} dy\}^{1/2} < \infty .$$

Due to the construction of $S_k$ as the completion of $C^N(0,1)$ with respect to (33) we see that $\overline{C^N(0,1)} = S_k$ , in the norm topology (33). Since the polynomials with zero constant term are dense in the space of continuous functions f on (0,1) with f(0) = 0, in the sup-norm topology, they are thus dense in $C^N(0,1)$ in the weaker topology (33), and thus also in $S_k$.

Choose an arbitrary $f \in S_k$. Then for each polynomial $r_\varepsilon$,

$$r_\varepsilon(y) = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y ,$$

where $r_\varepsilon \in S_k$ and $||f-r_\varepsilon||_k < \varepsilon$, it is easy to establish $Tr_\varepsilon \in S_k$ using (31). Thus $\overline{\mathcal{D}_T} = S_k$. Moreover, we need only note that the functions $q_i$ , $i = 1,2,\cdots$ , defined by $q_i(y) := y^i \log y$ belong to $S_k$. Thus $r_\varepsilon(y)\log y$ defines a function in $S_k$ , so $\overline{\mathcal{D}_M} = S_k$. QED.

The spaces we will use in the theorems to follow are defined by $(k = 1,2,\cdots)$

$$(34) \qquad H_k = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : f_1, f_2 \in S_k \right\} .$$

Give $H_k$ the inner product, $f_i \in S_k (i = 1,2,3,4)$

$$(35) \qquad \left( \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} , \begin{bmatrix} f_3 \\ f_4 \end{bmatrix} \right)_H := (f_1,f_3)_k + (f_2,f_4)_k .$$

Lemma 3: For integral $N \geq 2$, $k \geq 1$, and $x \geq 0$, let the map $B = B(\beta)$ be given by

(36) $$B = \begin{bmatrix} J_\beta & 0 \\ T & M \end{bmatrix},$$

where $J_\beta$ is given for each $f \in S_k$ by

(37) $$(J_\beta f)(y) := y^2 D^2 f(y) + (1-2\beta) y Df(y) \ .$$

Again, here $D^j$, $j = 1,2$, denote the distributional derivatives of f. For $N \geq 2$, B is a linear operator on $\mathcal{D}_B \subseteq H_k$, $\overline{\mathcal{D}}_B = H_k$.

Remark: Note that the definition of $S_k$ yields $H_1 \subset H_2 \subset H_3 \subset \cdots$. We note that the result below, which relates to a question of Montgomery mentioned in the introduction, tells us that for every $\beta > \frac{1}{2}$, $B(\beta)$, has no real eigenvalues in $H_1$ if and only if the Riemann hypothesis is true.

Theorem 4: (a) Fix $0 < \beta < 1$. Suppose for some $k \geq 1$ that $\lambda = -(\beta^2+\gamma^2)$, $\gamma \in \mathbb{R}$, is an eigenvalue of $B(\beta)$ with eigenvector $\vec{u} \in H_k$. Then both $\beta + i\gamma$, $\beta - i\gamma$ are zeros of $\zeta(s)$.

(b) Conversely, if $\beta + i\gamma$ is a non-trivial zero of $\zeta$, $\frac{1}{2k} < \beta$, then $-(\beta^2+\gamma^2)$ is an eigenvalue of $B(\beta): H_k \to H_k$.

Proof: Since $H_k \subset H_{k+1} \subset \cdots$, without loss of generality, we may let $k = [1/\beta] \geq 1$, where $[x]$ denotes the greatest integer less than $x$. As it's easy to verify $b[1/b] > \frac{1}{2}$ for all $0 < b \leq 1$, we have $\beta > 1/2k$.

One may verify the fundamental (since functions in $W^{N,2}(0,1)$ are absolutely continuous) solutions

$$(38) \qquad \begin{aligned} y^2D^2u_{\beta,\gamma}(y) + (1-2\beta)yDu_{\beta,\gamma}(y) + (\gamma^2+\beta^2)u_{\beta,\gamma}(y) &= 0 , \\ y^2D^2v_{\beta,\gamma}(y) + (1-2\beta)yDv_{\beta,\gamma}(y) + (\gamma^2+\beta^2)v_{\beta,\gamma}(y) &= 0 \end{aligned}$$

either by straightforward differentiation or by solving the second order Cauchy-Euler differential equation via the roots of the associated polynomial

$$(39) \qquad p(y) = b_0y(y-1) + b_1y + b_2 ,$$

where $b_1/b_0 = 1 - 2\beta$ and $b_2/b_0 = \beta^2 + \gamma^2$. In any case, it follows from (38) that

$$(40) \qquad \begin{aligned} (J_\beta u_{\beta,\gamma})(y) &= \lambda u_{\beta,\gamma}(y) , \\ (J_\beta v_{\beta,\gamma})(y) &= \lambda v_{\beta,\gamma}(y) . \end{aligned}$$

Moreover, since $\mathrm{Re}\{\zeta(\beta+i\gamma)(1-2^{1-\beta-i\gamma})\} = \sum_{n=1}^{\infty} (-1)^n n^{-\beta}\cos(\gamma \log n)$ $\mathrm{Im}\{\zeta(\beta+i\gamma)(1-2^{1-2})\} = -\sum_{n=1}^{\infty} (-1)^n n^{-\beta}\sin(\gamma \log n)$ may be considered as conditionally convergent sums we see that (31) gives

$$(41)\qquad \begin{aligned}(Tu_{\beta,\gamma})(y) &= 0\ ,\\ (Tv_{\beta,\gamma})(y) &= 0\ ,\end{aligned}$$

a.e.

It remains only to piece together (40), (41), and (36) to see that we have two linearly independent eigenvector solutions

$$B(\beta)\begin{bmatrix} u_{\beta,\gamma} \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} u_{\beta,\gamma} \\ 0 \end{bmatrix} \text{ and } B(\beta)\begin{bmatrix} v_{\beta,\gamma} \\ 0 \end{bmatrix} = \lambda\begin{bmatrix} v_{\beta,\gamma} \\ 0 \end{bmatrix}$$

as required.

Proof of (a): We suppose that for some $\begin{bmatrix} f \\ g \end{bmatrix} \in H_k$

$$B(\beta)\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} J_\beta f \\ Tf + Mg \end{bmatrix} = \lambda\begin{bmatrix} f \\ g \end{bmatrix},$$

where $\beta > \frac{1}{2k}$. From this evidently follows

$$(42)\qquad (J_\beta f)(y) = \lambda f(y).$$

and

$$(43)\qquad (Tf)(y) - (\log y)g(y) = \lambda g(y)$$

In particular, we get from (43)

$$\frac{(Tf)(y)}{\log y + \lambda} = g(y) \quad . \tag{44}$$

Reasoning from this relation, we note that it can be shown that this and (44) imply either $g \notin S_k$ or $g \equiv 0$ as follows. The solutions of (42), seen in (40), combined with the definition of T in (31) show that $Tf = C_1 u_{\beta,\gamma} + C_2 v_{\beta,\gamma}$, $C_1, C_2$ constants. By considering the first integral $(j = 0)$ in the norm-sum (33), we then obtain, since $\lambda < 0$, and if continuous, $g \notin S_k$ or $g \equiv 0$, as the resulting integral does not converge. From this it clearly follows $g \equiv 0$ (since g is a component of an eigenvector in $H_k$) and by (44),

$$(Tf)(y) \equiv 0 \quad . \tag{45}$$

We have mentioned that the solutions to (42) are generated by the functions (28) and (29). So it follows that because

$$(Tu_{\beta,\gamma})(y) - i(Tv_{\beta,\gamma})(y) = y^{\beta+i\gamma}\zeta(\beta+i\gamma) \quad , \quad \text{a.e.,}$$

the only possibility, in virtue of (45), is $\zeta(\beta+i\gamma) = 0$. Similarly,

$$(Tu_{\beta,-\gamma})(y) + i(Tv_{\beta,-\gamma})(y) = y^{\beta-i\gamma}\zeta(\beta-i\gamma)$$

yields $\zeta(\beta-i\gamma) = 0$. QED.

Although we shall not prove or need the result here, we note for the sake of completeness that for any complex number $\lambda = \gamma_1 + i\gamma_2$, $\gamma_1, \gamma_2 \in \mathbb{R}$, $\gamma_2 \neq 0$, we can find a $k \geq 1$ and $\beta > \frac{1}{2k}$ such that $\lambda$ is an eigenvalue for $B(\beta)$ on $H_k$. Worse yet, even if we replace the infinite sum (31) for $T$ by a finite one the operator will still not have an adjoint.

Theorem 5: Let $A$ denote the multiplicative semigroup generated by $\{1, a_1, a_2, a_3, \cdots : a_n > 1\}$. Assume $A$ has unique factorization and that for every $\varepsilon > 0$, $|p_n - a_n| \ll_\varepsilon n^{\frac{1}{2k}+\varepsilon}$, $k \geq 1$. The linear operator given by

$$(T_A f)(y) := \sum_{a \in A} f\left(\frac{y}{a}\right)$$

can be defined on a domain $\mathcal{D}_A \subseteq H_k$ analogous to (31). Moreover,

$$B = B(\beta) = \begin{bmatrix} J_\beta & 0 \\ T & M \end{bmatrix} \text{ and } B_A = B_A(\beta) = \begin{bmatrix} J_\beta & 0 \\ T_A & M \end{bmatrix} \tag{46}$$

have the same set of (real) negative eigenvalues $\lambda \leq -\beta^2 < -\frac{1}{4k^2}$.

Theorem 6: Define the operator $T_U$: $\mathcal{D}_U \to \mathcal{D}_U$ by

$$(T_U f)(y) := \sum f\left(\frac{y}{u}\right)$$

(where the summation ranges over all $u$ in the multiplicative semigroup $U$ generated by the $(1 < u_1 < u_2 < \cdots)$) analogous to (31) with domain $\mathcal{D}_U \subseteq H_1$, where the $u_n$'s are as in section 2. Then the operator $B_U(\beta)$: $\mathcal{D}_U \to \mathcal{D}_U$

$$B_U(\beta) := \begin{bmatrix} J & 0 \\ T_U & M \end{bmatrix}$$

has no real eigenvalues.

Remark: The latter two theorems will neither proven nor stated in fullest possible content. The procedures used in proving them can be carried over mutatis mutandis from Theorem 4 (followed by the Corollary) and Theorem 2 (specifically equation (12)), respectively.

This theorem, concerning a linear operator (associated with a class of arbitrary semigroups A) having an invariant set of negative eigenvalues, is intended to motivate the solution of the conjecture (also due to Montgomery [5]) that the pair correlation of the zeros of the zeta function behaves, on the average, like $1 - ((\sin \pi u)/\pi u)^2$. It is conjectured that, perhaps after certain transformation, the functions can be expanded in a suitable basis yielding a discrete complex Hermitian matrix corresponding to $B_H$, for every semigroup H satisfying the hypothesis of Theorem 4. If the n-dimensional restrictions of the resulting "random" matrix (corresponding to random generators) actually form a suitably general "unitary ensemble" (see Mehta [6], eqs. (2.49)-(2.51)) then the two-level eigenvalue correlation function is known to be given by $((\sin \pi u)/\pi u)^2$ (Mehta [6], eq. (6.13)), and Montgomery's conjecture will follow. At any rate, these results obviously offer further support of his heuristic analogy (Montgomery [5], p. 184) for the distribution conjecture.

I thank Paul Erdos for his inspiration; he has also pointed out that another random number theoretic approach, comparitive to our section 1, has been initiated by Hawkins. See [7] for a recent set of references and summary.

In conclusion, I gratefully acknowledge the encouraging and adroit suggestions of my teacher Professor Johan Belinfante, who effected many significant improvements in section 3.

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