# SM444 notes on algebraic graph theory

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These are notes[1] on algebraic graph theory for sm444.

This course focuses on "calculus on graphs" and will introduce and study the graph-theoretic analog of (for example) the gradient.

For some well-made short videos on graph theory, I recommend Sarada Herke's channel on youtube.

## Contents



[1] See Biggs [Bi93], Joyner-Melles [JM], Joyner-Phillips [JP] and the lecture notes of Prof Griffin [Gr17] for details.

# 1 Introduction

Algebraic graph theory is a field where one uses *algebraic* techniques to better understand properties of graphs. These techniques may come from matrix theory, the theory of polynomials, or topics from modern algebra such as group theory or algebraic topology.

For us, a *graph* $\Gamma = (V, E)$ is a pair comprising of a finite set of *vertices* $V$ and a set of *edges*

$$E \subset \{\{u, v\} \mid u, v \in V\},$$

consisting of pairs of vertices. We shall assume throughout that there are no *loops*, i.e., $u \neq v$, and no multiple edges. Such a graph is called a *simple graph*.

Sagemath can be used to do many graph-theoretic computations[2]. An introduction can be found in §9 below.

The *empty graph* is a graph having at least one vertex but no edges. The *null graph* is a graph having no vertices and no edges.

A graph $\Gamma_1 = (V_1, E_1)$ is called a *subgraph* of $\Gamma_2 = (V_2, E_2)$ if (a) $V_1 \subset V_2$, (b) $E_1 \subset E_2$. Obviously, since $\Gamma_1$ is a graph, the vertex subset $V_1$ must include all endpoints of the edge subset $E_1$.

A subgraph $\Gamma_1 = (V_1, E_1)$ of $\Gamma_2 = (V_2, E_2)$ is a *spanning subgraph* if $V_1 = V_2$. A subgraph of $\Gamma = (V, E)$ is an *induced subgraph* (or, to be more precise, a *vertex-induced subgraph*) if it has the following property: two vertices of $\Gamma_1$ are adjacent if and only if they are adjacent in $\Gamma_2$. Alternatively, a vertex-induced subgraph is obtained by removing a subset $S \subset V$ of vertices, and all edges in $E$ incident to at least one of them. In other words, a vertex-induced subgraph is a subgraph $\Gamma' = (V', E')$ of $\Gamma = (V, E)$ if $V' \subset V$ and if $E'$ consists of those edges in $E$ whose endpoints are in $V'$.

An *edge-induced subgraph* is a subgraph $\Gamma' = (V', E')$ of $\Gamma = (V, E)$ if $E' \subset E$ and if $V'$ consists of those vertices in $V$ are incident to edges in $E'$.

**Exercise 1.1.** *Find all subgraphs of the house graph (see Exercise 7, or Figure 12) and all spanning subgraphs.*

If $\Gamma_1 = (V_1, E_1)$ an $\Gamma_2 = (V_2, E_2)$ are two simple graphs then the *union* of these is the graph $\Gamma_1 \cup \Gamma_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

**Exercise 1.2.** *Show that* $\Gamma_1 \cup \Gamma_2 = \Gamma_2 \cup \Gamma_1$.

**Example 1.** *The union of the house graph, depicted in Figure 12, and the diamond graph, depicted in Figure 26, is depicted in Figure 1.*

The *intersection* of $\Gamma_1$ and $\Gamma_2$ is the graph $\Gamma_1 \cap \Gamma_2 = (V_1 \cap V_2, E_1 \cap E_2)$.

[2] Sagemath is a mathematics software package, much like Maple or Mathematica, or Matlab, except it is free to install on your computer (windows, mac, linux) or to use online as a web application at `https://cocalc.com` or `https://sagecell.sagemath.org`. The `sagecell` should only be used for short quick computations that you don't want to save or share.

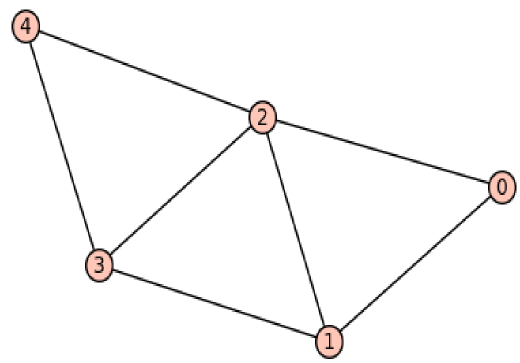


Figure 1: The union of the house graph and the diamond graph.

The *ring sum* of $\Gamma_1$ and $\Gamma_2$ is the graph $\Gamma_1 \oplus \Gamma_2$ whose vertex set is $V_1 \cup V_2$ and whose edges set consists of those edges in $\Gamma_1$ or $\Gamma_2$ but not in both, i.e., their symmetric difference $E_1 \Delta E_2$.

Let $\Gamma = (V, E)$ be a simple graph. If a vertex $v$ shares an edge with another vertex $w \neq v$ then we say $v$ and $w$ are *adjacent* or are *neighbors*. The set of all neighbors of $v$ is denoted

$$N_\Gamma(v) = \{w \in V \mid v, w \text{ adjacent}\}. \tag{1}$$

If a vertex $v$ is contained in an edge $e$ then we say $v$ and $e$ are *incident*. For $v \in V$, define

$$\Gamma - v$$

to be the subgraph with vertices $V - \{v\}$ and whose edges are those of $\Gamma$ except for all the edges incident to $v$. If $\{v_1, \dots, v_k\} \subset V$ is a set of distinct vertices then we define[3]

$$\Gamma - \{v_1, \dots, v_k\} = (\Gamma - \{v_1, \dots, v_{k-1}\}) - v_k$$

inductively. The *vertex connectivity* of $\Gamma$ is the size of the smallest set $S \subset V$ such that $\Gamma - S$ is disconnected. For example, the vertex connectivity of the friendship graph $F_n$, $n \geq 2$, is 1.

For $e \in E$, define

$$\Gamma - e$$

[3]It must be proven that this deinition is independent of the indexing choosen.

to be the subgraph with vertices $V$ and edges $E - \{e\}$. If $\{e_1, \dots, e_k\} \subset E$ is a set of distinct edges then we define[4]

$$\Gamma - \{e_1, \dots, e_k\} = (\Gamma - \{e_1, \dots, e_{k-1}\}) - e_k$$

inductively. The *edge connectivity* of $\Gamma$ is the size of the smallest set $S \subset E$ such that $\Gamma - S$ is disconnected. For example, the edge connectivity of the path graph $P_n$, $n \geq 3$, is 1.

Note that $\Gamma - v$ is an induced subgraph but $\Gamma - e$ is not.

The number of edges in $\Gamma$ incident to $v$ is called the *degree* (or *valency*) of $v$, denoted $\deg(v)$, or $\deg_\Gamma(v)$ to be more precise. A vertex of degree 1 is called a *leaf*.

As a matter of fixing notation, we shall usually set

$$V = \{0, 1, \dots, n-1\} \quad \text{or} \quad V = \{1, \dots, n\},$$

for some $n > 0$.

**Lemma 2.** *In any graph $\Gamma = (V, E)$, the sum of the degrees is equal to twice the number of edges, that is,*

$$\sum_{v \in V} \deg(v) = 2|E|.$$

*Proof.* The degree $\deg(v)$ counts the number of times $v$ appears as an endpoint of an edge. Since each edge has two endpoints, the sums $\sum_{v \in V} \deg(v)$ and $\sum_{e \in E} 2$ must agree. $\square$

The sequence of values of $\deg(v)$ ($v \in V$), sorted from least to most, is called the *degree sequence* of $\Gamma$. The lemma above implies that there are sequences of positive integers which do *not* arise as the degree sequence of a graph.

**Exercise 1.3.** *Prove that the number of odd numbers in a degree sequence is even.*

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[4] It must be proven that this deinition is independent of the indexing choosen.

# 2 Examples

In this section, we introduce specific examples to help make later concepts more concrete.

**Definition 3.** A graph with $n$ vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges $(v_1, v_2)$, $(v_2, v_3)$, ..., $(v_{n-1}, v_n)$ is called a *path graph* and denoted $P_n$.

The degree of every vertex of a path graph is either 1 (on the ends) or 2 (in the middle).

When, for some $n > 2$, a graph $\Gamma = (V, E)$ has $P_n$ as a subgraph, we say $\Gamma$ *contains a path of length* $n$. The *length* of a path in $\Gamma$ is the number of edges in the path. If $u, v \in V$ are distinct vertices and $\ell$ is the length of the shortest path $P$ in $\Gamma$ having $u, v$ as its leafs, then we call $\ell$ the *distance between* $u$ *and* $v$, written

$$\delta_\Gamma(u, v) = \ell.$$

**Example 4.** The adjacency matrix of $P_3$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

that of $P_4$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and that of $P_5$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We use Sagemath to compute characteristic polynomials (defined in §4.3 below).

Sagemath

```
sage: Gamma = graphs.PathGraph(3)
sage: Gamma.charpoly()
x^3 - 2*x
sage: Gamma = graphs.PathGraph(4)
sage: Gamma.charpoly()
x^4 - 3*x^2 + 1
sage: Gamma = graphs.PathGraph(5)
sage: Gamma.charpoly()
x^5 - 4*x^3 + 3*x
sage: Gamma = graphs.PathGraph(6)
sage: Gamma.charpoly()
x^6 - 5*x^4 + 6*x^2 - 1
```

**Definition 5.** A *star graph* having $n$ vertices, denoted $St_n$, is a graph in which the vertex set is $V = \{0, 1, 2 \ldots, n-1\}$, and the edge set is $E = \{(0,1), (0,2), \ldots, (0, n-1)\}$. In other words, it's a graph having one vertex of degree $n-1$ and all the other vertices are leaves (of degree 1).

Some people find it useful to regard the augmented neighborhood of $v$, $\{v\} \cup N_\Gamma(v)$, as the vertices of the star graph $St_d$ centered at $v$, where $d = \deg_\Gamma(v)$.

**Example 6.** The adjacency matrix of $St_3$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

that of $St_4$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and that of $St_5$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Sagemath

```
sage: Gamma = graphs.StarGraph(3)
sage: Gamma.charpoly()
x^4 - 3*x^2
sage: Gamma = graphs.StarGraph(4)
sage: Gamma.charpoly()
x^5 - 4*x^3
sage: Gamma = graphs.StarGraph(5)
sage: Gamma.charpoly()
x^6 - 5*x^4
```

Do you see any patterns?

**Example 7.** The *house graph* is depicted in Figure 2. The degrees are tabulated below.

| $v$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\deg(v)$ | 3 | 2 | 3 | 2 | 2 |

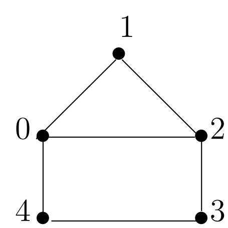


Figure 2: The house graph.

**Exercise 2.1.** *Let* $\Gamma$ *be the house graph depicted in Figure 2 and let* $\Gamma'$ *be the subgraph with the same vertices but missing the edges* $(0,2)$ *and* $(3,4)$. *Show that* $\Gamma'$ *is a spanning subgraph but not an induced subgraph.*

**Example 8.** Let $n$ be an integer with $n \geq 3$. A graph with $n$ vertices $V = \{v_1, v_2, \ldots, v_n\}$ and edges $(v_1, v_2)$, $(v_2, v_3)$, $\ldots$, $(v_{n-1}, v_n)$, $(v_n, v_1)$ is called a *cycle graph* and denoted $C_n$. The degree of every vertex of a cycle graph is 2.

The graph $C_5$ is shown in Figure 3.

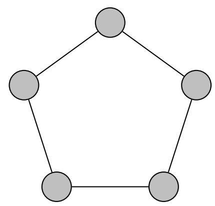

Figure 3: Cycle graph $C_5$

The adjacency matrix of $C_3$ is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

that of $C_4$ is

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

and that of $C_5$ is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We use Sagemath to compute characteristic polynomials (defined in §4.3 below).

Sagemath

```
sage: Gamma = graphs.CycleGraph(3)
sage: Gamma.charpoly()
x^3 - 3*x - 2
```

```
sage: Gamma = graphs.CycleGraph(4)
sage: Gamma.charpoly()
x^4 - 4*x^2
sage: Gamma = graphs.CycleGraph(5)
sage: Gamma.charpoly()
x^5 - 5*x^3 + 5*x - 2
```

When, for some $n > 2$, a graph $\Gamma$ has $C_n$ as a subgraph, we say $\Gamma$ *contains a cycle of length* $n$.

**Definition 9.** Let $G = \mathbb{Z}/n\mathbb{Z}$ denote the additive group of integers mod $n$ and let $C \subset G$ be a subset closed under negation (i.e., if $c \in C$ then $-c \in C$). Let $\Gamma = (V, E)$ be the graph defined by $V = G$ and, for $v, w \in V$, we say that $(v, w) \in E$ holds if and only if $w - v \in C$.

The *circulant graph* on $n$ vertices, also called an *additive Cayley graph*, is any graph constructed in the above fashion, for some $n$ and some $C \subset \mathbb{Z}/n\mathbb{Z}$. The notation $Cay(C, G)$ is sometimes used.

For example, the cycle graph on $n$ vertices is also a circulant graph (take $C = \{\pm 1\} \subset G = \mathbb{Z}/n\mathbb{Z}$).

**Definition 10.** A *bipartite graph* is a graph in which the vertex set $V$ can be partitioned into two sets, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, so that no two vertices in $V_1$ share a common edge and no two vertices in $V_2$ share a common edge.

For example, the cycle graph $C_4$ (which we can visualize as a square) is bipartite. However, the cycle graph $C_5$ is not. Other examples: for each $n > 1$, the star graph $S_n$ is bipartite.

**Exercise 2.2.** *Prove that the cycle graph $C_6$ is bipartite using only definitions of cycle graph and bipartite graph.*

**Exercise 2.3.** *Prove that, for each $k > 1$, the cube graph $Q_k$ (see Exercise 2.7) is bipartite.*

**Definition 11.** A *finite projective plane of order $n$* is a nonempty finite set $X$, whose elements are called *points*, and a nonempty finite set $L$, whose elements are called *lines*, of subsets of $X$ such that:

1. For every two distinct points, there is exactly one line that contains both points. We say this line is *incident* to these points.

2. The intersection of any two distinct lines contains exactly one point.
3. Each point is contained in $n + 1$ lines.
4. Each line contains in $n + 1$ points.
5. $|X| = |L| = n^2 + n + 1$.

**Example 12.** Consider the bipartite graph $\Gamma = (V, E)$ whose vertices are the $2(n^2+n+1)$ points and lines in the plane, and two vertices are connected by an edge if and only if one is a point and the other is line in $X$ that contains that point.

This is an example of a graph associated to a finite projective plane.

For instance, consider the "lines" of the form $\{x + 1, x + 2, x + 4\}$, where $x \in GF(7)$. This configuration forms a finite projective plane called the Fano plane, depicted in Figure 4.

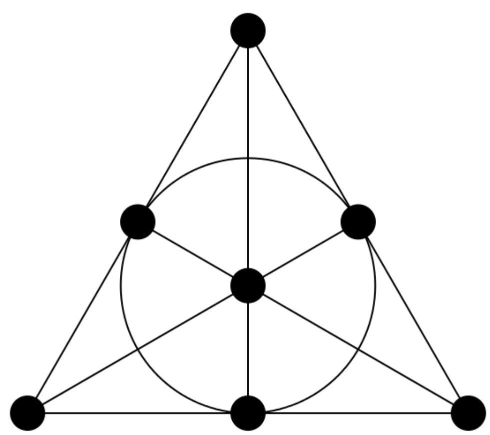

Figure 4: The Fano plane.

Sagemath

```
sage: for x in GF(7):
....:     L = [x+1,x+2,x+4]
....:     L.sort()
....:     print x, L
....:
0 [1, 2, 4]
```

```
1 [2, 3, 5]
2 [3, 4, 6]
3 [0, 4, 5]
4 [1, 5, 6]
5 [0, 2, 6]
6 [0, 1, 3]
sage: B = matrix(ZZ,[[0,0,0,1,0,1,1],[1,0,0,0,1,0,1],
  [1,1,0,0,0,1,0],[0,1,1,0,0,0,1],
  [1,0,1,1,0,0,0],[0,1,0,1,1,0,0],[0,0,1,0,1,1,0]])
sage: B

[0 0 0 1 0 1 1]
[1 0 0 0 1 0 1]
[1 1 0 0 0 1 0]
[0 1 1 0 0 0 1]
[1 0 1 1 0 0 0]
[0 1 0 1 1 0 0]
[0 0 1 0 1 1 0]
sage: A = block_matrix(2, 2, [ B*0, B, B.transpose(), B*0 ])
sage: Gamma = Graph(A, format='adjacency_matrix')
sage: Gamma.is_bipartite()
True
```

**Exercise 2.4.** *Find a labeling of the Fano plane in Figure 4.*

**Definition 13.** The *complete graph* on $n$ vertices, denoted $K_n$, is the graph in which every pair of distinct vertices is connected by an edge. The degree of every vertex of $K_n$ is $n$.

Note the graph $K_3$ is simply the cycle graph on 3 vertices, $C_3$. The complete graph on 4 vertices, $K_4$, is also called the *tetrahedron graph*.

**Example 14.** The adjacency matrix of $K_3$ is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

that of $K_4$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

and that of $K_5$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Sagemath

```
sage: Gamma = graphs.CompleteGraph(3)
sage: Gamma.charpoly()
x^3 - 3*x - 2
sage: Gamma = graphs.CompleteGraph(4)
sage: Gamma.charpoly()
x^4 - 6*x^2 - 8*x - 3
sage: Gamma = graphs.CompleteGraph(5)
sage: Gamma.charpoly()
x^5 - 10*x^3 - 20*x^2 - 15*x - 4
sage: Gamma = graphs.CompleteGraph(6)
sage: Gamma.charpoly()
x^6 - 15*x^4 - 40*x^3 - 45*x^2 - 24*x - 5
```

Do you see any patterns?

**Lemma 15.** *A graph $\Gamma = (V, E)$ is complete if and only if, for all $v \in V$, $N_\Gamma(v) = V - \{v\}$.*

*Proof.* ($\Rightarrow$) If $\Gamma$ is complete and $v \in V$ then $v$ is connected by an edge to every other vertex, so $N_\Gamma(v) = V - \{v\}$.

($\Leftarrow$) If $N_\Gamma(v) = V - \{v\}$, for each $v \in V$, then every vertex is connected to every other vertex. In other words, each pair of distinct vertices is connected by an edge, so $\Gamma$ is complete. □

**Definition 16.** If $\Delta$ is a subgraph of $\Gamma$ and if $\Delta$ is a complete graph then we call $\Delta$ a *clique* of $\Gamma$. In other words, a clique is a subgraph in which any two vertices are connected by an edge. The *clique number* of $\Gamma$, denoted $\omega(\Gamma)$, is the number of vertices contained in the largest clique of $\Gamma$.

For example, the largest clique of the house graph, depicted in Figure 2, is the subgraph defined by the roof, so to speak. Therefore, the clique number of house graph is 3.

**Exercise 2.5.** *Find the clique number of the tetrahedron graph.*

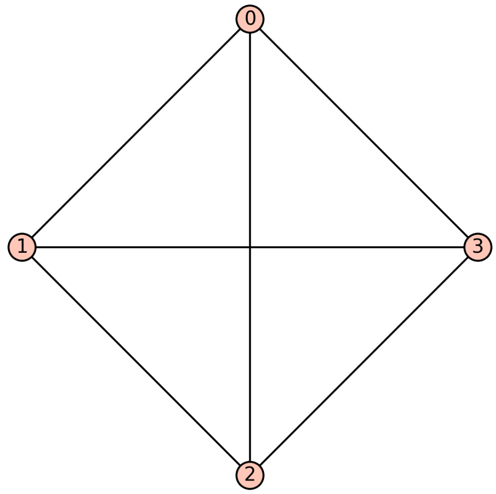


Figure 5: Complete graph $K_4$.

**Exercise 2.6.** *Let $\Gamma'$ be a clique of a graph $\Gamma$. Show that the subgraph of $\Gamma'$ obtained by removing a vertex from $\Gamma'$ is still a clique of $\Gamma$.*

**Lemma 17.** *Let $\Gamma = (V, E)$ be a connected graph. The graph $\Gamma$ is bipartite if and only if it has no cycles of odd length.*

In fact, connectedness is not needed for the lemma to hold. It is left as an exercise to extend this lemma to the disconnected case.

*Proof.* ($\Rightarrow$) Assume that $\Gamma$ is bipartite and that

$$v_0 \to v_1 \to \cdots \to v_{k-1} \to v_k = v_0$$

is a cycle of odd length (i.e., $k$ is odd). Decompose $V$ into a disjoint sum

$$V = V_1 \cup V_2, \qquad V_1 \cap V_2 = \emptyset,$$

and each edge $e \in E$ has one vertex in $V_1$ and the other in $V_2$. Call the vertices in $V_1$ "blue" and the vertices in $V_2$ "gold". Each edge must have a blue and a gold vertex. Assume without loss of generality (WLOG) that $v_0$ is blue. Then all the even-indexed vertices in the cycle are blue. The others much be gold. Since $k$ is odd, vertex $v_k$ much be gold. But $v_k = v_0$, so it must be blue. This is a contradiciton, so *if* $\Gamma = (V, E)$ is bipartite then it has no cycles of odd length.

($\Leftarrow$) Assume $\Gamma$ has no cycles of odd length. Fix a vertex $v_0 \in V$. Define

$$V_1 = \{w \in V \mid \delta(w, v_0) \text{ is even}\},$$
$$V_2 = \{w \in V \mid \delta(w, v_0) \text{ is odd}\},$$

so

$$V = V_1 \cup V_2, \qquad V_1 \cap V_2 = \emptyset.$$

We show that if $e = (v, w) \in E$ then $v, w \in V_1$ is impossible and that $v, w \in V_2$ is impossible.

Suppose $v, w \in V_1$. Let

$$v_0 \to v_1 \to \cdots \to v_{\ell-1} \to v_\ell = v$$

be a shortest path from $v_0$ to $v$ and let

$$v_0 \to w_1 \to \cdots \to w_{k-1} \to w_k = w$$

be a shortest path frrm $v_0$ to $w$. Both of these paths are even length.

These two path are not disjoint, since they share $v_0$. But $v_0$ might not be the only vertex they share. Let $u \in V$ be the last vertex in these paths that are in common. So, both paths go from $v_0$ to $u$, but then the first path goes to $v$ and the second one goes to $w$:

$$v_0 \to v_1 \to \cdots \to v_i = u \to \cdots \to v_{\ell-1} \to v_\ell = v,$$
$$v_0 \to w_1 \to \cdots \to w_j = u \to \cdots \to w_{k-1} \to w_k = w.$$

Since both these paths are shortest, we must have $i = j$. Therefore, the paths

$$u \to \cdots \to v_{\ell-1} \to v_\ell = v,$$
$$u \to \cdots \to w_{k-1} \to w_k = w,$$

are both even length (if $i = j$ is even) or both odd length (if $i = j$ is odd). In either case, the cycle

$$u \to \cdots \to v_{\ell-1} \to v_\ell = v \to w = w_k \to w_{k-1} \to \cdots \to u$$

has odd length. This is a contradition. Therefore, $v, w \in V_1$ is impossible.

The proof that $v, w \in V_2$ is impossible is similar. $\square$

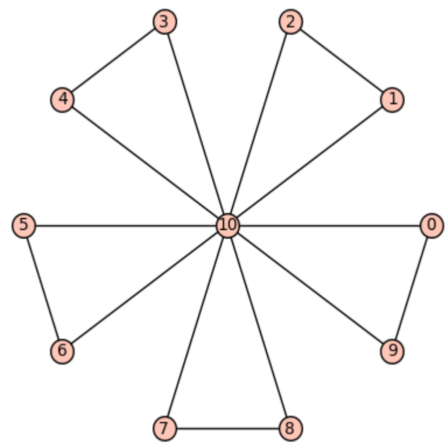

Figure 6: The friendship graph $F_5$

**Definition 18.** The *friendship graph* $F_n$ is the graph with $2n+1$ vertices and $3n$ edges constructed by joining $n$ copies of the cycle graph $C_3$ with a common vertex.

For example, $F_5$ is depicted in Example 6.

**Example 19.** The adjacency matrix of $F_3$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

that of $F_4$ is

$$\begin{pmatrix} 0&0&0&0&0&0&0&1&1\\ 0&0&1&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&1\\ 0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&1&0&0&1\\ 1&0&0&0&0&0&0&0&1\\ 1&1&1&1&1&1&1&1&0 \end{pmatrix},$$

and that of $F_5$ is

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&1&0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&1&0&0&1\\ 1&0&0&0&0&0&0&0&0&0&1\\ 1&1&1&1&1&1&1&1&1&1&0 \end{pmatrix}.$$

Sagemath

```
sage: Gamma = graphs.FriendshipGraph(3)
sage: Gamma.charpoly()
x^7 - 9*x^5 - 6*x^4 + 15*x^3 + 12*x^2 - 7*x - 6
sage: Gamma = graphs.FriendshipGraph(4)
sage: Gamma.charpoly()
x^9 - 12*x^7 - 8*x^6 + 30*x^5 + 24*x^4 - 28*x^3 - 24*x^2 + 9*x + 8
sage: Gamma = graphs.FriendshipGraph(5)
sage: Gamma.charpoly()
x^11 - 15*x^9 - 10*x^8 + 50*x^7 + 40*x^6 - 70*x^5 - 60*x^4 + 45*x^3 + 40*x^2 - 11*x - 10
```

Can you spot any patterns?

**Definition 20.** Let $n > 3$ be an integer. The *wheel graph* $W_n$ is the graph with $n$ vertices and $2n-2$ edges constructed by connecting a "center" vertex to all $n-1$ vertices of a cycle graph $C_{n-1}$ by edges ("spokes").

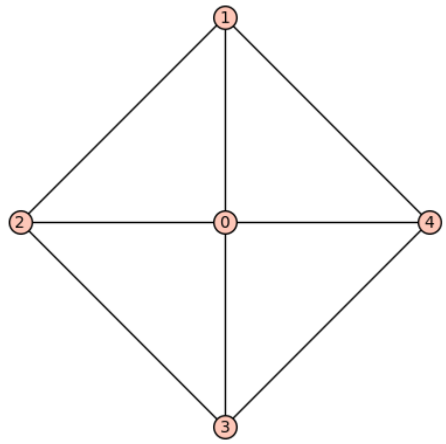


Figure 7: The wheel graph $W_5$

For example, $W_5$ is depicted in Figure 7.

**Example 21.** The adjacency matrix of the wheel graph $W_4$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

and that of $W_5$ is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

Sagemath

```
sage: Gamma = graphs.WheelGraph(4)
sage: Gamma.charpoly()
x^4 - 6*x^2 - 8*x - 3
sage: Gamma = graphs.WheelGraph(5)
sage: Gamma.charpoly()
x^5 - 8*x^3 - 8*x^2
sage: Gamma = graphs.WheelGraph(6)
sage: Gamma.charpoly()
x^6 - 10*x^4 - 10*x^3 + 10*x^2 + 8*x - 5
```

**Definition 22.** A *tree* is a connected graph with no cycles.

A spanning subgraph of $\Gamma$ which is a tree is called a *spanning tree* of $\Gamma$. For example, a spanning tree for the house graph in Example 7 is the subgraph formed by the edges of the roof and the two sides (but not the bottom).

**Lemma 23.** *If $\Gamma = (V, E)$ is a tree then any connected subgraph of $\Gamma$ is also a tree.*

*Proof.* Exercise (hint: use proof by contradiction). □

**Lemma 24.** *If $\Gamma = (V, E)$ is a tree then $\Gamma$ has a leaf.*

*Proof.* Let $P$ be a path[5] of maximal length $k > 1$ in $\Gamma$, denoted

$$v_0 \to v_1 \to \cdots \to v_k, \qquad v_i \in V.$$

The vertices in the interior of the path, $v_i$ for $i \leq i \leq k-1$, have degree at least 2: $\deg(v_i) \geq 2$ $(i \leq i \leq k-1)$. However, the vertices at the end do not, and here's why. Suppose $\deg(v_0) = 1$ is false. Then $v_0$ is connected by an edge to $v_1$ and another vertex, say $u \in V$. If $u \in \{v_2, \ldots, v_k\}$ then $\Gamma$ has a cycle. That's impossible, so we can assume $u \in V - \{v_2, \ldots, v_k\}$. But then

$$u \to v_0 \to v_1 \to \cdots \to v_k, \qquad v_i \in V$$

is a path in $\Gamma$ of length $k+1$. This contradicts the assumption that $P$ was a path of maximal length. □

**Lemma 25.** *If $\Gamma = (V, E)$ is a tree then $|V| - 1 = |E|$.*

*Proof.* Exercise (hint: use induction, Lemma 23 and Lemma 24). □

**Definition 26.** A *regular* graph is a graph where each vertex has the same degree. More precisely, if the degree of every vertex of a graph $\Gamma$ is $k > 0$ then we call $\Gamma$ a *$k$-regular* graph.

In a regular graph, every vertex has the same number of neighbors. For example, the cycle graph $C_n$ is regular or, more precisely, 2-regular.

The *cube graph* $Q_k = (V, E)$ is defined as follows[6]: $V = GF(2)^k$ and, for $v, w \in V$, we say $(v, w) \in E$ if and only if $v$ and $w$ differ in exactly one coordinate.

---

[5] Recall we can think of a path in $\Gamma$ is a subgraph which is a path graph.

[6] For any prime $p$, $GF(p) = \mathbb{Z}/p\mathbb{Z}$ is the finite field having $p$ elements, with addition and multiplication defined $\pmod p$. As a set, $GF(2) = \{0, 1\}$.

**Exercise 2.7.** *Let $k > 1$ be an integer. Prove that $Q_k$, defined above, is $k$-regular.*

**Example 27.** Let $f$ be a $GF(2)$-valued function on $GF(2)^N$. The *Cayley graph of $f$* is defined to be the edge-weighted digraph

$$\Gamma_f = (GF(2)^N, E_f), \tag{2}$$

whose vertex set is $V = V(\Gamma_f) = GF(2)^N$ and the set of edges is defined by

$$E_f = \{(u, v) \in GF(2)^N \mid f(u - v) \neq 0\}.$$

This is a $k$-regular graph, where $k = |\mathsf{supp}\,(f)|$ is the size of the support of $f$,

$$\mathsf{supp}\,(f) = \{x \in GF(2)^N \mid f(x) \neq 0\}.$$

**Definition 28.** A *distance-regular* graph is a regular graph $\Gamma = (V, E)$ such that, for any two vertices $v, w \in V$ with $i = \delta_\Gamma(v, w)$, the number of vertices at distance $j$ from $v$ and at distance $k$ from $w$ depends only on the three parameters $j$, $k$, and $i$.

More precisely, if the number of vertices of $\Gamma$ is $n$ and the diameter is $d$, a graph is distance-regular if there are integers (called the *intersection numbers* of $\Gamma$)

$$a_i \ 0 \leq i \leq d, \quad b_i \ 0 \leq i \leq d - 1, \quad c_i \ 1 \leq i \leq d + 1,$$

such that, for any pair $u, v \in V$, the number of neighbors of $v$ whose distance from $u$ is $i$, $i + 1$, and $i - 1$ is $a_i$, $b_i$, and $c_i$, respectively.

For example, the cycle graph $C_n$ is distance regular, for each $n > 2$. Moreover, the complete graph $K_n$ is distance regular, for each $n > 2$.

# 3 Basic definitions

## 3.1 Diameter, radius, and all that

**Definition 29.** A *walk* is an alternating sequence of vertices and edges,

$$(v_0, e_1, v_1, e_2, \ldots, e_k, v_k),$$

for some integer $k > 0$, starting and ending at a vertex, in which each edge $e_j = \{v_{j-1}, v_j\}$ is adjacent in the sequence to its two endpoints. If $\Gamma$ is a simple graph then we denote this walk more simply by

$$v_0 \to v_1 \to \cdots \to v_k.$$

The number of edges in the sequence is called the *length* of the walk.

We say $\Gamma$ is *connected* if, between any two vertices of $\Gamma$, there is a walk from one to the other.

If $\Gamma$ is not connected then the maximal connected subgraphs are called the *connected components* of $\Gamma$.

Recall, for $v, w \in V$, the distance from $v$ to $w$, denoted $\delta(v, w)$, is the number of edges in the shortest walk from $v$ to $w$. The *diameter*, denoted $\mathsf{diam}\,(\Gamma)$, of $\Gamma$ is the maximum value of the distance function:

$$\mathsf{diam}\,(\Gamma) = \max_{v,w \in V} \delta(v, w).$$

For example, the house graph in Example 7 has diameter 2.

The *eccentricity* $\epsilon(v)$ of a vertex $v \in V$ is the greatest distance between $v$ and any other vertex. In other words,

$$\epsilon(v) = \max_{w \in V} \delta(v, w).$$

**Exercise 3.1.** *Let $\Gamma = (V, E)$ be a connected graph. Prove that, for $v \in V$, $\epsilon(v) = 1$ if and only if $v$ is adjacent to every other vertext of $\Gamma$.*

**Exercise 3.2.** *Let $\Gamma = (V, E)$ be a connected graph with $n$ vertices. Prove that $\Gamma$ is the complete graph $K_n$ if and only if, for all $v \in V$, $\epsilon(v) = 1$.*

The *radius* $r(\Gamma)$ of a graph is the minimum eccentricity of any vertex. In other words,

$$r(\Gamma) = \min_{v \in V} \epsilon(v).$$

For example, the radius of the house graph is 2. Indeed, each vertex $v$ of the house graph satisfies $\epsilon(v) = 2$. A vertex $v \in V$ is a *central vertex of* $\Gamma$ if and only if

$$r(\Gamma) = \epsilon(v). \tag{3}$$

**Exercise 3.3.** *Show that the diameter of a graph is the maximum eccentricity of any vertex. In other words,*

$$\mathsf{diam}\,(\Gamma) = \max_{v \in V} \epsilon(v).$$

A vertex $v \in V$ is a *peripheral vertex of* $\Gamma$ if and only if

$$\mathsf{diam}\,(\Gamma) = \epsilon(v).$$

**Exercise 3.4.** *Let $\Gamma = (V, E)$ be a connected graph with $n$ vertices. Prove that $\Gamma$ is the complete graph $K_n$ if and only if* $\mathrm{diam}(\Gamma) = 1$.

**Lemma 30.** *Let $\Gamma$ be a connected graph. We have,*

$$r(\Gamma) \leq \mathsf{diam}\,(\Gamma) \leq 2r(\Gamma).$$

*Proof.* The inequality $r(\Gamma) \leq \mathsf{diam}\,(\Gamma)$ follows from the above exercise. The inequality $\mathsf{diam}\,(\Gamma) \leq 2r(\Gamma)$ follows from the triangle inequality. □

## 3.2 Treks, trails, paths

Let $\Gamma = (V, E)$ be a simple graph.

**Definition 31.** A *trek* in $\Gamma$ is a walk

$$(v_0, e_1, v_1, e_2, \ldots, e_k, v_k),$$

for which consecutive edges are distinct, $e_j \neq e_{j+1}$ (and, if the walk is closed, $e_1 \neq e_n$).

**Definition 32.** A *trail* is a walk $v_0 \to v_1 \to \cdots \to v_k$ with no repeated edges. If $v_0 = v_k$, and the length is at least 3, then the trail is said to be *closed*, also called a *circuit*.

An *Eulerian trail* is a trail in a graph which visits every edge exactly once. An *Eulerian circuit*[7] is a path in a finite graph which visits every edge exactly once. If such a circuit exists, the graph is called *Eulerian*.

[7] Also called an Eulerian tour.

**Definition 33.** A *path* is a walk

$$v_0 \to v_1 \to \cdots \to v_k$$

without repeated vertices.

The one exception to this definition is when the endpoints are the same: if $v_0 = v_k$ then the path is said to be *closed*, also called a *cycle*.

The following lemma is "obvious" but its proof is a good exercise in these ideas.

**Lemma 34.** *If $\Gamma = (V, E)$ is a graph with distinct vertices $u, v \in V$, and if there is a walk from $u$ to $v$ then there is a path from $u$ to $v$.*

In fact, we prove that the walk of minimum length from $u$ to $v$ is a path.

*Proof.* If there is a walk from $u$ to $v$ then[8] there is a walk of shortest length. Assume this shortest length is $k$. We may assume that $k \geq 1$, since $k = 0$ means $u = v$, which is impossible, and $k = 1$ means $u, v$ are connected by an edge, which is a path. Assume that this shortest walk, denoted

$$u = v_0 \to v_1 \to \cdots \to v_{k-1} \to v_k = v,$$

is not a path. This means there is a repeated vertex in this walk. Suppose that $v_i = v_j$, for some $i \neq j$. Skipping over this closed walk, $v_i \to v_{i+1} \to \cdots \to v_j (= v_i)$, we then have a shorter walk,

$$u = v_0 \to v_1 \to \cdots \to v_{i-1} \to v_i \to v_{j+1} \to \cdots \to v_k = v.$$

But this is impossible, since we selected a shortest walk to begin with. $\square$

A *Hamiltonian path* in a connected graph is a path that visits each vertex exactly once. A *Hamiltonian cycle* is a Hamiltonian path that is a cycle. A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph.*

**Lemma 35.** *A connected graph is Eulerian if and only if every vertex has even degree.*

[8] Here we use the well-ordering principle, which states that every non-empty set of positive integers contains a least element. This is not to be confused with the "well-ordering theorem," actually an axiom in set theory: For any set $X$, there is a binary relation $R$ which is a total order on $X$ with the property that every nonempty subset of $X$ has a member which is minimal under $R$.

*Proof.* ($\Rightarrow$) Let $\Gamma = (V, E)$ be an Eulerian graph and let

$$v_0 \to v_1 \to \cdots \to v_{n-1} \to v_n = v_0$$

be an Euler cycle. There can be repeated vertices but there cannot be any repeated edges. To compute the degree of a vertex $v \in V$, each time a $v$ occurs in this cycle, we add 2 to its degree (one for each incident edge). This is an even number.

($\Leftarrow$) Let $\Gamma = (V, E)$ be a graph for which each vertex has even degree and let

$$W : v_0 \to v_1 \to \cdots \to v_{k-1} \to v_k$$

be a longest walk with no repeated edges. Note that every edge incident to $v_k$ occurs in this walk, for otherwise we could add it to the end of the walk, making it longer. Now we claim $v_0 = v_k$. If not, then the degree of $v_k$ is odd, contradicting the hypothesis. Therefore, the longest walk in $\Gamma$ with no repeated edges is a cycle.

Suppose that it is not an Eulerian tour. There is a edge $E$ in $\Gamma$ not in $W$. If this edge is not incident to a vertex in $W$ then, since $\Gamma$ is connected, there is a path $P$ containing no edges in $W$ from $e$ to a vertex $v_i$ in $W$. The walk that goes along $P$ to $v_i$, then around $W$ back to $v_i$. This is a walk with distinct edges that is longer than $W$. This contradicts the assumption that $W$ is a longest walk with no repeated edges. Therefore, $W$ must be an Eulerian cycle. □

**Definition 36.** The *line graph* $L(\Gamma)$ is a graph such that (a) the vertices of $L(\Gamma)$ are the edges of $\Gamma$, (b) two vertices of $L(\Gamma)$ are adjacent if and only if the corresponding edges are incident in $\Gamma$.

For example, the line graph of a cyclic graph with $n$ vertices is a cyclic graph with $n$ vertices.

**Exercise 3.5.** *Find the line graph of the house graph.*

**Exercise 3.6.** *Prove that an induced subgraph of a line graph is a line graph.*

**Lemma 37.** *To each walk $W$ in $\Gamma$,*

$$W : (v_0, e_1, v_1, e_2, \ldots, e_k, v_k),$$

*for some integer $k > 0$, there is associated a unique walk $W'$ in the line graph $L(\Gamma)$,*

$$W' : (e_1, v_1, e_2, \dots, v_{k-1}, e_k),$$

*Proof.* Exercise. □

**Lemma 38.** *The line graph of every Eulerian graph is Hamiltonian.*

*Proof.* Let $\Gamma = (V, E)$ be an Eulerian graph and let $\Gamma' = (V', E')$ denote its line graph. Let $W$ denote an Euler cycle in $\Gamma$,

$$v_0 \to v_1 \to \cdots \to v_{k-1} \to v_k = v_0,$$

in which every edge of $\Gamma$ occurs exactly once. The corresponding walk $W'$ in $\Gamma'$ passes through every vertex of $\Gamma'$ exactly once. Hence $\Gamma'$ is Hamiltonian. □

Two paths $P_1$ and $P_2$ in a graph $\Gamma = (V, E)$ are *disjoint* if they have no vertices in common. Given $u, v \in V$, two paths $P_1$ and $P_2$ in $\Gamma$, each from $u$ to $v$, are *openly disjoint* if they have no vertices in common, except for $u, v$.

For distinct vertices $v, w$ in a graph $\Gamma = (V, E)$, define $v \sim w$ if and only if there is a path in $\Gamma$ connecting $v$ and $w$.

**Exercise 3.7.** *Prove that this relation $\sim$ on $V$ is an equivalence relation.*

Each equivalence class corresponds to an induced subgraph $\Gamma$; these subgraphs are the connected components of $\Gamma$. The number of connected components is denoted $c(\Gamma)$.

**Definition 39.** A *tour* is a closed trail.

**Example 40.** From[9] James Tantons *A Dozen Questions About a Triangle,* **Math Horizons** 9(2002)23-28: King Tricho lives in a palace in which every room is a triangle, as depicted in Figure 8.

Is there a path that will let the King visit each room once and only once? He does not have to return to his starting point (i.e., his path need not be a cycle).

[9] See also `https://www.futilitycloset.com/2017/07/11/patrolling-the-palace/`.

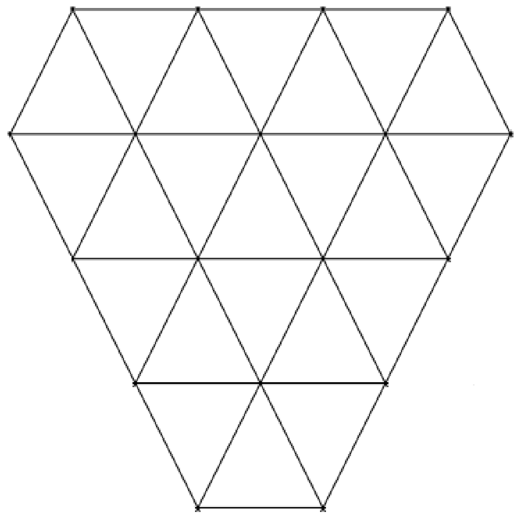

Figure 8: King Tricho's palace of triangular rooms.

The *girth* of $\Gamma$ is the number of edges in the shortest cycle of $\Gamma$, if it exists. For example, the house graph in Example 7 has girth 3. The *circumference* of $\Gamma$ is the number of edges in the longest cycle of $\Gamma$, if it exists. For example, the house graph in Example 7 has circumference 5.

For $v, w \in V$, define $v \sim w$ iff there is a walk from $v$ to $w$.

**Lemma 41.** *(a) $v \sim w$ iff there is a path from $v$ to $w$.*

*(b) $v \sim w$ is an equivalence relation.*

*(c) The equivalence classes of $\sim$ are in bijection with the set of connected components of $\Gamma$.*

The *order* of $\Gamma$ is the number of vertices, $|V|$.

The *size* of $\Gamma$ is the number of edges, $|E|$.

## 3.3 Maps between graphs

**Definition 42.** A *homomorphism* from $\Gamma_2 = (V_2, E_2)$ to $\Gamma_1 = (V_1, E_1)$, $f : \Gamma_2 \to \Gamma_1$, is a pair of functions $f_V : V_2 \to V_1$ such that $f_V$ induces a map

$$\begin{aligned} f_E : E_2 &\to E_1, \\ e = \{v, w\} &\mapsto f_E(e) = \{f_V(v), f_V(w)\}. \end{aligned}$$

Two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are *isomorphic*, denoted

$$\Gamma_1 \cong \Gamma_2,$$

if there exists a homomorphism $f : \Gamma_2 \to \Gamma_1$ such that (a) $f_V : V_2 \to V_1$ is a bijection, (b) the induced map

$$\begin{aligned} f_E : E_1 &\to E_2, \\ e = \{v, w\} &\mapsto f_E(e) = \{f_V(v), f_V(w)\}, \end{aligned}$$

is also a bijection. An isomorphism of a graph with itself is called an *automorphism.*

For example, in Figure 9, the same graph has two different labelings. We use the idea of an isomorphism to regard them as "the same".

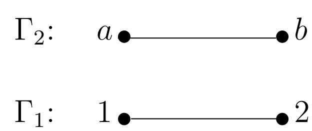


Figure 9: A graph with two different labeling of the vertices.

**Exercise 3.8.** *Prove that if two graphs $\Gamma_1, \Gamma_2$ are isomorphic then they have the same size and the same order.*

**Exercise 3.9.** *Suppose $\Gamma_1 \cong \Gamma_2$ and that $P_1$ is a (possibly closed) path in $\Gamma_1$. Prove that $\Gamma_2$ also has a path $P_2$ of length $k$. Also, show that $P_1$ is closed if and only if $P_2$ is closed.*

**Lemma 43.** *If $f : \Gamma \to \Gamma$, is an automorphism of $\Gamma = (V, E)$ then the inverse map $g : \Gamma \to \Gamma$, defined by (a) the associated inverse map on vertices,*

$$g_V : V \to V$$
$$g_V(v) = f_V^{-1}(v), \qquad v \in V,$$

*and (b) the induced map on edges*

$$g_E : E \to E,$$

$$g_E : e = \{v, w\} \mapsto g_E(e) = \{g_V(v), g_V(w)\},$$

*is an automorphism.*

The automorphism in the above lemma defines $f^{-1} = g$. Note that $g_V(v) = f_V^{-1}(v)$ is defined since $f_V$ is assumed to be a bijection.

*Proof.* The inverse of a bijection is a bijection. □

**Lemma 44.** *If $f : \Gamma \to \Gamma$ and $g : \Gamma \to \Gamma$ are automorphisms of $\Gamma = (V, E)$ then the composite map $f \circ g : \Gamma \to \Gamma$, defined by (a) the compotion on vertices*

$$(f \circ g)_V : V \to V$$
$$(f \circ g)_V(v) = f_V(g_V(v)), \qquad v \in V,$$

*(b) the induced map*

$$(f \circ g)_E : E \ \to \ E,$$
$$(f \circ g)_E : e = \{v, w\} \mapsto (f \circ g)_E(e) = \{f_V(g_V(v)), f_V(g_V(w))\},$$

*is an automorphism.*

*Proof.* The composition of bijections is a bijection. □

The collection of all automorphisms of $\Gamma$ is called the *automorphism group*, $Aut(\Gamma)$.

For example, the automorphism of $F_5$ (in Figure 6) contains the cyclic group of order 5 formed by "rotations" of the outer (i.e., degree 2) vertices. In fact, $Aut(F_5)$ is a group of order 3840, so these 5 automorphisms are far from all of them.

**Exercise 3.10.** *Consider two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, and assume $f : \Gamma_2 \to \Gamma_1$ is an isomorphism. Show that, for all $v \in V_2$, $\deg_{\Gamma_1}(v) = \deg_{\Gamma_2}(f_V(v))$.*

## 3.4 Colorings

**Definition 45.** A labeling, or coloring, of the vertices of $\Gamma$ such that no two adjacent vertices share the same label/color is called a *(vertex) coloring*. Similarly, a labeling, or coloring, of the edges so that no two adjacent edges share the same label, or color, is called ad *edge coloring*.

A coloring using at most $k$ colors, if it exists, is called a *$k$-coloring*.

The smallest number of colors needed to (vertex) color a graph $\Gamma$ is called its *chromatic number*, and is often denoted $\chi_v(\Gamma)$. The smallest number of colors needed for an edge coloring of $\Gamma$ is the *chromatic index*, or *edge chromatic number*, $\chi_e(\Gamma)$.

For example, the house graph in Example 7 has a 3-coloring.

An edge coloring of a graph is just a vertex coloring of its line graph.

**Lemma 46.** *A graph $\Gamma$ is bipartite*[10]*) if and only if $\chi_v(\Gamma) = 2$.*

*Proof.* Exercise. □

A subset $S \subset V$ of vertices of $\Gamma = (V, E)$ is called an *independent set* if no two elements are neighbors. In other words, $S$ is an independent set if, for each distinct $s_1, s_2 \in S$, $N_\Gamma(s_1) \cap N_\Gamma(s_2) = \emptyset$. The maximum size of an independent set is called the *independence number* of $\Gamma$, denoted $\alpha(\Gamma)$. For example, the independence number of the house graph is 2.

**Lemma 47.** *Consider the set of all independent subsets of $\Gamma$, $\mathcal{I}$. This collection satisfies*

*I1:* $\emptyset \in \mathcal{I}$,

*I2:* $I_1 \in \mathcal{I}$ *and* $I_2 \subset I_1$ *implies* $I_2 \in \mathcal{I}$.

*I3: (exchange property)* $I_1, I_2 \in \mathcal{I}$ *and* $|I_2| > |I_1|$ *implies there exists* $e \in I_2 - I_1$ *with* $I_1 \cup \{e\} \in \mathcal{I}$,

*Proof.* ... □

[10] Recall biparite is defined in Definition 10 above.

## 3.5 Transitivity

**Definition 48.** A *vertex-transitive graph* is a graph $\Gamma = (V, E)$ such that, given any two vertices $v_1$ and $v_2$ of $\Gamma$, there is some automorphism

$$f : V \to V,$$

such that $f(v_1) = v_2$.

**Lemma 49.** *A complete graph is vertex-transitive.*

*Proof.* Exercise. □

**Definition 50.** A distance-transitive graph is a graph $\Gamma = (V, E)$ such that, given any two vertices $v, w \in V$ with $\delta(v, w) = i$, and any other two vertices $x, y \in V$ with $\delta(x, y) = i$, there is an automorphism $g \in Aut(\Gamma)$ such that $g(v) = x$ and $g(w) = y$.

Are all complete graphs distance-transitive? (Is $S_n$ a 2-transitive group?)

**Lemma 51.** *A distance-transitive graph is distance-regular.*

*Proof.* Omitted. □

**Definition 52.** A *edge-transitive graph* is a graph $\Gamma = (V, E)$ such that, given any two edges $e_1 = (v_1, w_1) \in E$ and $e_2 = (v_2, w_2) \in E$ of $\Gamma$, there is some automorphism

$$f : V \to V,$$

such that either (a) $f(v_1) = v_2$ and $f(w_1) = w_2$ (so $f$ sends $e_1 \mapsto e_2$), or (b) $f(v_1) = w_2$ and $f(w_1) = v_2$ (so, again, $f$ sends $e_1 \mapsto e_2$).

The concept of edge-transitive is a special case of distance-transitive.

**Definition 53.** An *arc-transitive graph* is a graph $\Gamma = (V, E)$ such that, given any two edges $e_1 = (v_1, w_1) \in E$ and $e_2 = (v_2, w_2) \in E$ of $\Gamma$, there is some automorphism

$$f : V \to V,$$

such that either $f(v_1) = v_2$ and $f(w_1) = w_2$ (so $f$ sends $e_1 \mapsto e_2$).

# 4 Adjacency matrix

Let $\Gamma = (V, E)$ be an undirected graph with vertices $V = \{v_1, \dots, v_n\}$ and edge set $E$.

## 4.1 Definition

Let $\Gamma = (V, E)$ be an undirected graph with vertices $V = \{v_1, \dots, v_n\}$ and edge set $E$.

The *adjacency matrix* of $\Gamma$ is the $n \times n$ matrix $A(\Gamma) = A = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

As $\Gamma = (V, E)$ is an undirected simple graph, we can make some basic observations:

- Observe, $\{v_i, v_j\} \in E$ is true iff $\{v_j, v_i\} \in E$ is true. Thus, $A$ is a symmetric matrix, and, by a well-known result in matrix theory, we therefore know that $A$ is diagonalizable.
- Since $\Gamma$ has no loops, the diagonal entries of $A$ are all 0.
- There is no simple formula for the rank of the adjacency matrix.

The adjacency matrix completely determines a graph, up to an indexing of the vertices. The following Lemma explains what happens in the case of a disconnected graph with carefully labeled vertices.

**Lemma 54.** *If $\Gamma = (V, E)$ is disconnected having two connected components, $\Gamma_1$ and $\Gamma_2$ say, then $V$ can be indexed in such a way that $A$ is a block diagonal matrix of the form*

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

*where $A_i$ is the adjacency matrix of $\Gamma_i$ $(i = 1, 2)$. A similar decomposition holds in the case where $\Gamma$ has $k$ connected components.*

*Proof.* Exercise. □

## 4.2 Basic results

The adjacency matrix obviously depends on the ordering (i.e., labeling) of the vertices. The next result explains what happens to the adjacency matrix when you label the same graph two different ways.

**Lemma 55.** *Let $\Gamma_1 = (V_1, E_1)$ be a graph with adjacency matrix $A_1$ whose row and columns are indexed by $V_1 = \{v_1, v_2, \ldots, v_n\}$. Let $\Gamma_2 = (V_2, E_2)$ be a graph with adjacency matrix $A_2$ whose row and columns are indexed by $V_2 = \{w_1, w_2, \ldots, w_n\}$, where*

$$w_k = \begin{cases} v_k, & k \notin \{i, j\}, \\ v_i, & k = j, \\ v_j, & k = i, \end{cases}$$

*and where $E_2$ is the set of edges $(w_k, w_\ell)$, $k \neq \ell$, for which*

*(a) $(v_i, v_j) \in E_1$, if $\{k, \ell\} = \{i, j\}$,*
*(b) $(v_i, v_\ell) \in E_1$, if $\ell \notin \{i, j\}$ and $k = j$,*
*(c) $(v_j, v_\ell) \in E_1$, if $\ell \notin \{i, j\}$ and $k = i$,*
*(d) $(v_k, v_\ell) \in E_1$, if $\ell \notin \{i, j\}$ and $k \notin \{i, j\}$.*

*In other words, $E_2$ is the same as $E_1$, except we swap $v_i$ and $v_j$. Then $A_2$ is obtained from $A_1$ by swapping the $i$th row and column with the $j$-th row and column.*

*Proof.* Let $M$ denote any $n \times n$ matrix. Note that the operation which swaps rows $i$ and $j$, and simultaneously columns $i$ and $j$, of $M$ is the map $M \mapsto P^{-1}MP$, where $P = P(i, j)$ is the $(0, 1)$-matrix having 1s along the diagonal, except in the $i$th and $j$th positions, and 0s elsewhere, except for 1s in the $(i, j)$-position and the $(j, i)$-position. Now, the lemma above is a straightforward consequence of the definition of $A$. $\square$

Since every permutation is composed of a product of such simple transpositions, one may use Lemma 55 and mathematical induction to prove a more general result (see Theorem 73, Theorem 75 below).

**Exercise 4.1.** *Prove the following: If $A$ is the adjacency matrix of $\Gamma = (V, E)$, where $V = \{1, 2, \ldots, n\}$, then the diagonal entries of $A^2$ is the list of degrees,* $\deg_\Gamma(1)$, $\deg_\Gamma(2)$, $\ldots$, $\deg_\Gamma(n)$.

**Lemma 56.** *If $A$ is the adjacency matrix of $\Gamma = (V, E)$, where $V = \{1, 2, \ldots, n\}$, then, for all $i \neq j$, the $ij$-th entry of $A^2$ is the number of common neighbors of vertex $i$ and vertex $j$.*

*Proof.* By definition of matrix multiplication,

$$\sum_{\ell=1}^{n} A_{i,\ell}A_{\ell,j}$$

is the $(i,j)$-th entry of the product $A^2 = A \cdot A$. This sum counts the vertices $\ell$ which is a neighbor of both $i$ and of $j$. $\square$

**Lemma 57.** *Consider a bipartite graph* $\Gamma = (V,E)$*,* $V = V_1 \cup V_2$*, where* $V_1, V_2$ *are disjoint and* $|V_1| = r$ *and* $|V_2| = s$*. If the vertices* $V$ *are indexed so that the vertices of* $V_1$ *are before all the vertices of* $V_2$*, then the adjacency matrix* $A$ *of* $\Gamma$ *has the form*

$$A = \begin{pmatrix} 0_{r,r} & A_0 \\ A_0^T & 0_{s,s} \end{pmatrix}$$

*where* $A_0$ *is an* $r \times s$ *matrix, and* $0_{k,k}$ *represents the* $k \times k$ *zero matrix (*$k = r, s$*).*

In this case, the matrix $A_0$ is called the *biadjacency matrix*.

*Proof.* Exercise. (Hint: When constructing $A$, order the vertices $V$ so that the vertices of $V_1$ are before all the vertices of $V_2$.) $\square$

*A function-theoretic interpretation*: Let

$$C_0(\Gamma) = \{f : V \to \mathbb{R}\ \}$$

denote the space of functions on the vertices of $\Gamma$.

**Exercise 4.2.** *For each* $w \in V$*, let*

$$\delta_w(v) = \begin{cases} 1, & \textit{if } \{v,w\} \in E, \\ 0, & \textit{otherwise.} \end{cases}$$

*Show that* $C_0(\Gamma)$ *is a vector space, under point-wise addition of functions, of dimension* $|V|$ *with basis* $\{\delta_w\}_{w \in V}$*.*

Consider the map $\mathcal{A} : C_0(\Gamma) \to C_0(\Gamma)$ defined, for all $v \in V$, by

$$\mathcal{A}f(v) = \sum_{w \in V,\ \{v,w\} \in E} f(w) = \sum_{w \in N_\Gamma(v)} f(w),$$

where (recall from (1) above) $N_\Gamma(v)$ denotes the neighborhood of $v \in V$. The sum runs over all vertices in $\Gamma$ which are adjacent to $v$. The matrix representation of $\mathcal{A}$ with respect to the basis $\{\delta_w\}_{w\in V}$ of $C_0(\Gamma)$ is the adjacency matrix.

**Exercise 4.3.** 1. *Compute the adjacency matrix of the house graph.*

2. *Compute the adjacency matrix of the complete graph $K_4$.*

3. *Let $A$ be an $n \times n$ matrix having all of its entries taken from $\{0, 1\}$. (Such a matrix is called a $(0, 1)$-matrix.) Prove that $A$ is the adjacency matrix of an undirected simple graph iff $A$ is symmatric and has 0's on the diagonal.*

**Proposition 58.** *Let $N > 0$ be an integer. If $\Gamma = (V, E)$ is a graph with $V = \{1, 2, \dots, n\}$ then, for each $i \neq j$, the $(i, j)$ entry in the matrix $A(\Gamma)^N$ is the number of walks of length $n$ from $v_i$ and $v_j$.*

*Proof.* We prove this by mathematical induction.

Case $N = 1$: The statement is clearly true for $N = 1$.

Case $N = k$: Assume the statement is true for $N = k > 1$.

The set of walks of length $k+1$ from $v_i$ and $v_j$ is in bijection with the set of walks of length $k$ from $v_i$ and $v_\ell$, where $v_\ell$ is a neighbor of $v_j$. Therefore, by the induction hypothesis, the number of walks of length $k+1$ from $v_i$ and $v_j$ is

$$\sum_{v_\ell \in N_\Gamma(v_j)} (A^k)_{i,\ell}.$$

Note that $A_{\ell,j} = 1$ if and only if $v_\ell \in N_\Gamma(v_j)$. Therefore,

$$\sum_{v_\ell \in N_\Gamma(v_j)} (A^k)_{i,\ell} = \sum_{\ell=1}^{n} (A^k)_{i,\ell} A_{\ell,j}.$$

By definition of matrix multiplication,

$$\sum_{\ell=1}^{n} (A^k)_{i,\ell} A_{\ell,j}$$

is the $(i, j)$-th entry of the product $A^k \cdot A$.

Putting all these together, we see that the number of walks of length $k+1$ from $v_i$ and $v_j$ is the $(i, j)$-th entry of the product $A^k \cdot A = A^{k+1}$. The result follows by induction. $\square$

Lemma 56 is a corollary of the result above. A corollary of this corollary is the following result.

**Corollary 59.** *The trace of the matrix* $A(\Gamma)^2$ *is twice the number of edges in* $\Gamma$.

*Proof.* Exercise. (Hint: use Lemma 2.) $\square$

Another corollary.

**Corollary 60.** *If* $V = \{v_1, v_2, \ldots, v_n\}$ *then*

$$\delta(v_j, v_j) = \min\{k \geq 0 \mid A(\Gamma)^k_{ij} \neq 0\}.$$

**Exercise 4.4.** *Let* $\Gamma$ *denote the house graph. Using* $A^4$*, find the number of walks of length* 4 *from each vertex to itself.*

**Exercise 4.5.** *Let* $\Gamma$ *denote the diamond graph. Using* $A^4$*, find the number of walks of length* 4 *from each vertex to itself.*

**Exercise 4.6.** *Let* $\Gamma$ *denote* $K_5$*. Using* $A^4$*, find the number of walks of length* 4 *from each vertex to itself.*

## 4.3 The spectrum

The characteristic polynomial of the adjacency matrix of $\Gamma$ is called the *characteristic polynomial of* $\Gamma$ denoted

$$\chi_\Gamma(x) = \det(xI - A).$$

(Some people use instead $\det(A - xI)$.)

**Lemma 61.** *If* $\Gamma = (V, E)$ *is disconnected having two connected components,* $\Gamma_1$ *and* $\Gamma_2$ *say, then*

$$\chi_\Gamma(x) = \chi_{\Gamma_1}(x)\chi_{\Gamma_2}(x).$$

*A similar decomposition holds in the case where* $\Gamma$ *has* $k$ *connected components.*

*Proof.* Exercise. (Hint: use Lemma 54.) □

The eigenvalues of $A(\Gamma)$, that is the roots of $\chi_\Gamma(x)$, often denoted

$$\lambda_n(\Gamma) \leq \cdots \leq \lambda_1(\Gamma),$$

are the spectrum of $\Gamma$ (or *adjacency spectrum*, if you want to be more precise). A general problem in algebraic graph theory is the following.

**Problem:** What properties of the graph can be determined from its spectrum?

For example, can you determine the number of vertices and edges from the spectrum? (Yes, by Lemma 64 and Corollary 59.)

Can you determine if a graph is bipartite or not knowing only its spectrum? (Yes, by Lemma 62.)

**Lemma 62.** *The following are equivalent for a connected graph* $\Gamma$*:*

*(1) The (adjacency) spectrum of* $\Gamma$ *is symmetric about* $0$*.*

*(2) The graph* $\Gamma$ *is bipartite.*

*Proof.* We only prove the direction (2) $\Longrightarrow$ (1).

According to Lemma 57, if $(\vec{x}, \vec{y})^T$ is an eigenvector of $A$ having eigenvalue $\lambda$ (where $\vec{x}$ is a row vector of length $s$ and $\vec{y}$ is a row vector of length $r$), then $(\vec{x}, -\vec{y})^T$ is an eigenvector of $A$ having eigenvaue $-\lambda$. Since all eigenvectors of $A$ have this partitioned form, this shows that $\lambda$ is an eigenvalue of $A$ (of multiplicity $m$) if and only if $-\lambda$ is an eigenvalue of $A$ (of multiplicity $m$). □

We know that the adjacency matrix determines the graph (up to isomorphism[11]). But does the spectrum of determine the graph? The answer is no. The counter-example below is from Biggs [Bi93], page 12.

**Example 63.** Imagine a graph $\Gamma$ with a boxy $w$ shape, with vertices

$$V = \{1, 2, 3, 4, 5, 6\},$$

$(1, 2, 3$ across the top, $4, 5, 6$ along the bottom), and edges

$$E = \{(1, 4), (2, 5), (3, 6), (4, 5), (5, 6)\}.$$

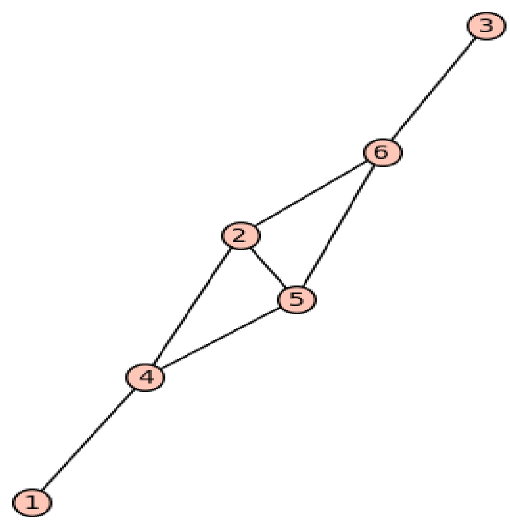


Figure 10: A graph with 6 vertices and 7 edges.

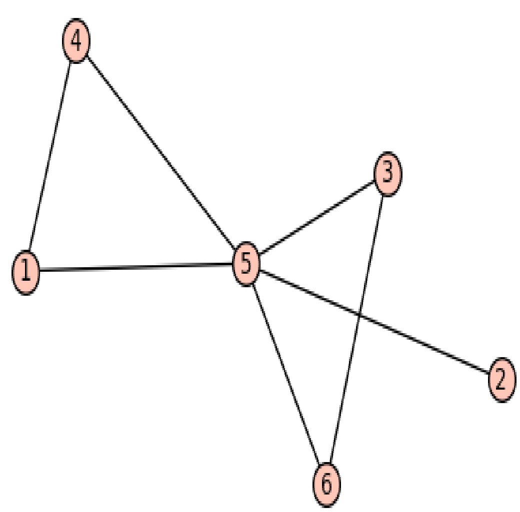


Figure 11: A graph with 6 vertices and 7 edges.

We add two edges onto $\Gamma$ to get $\Gamma_1$: $(2,4)$ and $(2,6)$. See Figure 10.

We add two edges onto $\Gamma$ to get $\Gamma_2$: $(1,5)$ and $(3,5)$. See Figure 11.

These graphs $\Gamma_1$ and $\Gamma_2$ are not isomorphic but they do have the same characteristic polynomial:

$$\chi_{\Gamma_1}(x) = \chi_{\Gamma_1}(x) = x^6 - 7x^4 - 4x^3 + 7x^2 + 4x - 1.$$

**Lemma 64.** *If*

$$\chi_\Gamma(x) = x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0,$$

*then (a) the degree of* $\chi_\Gamma$ *is the size of* $\Gamma$*, (b)* $c_{n-1} = 0$*, (c)* $c_0 = (-1)^n \det(A)$.

We shall not prove it, but is also known[12] that $-c_{n-2}$ is the number of edges of $\Gamma$.

*Proof.* Parta (a) and (c) follow from the definition of $\chi_\Gamma$. It suffices to prove (b). Diagonalize $A$. Note $c_{n-1}$ is the negative of the sum of the eigenvalues of $A$. This is the negative of the sum of the diagonal entries of $A$. Since all the diagonal entries of $A$ are 0, $c_{n-1} = 0$. $\square$

Another helpful fact about computing characteristic polynomials is stated below (see Biggs [Bi93]).

**Lemma 65.** *If* $\Gamma = (V, E)$ *is a graph then*

$$\chi'_\Gamma(x) = \sum_{v \in V} \chi_{\Gamma - v}(x).$$

Instead of a proof, we give an example.

**Example 66.** Let $\Gamma$ be the diamond graph, as depicted in Figure 26. Its adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

According to Lemma 65, we have

[11] For more details, see Theorems 73 and 75 below.

[12] For example, see Biggs, [Bi93], Proposition 2.3.

$$\chi'_\Gamma(x) = \chi_{\Gamma-0}(x) + \chi_{\Gamma-1}(x) + \chi_{\Gamma-2}(x) + \chi_{\Gamma-3}(x) = 2\chi_{C_3}(x) + 2\chi_{P_3}(x),$$

where $C_3$ is the cyclic graph of 3 vertices and $P_3$ is the path graph of length 3. We have

$$\chi_{C_3}(x) = \det \begin{pmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{pmatrix} = x^3 - 3x - 2,$$

and

$$\chi_{P_3}(x) = \det \begin{pmatrix} x & -1 & 0 \\ -1 & x & -1 \\ 0 & -1 & x \end{pmatrix} = x^3 - 2x.$$

Therefore,

$$\chi'_\Gamma(x) = 2\chi_{C_3}(x) + 2\chi_{P_3} = 2(x^3 - 3x - 2) + 2(x^3 - 2x) = 4x^3 - 10x - 4.$$

Integrating, we get

$$\chi_\Gamma(x) - \chi_\Gamma(0) = x^4 - 5x^2 - 4x.$$

Since $A$ has identical columns, $\chi_\Gamma(0) = (-1)^4 \det(A) = 0$, thus $\chi_\Gamma(x) = x^4 - 5x^2 - 4x$.

**Exercise 4.7.** *Use Lemma 65 to compute*
*(a)* $\chi_{C_4}(x)$,
*(b)* $\chi_{P_4}(x)$.

For a graph $\Gamma = (V, E)$, we denote by $\Delta(\Gamma)$ and $\delta(\Gamma)$, the maximum and the minimum of the vertex degrees of $\Gamma$, respectively,

$$\Delta(\Gamma) = \max_{v \in V} \deg_\Gamma(v), \quad \delta(\Gamma) = \min_{v \in V} \deg_\Gamma(v).$$

Let $\lambda_1, \dots, \lambda_n$ be the (real or complex) eigenvalues of an $n \times n$ matrix $M$. Then its *spectral radius* $\rho(M)$ is defined as:

$$\rho(M) = \max\{|\lambda_1|, \dots, |\lambda_n|\}.$$

One of the basic facts about the spectral radius is the following fact which I think was probably known to I. Schur[13]

**Proposition 67.** *For any symmetric, square, real matrix $M$, we have*

$$\lambda_{max} = \max_{v \in \mathbb{R}^n - \{0\}} \frac{v \cdot Mv}{||v||^2},$$

*where $\lambda_{max}$ is the largest (real) eigenvalue of $M$.*

*Proof.* Since $M$ is symmetric, it's diagonalizable. In fact, we can write $M = QDQ^T$, for a diagonal matrix $D$ and an orthogonal matrix $Q$. Indeed, $Q$ is the matrix of eigenvectors of $M$, all of which have be normalized to be unit vectors. Note that $M$ and $D$ have the same (real) eigenvalues and that $||Qv|| = ||v||$, for all $v \in \mathbb{R}^n$. Since $Q$ is non-singular, it follows that

$$\max_{v \in \mathbb{R}^n - \{0\}} \frac{v \cdot Mv}{||v||^2} = \max_{||v||=1} v^T Mv = \max_{||v||=1} v^T Dv. \tag{4}$$

Let

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

denote the eigenvalues of $M$. If $v = (v_1, \dots, v_n)$ then we have

$$v^T Dv = \sum_{i=1}^n \lambda_i v_i^2 \le \lambda_1 ||v||^2,$$

with equality if $v = (1, 0, \dots, 0)$. The proposition follows. □

**Proposition 68.** *For a graph $\Gamma$,*

$$\delta(\Gamma) \le \lambda_1(\Gamma) \le \Delta(\Gamma).$$

*Proof.* Let $A$ be the adjacency matrix of $\Gamma$. For the claimed lower bound, Proposition 67 gives

$$\lambda_1(\Gamma) = \max_{v \in \mathbb{R}^n - \{0\}} \frac{v \cdot Av}{||v||^2} \ge \frac{\mathbf{1} \cdot A\mathbf{1}}{||\mathbf{1}||^2},$$

[13]Not to be confused with the mathematician Fredrich Schur, Issai Schur (1875-1941) was a Russian-born Jewish mathematician who worked in Germany for most of his life. As a student of Frobenius, he worked on group representations, combinatorics, number theory and theoretical physics. He died in Tel Aviv at the age of 66. Reference: Wikipedia.

where $\mathbf{1} \in \mathbb{R}^n$ is the all 1s vector. Note that, by definition of $A$, $\mathbf{1} \cdot A\mathbf{1} = \sum_i \deg_\Gamma(i)$. By Lemma 2, this is twice the number $m$ of edges of $\Gamma$. Therefore,

$$\lambda_1(\Gamma) \geq \frac{2m}{n}.$$

Since $n\delta(\Gamma) \leq \sum_i \deg_\Gamma(i) = 2m$, we have $\lambda_1(\Gamma) \geq \delta(\Gamma)$.

For the upper bound, let $x$ be an eigenvector of $A$ corresponding to $\lambda_1(\Gamma)$, so $Ax = \lambda_1(\Gamma)x$. If $V = \{v+1, \dots, v_n\}$ then the $i$th coordinate of $\lambda_1(\Gamma)x$ is

$$\lambda_1(\Gamma)x_i = \sum_{v_j \in N_\Gamma(v_i)} x_j.$$

If $x_k = \max_i x_i$ then this gives $\lambda_1(\Gamma)x_k \leq \Delta(\Gamma)x_k$. Cancelling out $x_k$ gives the upper bound. □

Recall form Definition 26 that a graph is called $k$-regular if all it's vertices have the same degree, $k$. We have the following amusing consequence of the above proposition.

**Corollary 69.** *If $\Gamma$ is a $k$-regular graph then $\lambda_1(\Gamma) = k$.*

The *minimal polynomial of* $\Gamma$ is the minimal polynomial of the adjacency matrix of $\Gamma$, i.e., the monic polynomial of least degree such that

$$m_\Gamma(A) = 0,$$

where 0 denotes the 0 matrix. The Cayley-Hamilton theorem from linear algebra says that $m_\Gamma(x) | \chi_\Gamma(x)$, i.e, the minimal polynomial divides the characteristic polynomial.

**Corollary 70.** (to Proposition 58) *If $\Gamma$ is a graph then*

$$\deg(m_\Gamma) \geq 1 + \mathsf{diam}\,(\Gamma).$$

*Proof.* Let $\Gamma = (V, E)$ have diameter $d > 0$. By definition of diameter, there are vertices $u, v \in V$ such that there is a path $u, v$-path of length $d$, but no $u, v$-path of length $d-1$ or less. If $V = \{1, \dots, n\}$ then we may suppose $u = i$ and $v = j$, for some distinct $i, j$. By Prop. 58, the $i, j$-entry of the matrices $A$, $A^2$, ..., $A^{d-1}$ is 0. On the other hand, if this shortest path is denoted

$$i = i_0 \to i_1 \to \cdots \to i_d = j,$$

then

- the $i_0, i_1$-entry of the matrix $A$ is non-zero,
- the $i_0, i_2$-entry of the matrix $A^2$ is non-zero but that entry of $A$ is 0,
- the $i_0, i_3$-entry of the matrix $A^3$ is non-zero but that entry of $A$ and of $A^2$ is 0, ...,
- the $i_0, i_d$-entry of the matrix $A^d$ is non-zero but that entry of $A^k$ is 0 $(k < d)$.

In particular, for each integer $k$ with $k \leq d$, any linear combination

$$A^k + c_{n-1}A^{k-1} + c_{k-2}x^{k-2} + \cdots + c_1 A + c_0 I$$

is non-zero (in the $i, j$-position, and therefore non-zero as a matrix). This proves that $d < \deg(m_\Gamma)$. □

**Proposition 71.** *Let $\ell > 0$ be an integer and let $\{\lambda_1, \ldots, \lambda_n\}$ denote the spectrum of a graph $\Gamma$. The number of closed walks of length $\ell$ in $\Gamma$ is $\sum_{i=1}^n \lambda_i^\ell = tr(A^\ell)$.*

*Proof.* Exercise. (Hint: diagonalize $A$ and use Prop. 58.) □

**Exercise:** Verify Corollary 59 for the house graph.

**Example 72.** Let $\Gamma = W_5$ denote the wheel graph in Figure 7. The first few powers of the adjacency matrix are

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix},$$

$$A^2 = \begin{pmatrix} 4 & 2 & 2 & 2 & 2 \\ 2 & 3 & 1 & 3 & 1 \\ 2 & 1 & 3 & 1 & 3 \\ 2 & 3 & 1 & 3 & 1 \\ 2 & 1 & 3 & 1 & 3 \end{pmatrix},$$

and

$$A^3 = \begin{pmatrix} 8 & 8 & 8 & 8 & 8 \\ 8 & 4 & 8 & 4 & 8 \\ 8 & 8 & 4 & 8 & 4 \\ 8 & 4 & 8 & 4 & 8 \\ 8 & 8 & 4 & 8 & 4 \end{pmatrix}.$$

**Exercise 4.8.** *Verify this in the case where $\Gamma_1$ is the house graph as indexed in Figure 2, and $\Gamma_2$ is the same graph but with vertex 0 and 1 swapped.*

**Theorem 73.** *Consider two graphs $\Gamma_1$ and $\Gamma_2$ with the same vertex set $V = \{1, 2, \dots, n\}$. If $\sigma : V \to V$ is a permutation (i.e., is a bijection) and if this map induces an isomorphism $\Gamma_1 \cong \Gamma_2$ then there is a permutation matrix $P = P(\sigma)$ such that $A(\Gamma_1) = P^{-1}A(\Gamma_2)P$.*

*Proof.* An isomorphism $\Gamma_1 \to \Gamma_2$ induced by $\sigma$ gives a bijections of the vertices $V_1 \to V_2$ and the edges $E_1 \to E_2$, This implies the permutation matrix $P = P(\sigma)$ associated to $\sigma$ satisfies $A(\Gamma_1) = P^{-1}A(\Gamma_2)P$. □

**Example 74.** Consider two labelings of the house graph. Let $\Gamma_1 = (V_1, E_1)$ have vertices $V_1 = \{0, 1, 2, 3, 4\}$, and edges $E_1 = \{(0, 1), (0, 2), (0, 4), (1, 2), (2, 3), (3, 4)\}$. It has adjacency matrix

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\Gamma_2 = (V_2, E_2)$ have vertices $V_2 = \{0, 1, 2, 3, 4\}$, and edges

$$E_2 = \{(0, 1), (0, 2), (0, 3), (1, 4), (2, 3), (2, 4)\}.$$

It has adjacency matrix

$$A_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

To pass from $\Gamma_1$ to $\Gamma_2$, we have swapped 1 with 3 and 0 with 2, associated to the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to check that $P^{-1}A_1P = A_2$, as predicted by Theorem 73.

**Theorem 75.** *Two graphs $\Gamma_1$ and $\Gamma_2$ are isomorphic if and only if there is a permutation matrix $P$ such that $A(\Gamma_1) = P^{-1}A(\Gamma_2)P$.*

*Proof.* Every graph isomorphic to $\Gamma$ having the same vertex set is obtained by applying a suitable sequence of swapping maps, as described in Lemma 55. □

**Corollary 76.** *If two graphs are isomorphic then they have the same spectrum.*

**Exercise 4.9.** *Prove this:*
**Lemma***: If $\Gamma$ is a graph with characteristic polynomial $\chi_\Gamma(x)$ then*
*(a) $\chi_\Gamma(x)$ has integer coefficients,*
*(b) if $r \in \mathbb{Q}$ is a rational root of $\chi_\Gamma(x)$ then $r$ must be an integer.*

## 4.4 Application to the Friendship Theorem

In 1966 Paul Erdös, Alfréd Rényi, and Vera T. Sós proved a result called the friendship theorem. Before stating, and proving, it, we give a definition.

**Definition 77.** Let $\Gamma = (V, E)$ be a graph and let $v \in V$ be any vertex. If, for each $w \in V$ with $v \neq w$, the vertices $v$ and $w$ are neighbors (i.e., $v \in N_\Gamma(w)$ and $w \in N_\Gamma(v)$) then $v$ is called a *politician*.

**Lemma 78.** *Let $\Gamma = (V, E)$ be a graph. If,*

*(a) for each $v, w \in V$ with $v \neq w$, the vectors $v$ and $w$ share exactly one neighbor,*

*(b) $\Gamma$ has no politicians,*

*then $\Gamma$ must be regular.*

*Proof.* First, let $u, v \in V$ with $v \neq u$, be non-adjacent vertices and let $\deg(u) = k$. These vertices exist since $\Gamma$ has no politicians. Suppose

$$N(u) = \{w_1, \ldots, w_k\}, \quad w_i \in V,$$

are the neighbors of $u$. By hypothesis, exactly one of these, say $w_1$, is adjacent to $v$. Similarly, exactly one of these, say $w_2$, is adjacent to $u$ and $w_1$. By hypothesis, there is a (unique) vertex $z_1$ which is a common neighbor of $v$ and $w_2$. Similarly, there is a (unique) vertex $z_2$ which is a common neighbor of $v$ and $w_3$. If $z_1 = z_2$ then there is a 4-cycle, $v \to w_2 \to z_1 = z_2 \to w_3 \to v$, a contradiction (since then $w_2 \neq w_3$ share 2 neighbors). So, $z_1 \neq z_2$. Similarly, we construct $z_3$ (as a common neighbor of $v$ and $w_4$), $\ldots$, $z_{k-1}$ (as a common neighbor of $v$ and $w_k$). Lastly, we construct $z_k$ (as a common neighbor of $v$ and $w_1$). As above, if $z_i = z_j$ $(i \neq j)$ then there is a 4-cycle, $v \to w_{i+1} \to z_i = z_j \to w_{j+1} \to v$, a contradiction. Therefore, there are at least $k$ neighbors of $v$ (namely, the $z_i$s). This proves $\deg(v) \geq \deg(u)$. Since $u \neq v$ were non-neighbors selected arbitrarily, we may replace $u$ by $v$ to get $\deg(u) \geq \deg(v)$. This proves $\deg(v) = \deg(u)$ in this case.

First, let $u, v \in V$ with $v \neq u$, be adjacent vertices and let $\deg(u) = k$. Again, suppose $N(u) = \{w_1, \ldots, w_k\}, \quad w_i \in V$, are the neighbors of $u$. Again, by hypothesis, exactly one of these, say $w_1$, is adjacent to $v$. Any vertex distinct from $u, v, w_1$ is not adjacent to $u$ or to $v$ and therefore, by the previous part of this proof, has degree $k$. Since $u \neq v$ were neighbors selected arbitrarily, each vertex must have degree $k$. □

**Theorem 79.** *Let $\Gamma = (V, E)$ be a connected graph. If, for each $v, w \in V$ with $v \neq w$, the vectors $v$ and $w$ share exactly one neighbor (i.e., $|N_\Gamma(w) \cap N_\Gamma(v)| = 1$) then $\Gamma$ has a politician.*

Note that the friendship graphs (Defn 18) have this property.

We follow the presentation in [AZ04], chapter 34, who give the original proof of Erdös, Rényi, and Sós

Before starting the proof, we need on small lemma.

**Lemma 80.** *If $m$ is an integer such that $\sqrt{m}$ is a rational number, then $\sqrt{m}$ is an integer (and $m$ is the square of an integer).*

*Proof.* This is left as an exercise. □

*Proof.* Assume the theorem is false. Let $\Gamma$ be a graph which satisfies the hypothesis but has no politician. By Lemma 78, $\Gamma$ is a $k$-regular graph.

The theorem can be verified directly for the only 1-regular graph ($K_1$) and 2-regular graph (the cycle graph on 3 vertices). Therefore, $k > 2$.

Let $A = (A_{ij})$ denote the adjacency matrix of $\Gamma$. By Lemma 78, any two distinct rows of $A$ have exactly $k$ 1s and the remainder of the entries are 0. By hypothesis, any two distinct rows of $A$ have exactly one column where both rows have a 1 (and, for every other column, either one row has a 0 and the other a 1, or both rows have a 0). As a consequence of this, note that[14]

$$A^2 = \begin{pmatrix} k & 1 & & 1 \\ 1 & k & & 1 \\ \vdots & \ddots & \vdots & \\ 1 & \dots & 1 & k \end{pmatrix},$$

so $A^2 = (k-1)I_n + J_n$ (where $I_n$ is the $n \times n$ identity matrix and $J_n$ is the $n \times n$ all-1s matrix). It is easy to see that the rank of $J_n$ is 1, so the kernel is $n-1$-dimensional, so 0 is an eigenvalues with multiplicity $n-1$. The remaining eigenvalue is $n$ (with eigenvector the all-1s vector), of multiplicity 1. Therefore, the eigenvalues of $J_n = A^2 - (k-1)I_n$ are known. Adding $(k-1)I_n$ to $J_n$ simply shifts the eigenvalues of $J_n$ by $k-1$, so we also know the eigenvalues of $A^2$:

$$k-1 \quad (\text{mult. } n-1), \quad n+k-1 \quad (\text{mult. } 1).$$

Since $A$ is symmetric, it's diagonalizable, so $A = P^{-1}DP$, for some invertible matrix $P$ and diagonal matrix $D$ (whose diagonal consists of the eigenvalues of $A$). Therefore, $A^2 = P^{-1}D^2P$, so the eigenvalues of $A^2$ are the squares of the eigenvalues of $A$. Therefore, the eigenvalues of $A$ are

$$\pm\sqrt{k-1}, \quad \pm\sqrt{n+k-1}.$$

[14] We do not need Proposition 58 for this, just the definition of matrix multiplication and the facts about $A$ given above.

Since one of the eigenvalues of $A$ is $k$ (with eigenvector the all-1s vector), we must have $\pm\sqrt{n+k-1} = k$, so

$$n + k - 1 = k^2.$$

Suppose that $r$ of the eigenvalues of $A$ are $\sqrt{k-1}$ and $s$ of the eigenvalues of $A$ are $-\sqrt{k-1}$, where $r + s = n - 1$. Since the sum of the eigenvalues is the trace of $A$, which is 0, we have

$$r\sqrt{k-1} - s\sqrt{k-1} + k = 0,$$

so ($s \neq r$ and)

$$\frac{k}{s-r} = \sqrt{k-1}.$$

By Lemma 80, there is an $h \in \mathbb{Z}$ such that $h > 0$ and $k - 1 = h^2$. This forces $h^2 + 1 = (s - r)h$, so $h$ divides $h^2 + 1$, which implies $h = 1$. This means $k = 2$, a contradiction. □

## 4.5 Eigenvector centrality

Let $\Gamma = (V, E)$ be a connected graph, let $V = \{1, 2, \ldots, n\}$, and let $A$ denote it's $n \times n$ adjacency matrix.

One way to determine which vertexrates as more "important" than another is to list the central vertices, in the sense of (3). One could stipulate that those vertices which are central in this sense are to be rated the most important. Another way, which some call *degree centrality*, is to simply rank vertices by degree, from smallest to largest.

In this section, we discuss another way to rank vertices, called *eigenvalue centraility*.

The *relative centrality score* of vertex $i \in V$ can be defined as the $i$th coordinate $x_i$ of an eigenvector

$$Ax = \lambda x,$$

where $\lambda > 0$ is selected so that all the coordinates of $x$ are positive. According to the Perron-Frobenius theorem, we should select $\lambda = \rho$ to be the largest eigenvalue of $A$ and, in that case, we can find positive eigenvector as above.

The following result is central (pun intended) for the material in this section. It is known as the *Perron-Frobenius theorem for irreducible matrices*

[Sm06]. A square matrix is *irreducible* if it is not similar via a permutation to a block upper triangular matrix (that has more than one block of positive size). In case all the entries of a square matrix $A$ are non-negative, it's known that $A$ is irreducible iff, for every pair of indices $i$ and $j$, there exists a natural number $m$ such that $(A^m)_{ij} \neq 0$.

**Theorem 81.** *Let $A$ be an irreducible non-negative $n \times n$ matrix with spectral radius $\rho$. Then the following statements hold.*

- *The number $\rho$ is a positive real number and it is an eigenvalue*[15] *of the matrix $A$.*
- *The Perron-Frobenius eigenvalue $\rho$ is simple. Both right and left eigenspaces associated with $r$ are one-dimensional.*
- *$A$ has a left eigenvector $\mathbf{v}$ with eigenvalue $\rho$ whose components are all positive. Likewise, $A$ has a right eigenvector $\mathbf{w}$ with eigenvalue $\rho$ whose components are all positive.*
- *The only eigenvectors whose components are all positive are those associated with the eigenvalue $\rho$.*

**Example 82.** For the house graph, depicted as in Figure 12, we have the following Sagemath commands.

Sagemath

```
sage: Gamma = graphs.HouseGraph()
sage: A = Gamma.adjacency_matrix()
sage: A.right_eigenvectors()

[(0, [
  (1, -1, 1, -1, 0)
  ], 1), (-2, [
  (1, -1, -1, 1, 0)
  ], 1), (-1.170086486626034?,
  [(1, 1, -2.170086486626034?, -2.170086486626034?, 3.709275359436923?)],
  1), (0.6888921825340181?,
  [(1, 1, -0.3111078174659819?, -0.3111078174659819?, -0.903211925911554?)],
  1), (2.481194304092016?,
  [(1, 1, 1.481194304092016?, 1.481194304092016?, 1.193936566474631?)],
  1)]
```

Therefore, $\rho = 2.481...$ and the highest scoring ("most central") vertices are the two vertices having degree 3, vertex 2 and vertex 3.

[15]This eigenvalue is called the *Perron-Frobenius eigenvalue*.

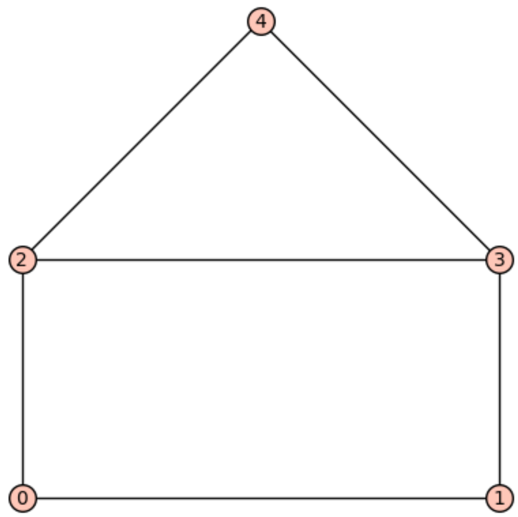


Figure 12: House graph $\Gamma$.

**Example 83.** For the path graph $P_5$, we have the following Sagemath commands.

Sagemath

```
sage: Gamma = graphs.PathGraph(5)
sage: A = Gamma.adjacency_matrix()
sage: A.right_eigenvectors()

[(1, [
  (1, 1, 0, -1, -1)
  ], 1), (0, [
  (1, 0, -1, 0, 1)
  ], 1), (-1, [
  (1, -1, 0, 1, -1)
  ], 1), (-1.732050807568878?,
  [(1, -1.732050807568878?, 2, -1.732050807568878?, 1)],
  1), (1.732050807568878?,
  [(1, 1.732050807568878?, 2, 1.732050807568878?, 1)],
  1)]
```

Therefore, $\rho = 1.732...$ and the highest scoring ("most central") vertex is the middle vertex, vertex 2.

**Example 84.** For the $(5, 1)$-barbell graph, depicted in Figure 13, we have the following Sagemath commands.

Sagemath

```
sage: Gamma = graphs.BarbellGraph(5,1)
```

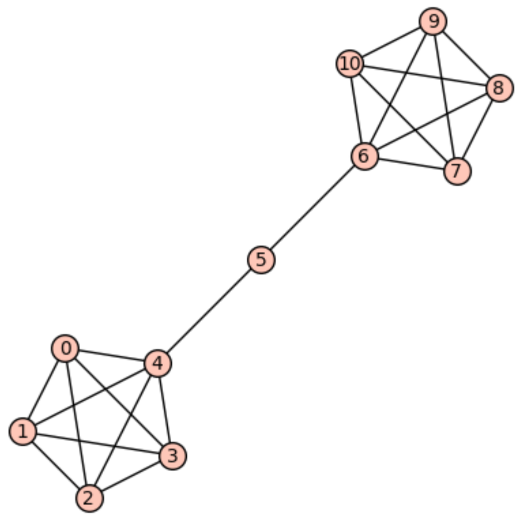


Figure 13: The $(5,1)$-barbell graph.

```
sage: A = Gamma.adjacency_matrix()
sage: A.right_eigenvectors()

[(4, [
  (1, 1, 1, 1, 1, 0, -1, -1, -1, -1, -1)
  ], 1), (-1, [
  (1, 0, 0, 0, -1, 0, 1, 0, 0, 0, -1),
  (0, 1, 0, 0, -1, 0, 1, 0, 0, 0, -1),
  (0, 0, 1, 0, -1, 0, 1, 0, 0, 0, -1),
  (0, 0, 0, 1, -1, 0, 1, 0, 0, 0, -1),
  (0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -1),
  (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, -1),
  (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -1)
  ], 7), (-1.882020544857069?,
  [(1, 1, 1, 1, -4.882020544857069?, 5.188062965835303?, -4.882020544857069?, 1, 1, 1, 1)],
  1), (0.7765379283307647?,
  [(1, 1, 1, 1, -2.223462071669236?, -5.726602630856058?, -2.223462071669236?, 1, 1, 1, 1)],
  1), (4.105482616526304?,
  [(1, 1, 1, 1, 1.105482616526304?, 0.5385396650207548?, 1.105482616526304?, 1, 1, 1, 1)],
  1)]
```

Therefore, $\rho = 4.105...$ and the highest scoring ("most central") vertices are the two vertices having highest degree 3, vertex 4 and vertex 6.

The next example shows that eigenvalue centrailty is not always consistent with degree centrality.

**Example 85.** For the Kittel graph $\Gamma$, depicted as in Figure 14, we have the following Sagemath commands.

Sagemath

```
sage: Gamma =  graphs.KittellGraph()
sage: degs = [Gamma.degree(v) for v in Gamma.vertices()]; degs
[6, 6, 5, 5, 5, 5, 7, 6, 5, 5, 6, 7, 5, 5, 5, 7, 5, 5, 5, 6, 5, 5, 5]
```

The Perron-Frobenius eigenvalue is $\rho = \lambda_1 = 5.583...$, having eigenvector

$$v_1 = (1, 1.0347..., 0.8424.., 0.712.., 0.7381.., 0.7981.., 1.1666...,$$
$$1.003.., 0.7593.., 0.7422.., 0.9632.., 1.1620.., 0.8337.., 0.8064..,$$
$$0.7681.., 1.1733.., 0.8390.., 0.866.., 0.8329.., 0.8547.., 0.7600.., 0.8722.., 0.8109...).$$

Look at the last 4 vertices listed to see that there is a degree 5 vertex with higher eigenvalue centrality than a degree 6 vertex.

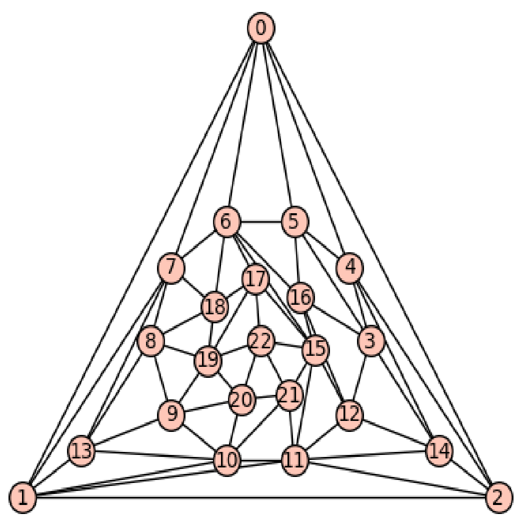


Figure 14: Kittel graph $\Gamma$, having 23 vertices and 63 edges.

### 4.5.1 Keener ranking

In a 1993 paper[16], Keener modified the eigenvector centrality idea to derive a ranking method for sports teams. In this section, we present his idea via an example using the USNA men's baseball team in the 2016 Patriot League.

| x\y | Army | Bucknell | Holy Cross | Lafayette | Lehigh | Navy |
|---|---|---|---|---|---|---|
| Army | × | 14-16 | 14-13 | 14-24 | 10-12 | 8-19 |
| Bucknell | 16-14 | × | 27-30 | 18-16 | 23-20 | 28-42 |
| Holy Cross | 13-14 | 30-27 | × | 19-15 | 27-13 | 43-53 |
| Lafayette | 24-14 | 16-18 | 15-19 | × | 12-23 | 17-39 |
| Lehigh | 12-10 | 20-23 | 43-53 | 23-12 | × | 12-18 |
| Navy | 19-8 | 42-28 | 30-12 | 39-17 | 18-12 | × |

Figure 15: Sorted/ordered as Army vs Bucknell, Army vs Holy Cross, Army vs Lafayette, . . . , Lehigh vs Navy.

The win-loss graph is depicted in Figure 16.

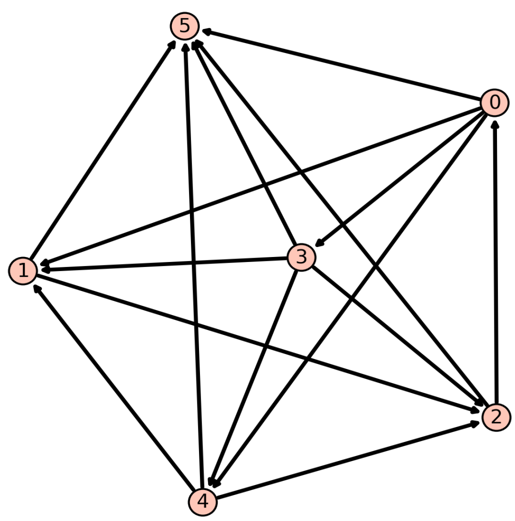


Figure 16: Win-loss digraph based on Figure 15.

There are exactly 15 pairing between these teams. These pairs are sorted lexicographically, as follows:

[16] J.P. Keener, "The Perron-Frobenius theorem and the ranking of football," SIAM Review 35 (1993)80-93.

$$(1,2),(1,3),(1,4), \ldots, (5,6).$$

Suppose $n$ teams play each other. Let $A = (a_{ij})_{1\le i,j\le n}$ be a non-negative square matrix determined by the results of their games, called the *preference matrix*. In his 1993 paper, Keener defined the *score* of the $i$th team to be given by

$$s_i = \frac{1}{n_i}\sum_{j=1}^{n} a_{ij} r_j,$$

where $n_i$ denotes the total number of games played by team $i$ and $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ is the rating vector (so $r_i \ge 0$ denotes rating of team $i$).

One possible preference matrix the matrix $S$ of total scores obtained from the table of scores was given in Figure 15.

$$S = \begin{pmatrix} 0 & 14 & 14 & 14 & 10 & 8 \\ 16 & 0 & 27 & 18 & 23 & 28 \\ 13 & 30 & 0 & 19 & 27 & 43 \\ 24 & 16 & 15 & 0 & 12 & 17 \\ 12 & 20 & 43 & 23 & 0 & 12 \\ 19 & 42 & 30 & 39 & 18 & 0 \end{pmatrix},$$

(In this case, $n_i = 4$ so we ignore the $1/n_i$ factor.)

In his paper, Keener proposed a ranking method where the ranking vector $\mathbf{r}$ is proportional to its score. The score is expressed as above in terms of a matrix product $A\mathbf{r}$, where $A$ is a square preference matrix. In other words, there is a constant $\rho > 0$ such that $s_i = \rho r_i$, for each $i$. This is the same as saying $A\mathbf{r} = \rho\mathbf{r}$.

The above Frobenius-Perron theorem implies that $S$ has an eigenvector $\mathbf{r} = (r_1, r_2, r_3, r_4, r_5, r_6)$ having positive entries associated to the largest eigenvalue $\lambda_{max}$ of $S$, which has (geometric) multiplicity 1. Indeed, $S$ has maximum eigenvalue $\lambda_{max} = 110.0385...$, of multiplicity 1, with eigenvector

$$\mathbf{r} = (1, 1.8313\ldots, 2.1548\ldots, 1.3177\ldots, 1.8015\ldots, 2.2208\ldots).$$

Therefore the teams, according to Kenner's method, are ranked,

Army < Lafayette < Lehigh < Bucknell < Holy Cross < Navy.

## 4.6 Strongly regular graphs

**Definition 86.** A *strongly regular* graph[17] is a regular graph $\Gamma = (V, E)$, where every pair of neighboring vertices has the same number $\lambda$ of neighbors in common, and every pair of vertices which are not neighbors has the same number $\mu$ of neighbors in common. If $\Gamma$ is $k$-regular and has $n$ vertices, then the *parameters* of the SRG are $(n, k, \lambda, \mu)$.

**Lemma 87.** *If $A$ is the adjacency matrix of a strongly regular graph having parameters $(n, k, \lambda, \mu)$ then*

$$A^2 = kI + \lambda A + \mu(J - I - A), \tag{5}$$

*where $J$ is the all $1$s matrix and $I$ is the identity matrix.*

*Proof.* Sketch: Using Lemma 56, compute $(A^2)_{ij}$ in the three separate cases (a) $i = j$, (b) $i \neq j$ and $i, j$ adjacent, (c) $i \neq j$ and $i, j$ non-adjacent. □

### 4.6.1 The Petersen graph

An example of a strongly regular graph is the Petersen graph below.

**Definition 88.** The *Petersen graph*[18] is a connected graph with 10 vertices and 15 edges depicted in Figure 17.

The Petersen graph is strongly regular, and it is both edge transitive and vertex transitive. It's also 3-arc-transitive[19]. It's a distance-regular graph.

The Petersen graph has a Hamiltonian path but no Hamiltonian cycle.

The Petersen graph has chromatic number 3, meaning that its vertices can be colored with three colors, but not with two. The Petersen graph has chromatic index 4, meaning that its edges can be colored with four colors, but not three.

The Petersen graph has girth 5, radius 2 and diameter 2. It's graph spectrum is $-2, -2, -2, -2, 1, 1, 1, 1, 1, 3$.

[17]Often abbreviated in the literature as SRG.

[18]First constructed by Kempe in 1886 then, independently, by Petersen in 1898.

[19]Every directed three-edge path in the Petersen graph can be transformed into every other such path by an element of the automorphism group of the graph.

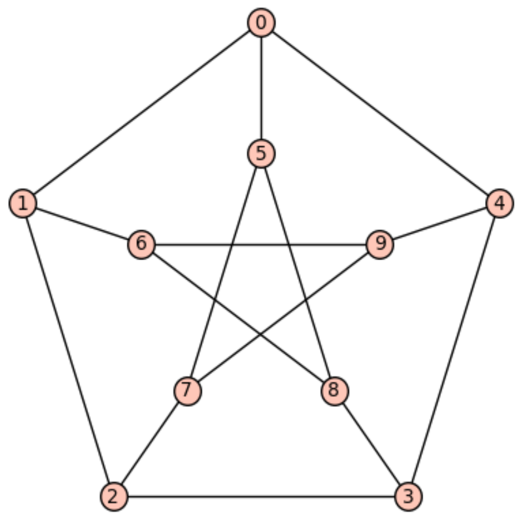


Figure 17: The Petersen graph.

## 4.7 Desargues graph

This graph, depicted[20] in Figure 18, is a distance-transitive 3-regular graph with 20 vertices and 30 edges. Its automorphism group acts regularly and is order 240.

The characteristic polynomial of the Desargues graph is

$$(x-3)(x-2)^4(x-1)^5(x+1)^5(x+2)^4(x+3).$$

Sagemath

```
sage: Gamma = graphs.DesarguesGraph()
sage: len(Gamma.edges())
30
sage: len(Gamma.vertices())
20
sage: Gamma.is_regular(3)
True
sage: Gamma.is_distance_regular()
True
sage: Gamma.is_edge_transitive()
True
sage: Gamma.is_vertex_transitive()
True
```

[20] This plot by David Eppstein found on the Wikipedia page for the Desargues graph is in the public domain.

```
sage: Gamma.automorphism_group().order()
240
```

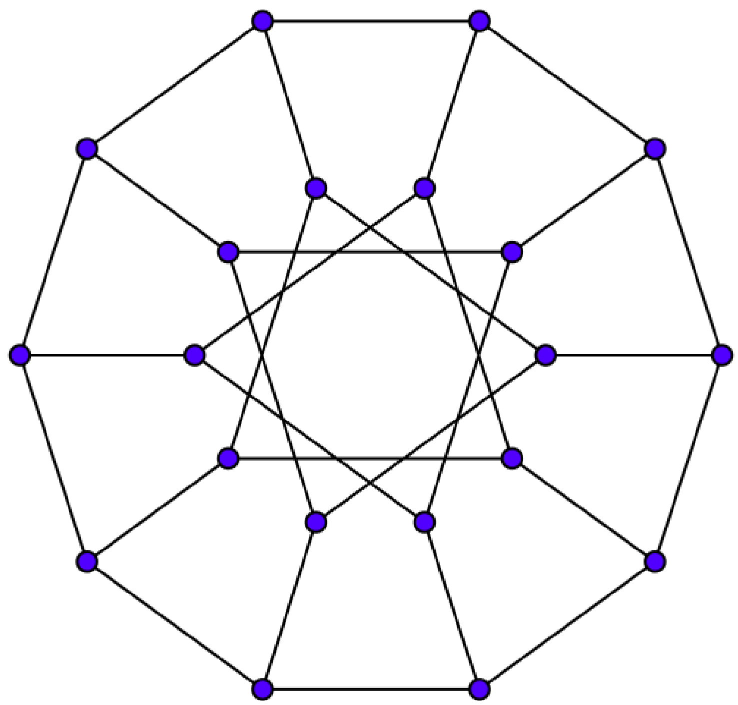

Figure 18: A Desargues graph example.

**Exercise 4.10.** *Use* Sagemath *to verify that the diameter of the Desargues graph is* 5*.*

## 4.8 Dürer graph

A graph, depicted in Figure 19 with 12 vertices and 18 edges, named after Albrecht Dürer, whose 1514 engraving Melencolia I includes a depiction of a solid convex polyhedron having the Dürer graph as its skeleton.

It is a 3-vertex-connected simple planar graph. The automorphism group of the Dürer graph is isomorphic to the dihedral group of order 12, $D_{12}$, and the characteristic polynomial is

$$(x-3)\cdot(x-1)\cdot x^2\cdot(x+2)^2\cdot(x^2-5)\cdot(x^2-2)^2.$$

Sagemath

```
sage: Gamma = graphs.DurerGraph()
sage: Gamma.is_edge_transitive()
False
sage: Gamma.is_regular(4)
False
sage: Gamma.is_regular(3)
True
sage: Gamma.is_distance_regular()
False
sage: Gamma.is_vertex_transitive()
False
sage: Gamma.is_perfect()
False
sage: Gamma.automorphism_group().order()
12
```

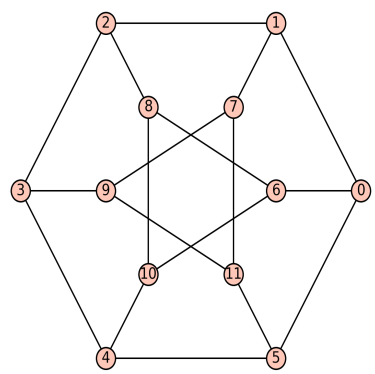


Figure 19: A Dürer graph example.

**Exercise 4.11.** *Use* Sagemath *to verify that the diameter of the Dürer graph is* 4.

## 4.9 Orientation on a graph

Let $\Gamma$ be a graph. The cardinality $|V|$ of the vertex set $V$ is called the *order* of $\Gamma$, and the cardinality $|E|$ of the edge set $E$ is called the *size* of $\Gamma$.

An *orientation* of the edges is an injective function $\sigma : E \to V \times V$. If $e = \{u, v\}$ and $\sigma(e) = (u, v)$ then we call $v$ the *head* of $e$ and $u$ the *tail* of $e$. We define the head and tail functions $h : E \to V$ and $t : E \to V$ by $h(e) = v$ and $t(e) = u$, i.e., $h(e)$ is the head of $e$ and $t(e)$ is the tail of $e$. If the vertices of a graph is the set $V = \{0, 1, \ldots, n-1\}$ (or $V = \{1, 2, \ldots, n\}$) then the *default orientation* $\sigma_0 = (h_0, t_0)$ of an edge $e = \{u, v\}$ is defined by $h_0(e) = \max\{u, v\}$, $t_0(e) = \min\{u, v\}$ and the *default labeling* of the edges is $E = \{e_1, e_2, \ldots, e_m\}$, where the ordering is the lexicographical ordering. In other words, we define $<$ on $E$ by: $e < e'$ if and only if $t_0(e) < t_0(e')$ or $t_0(e) = t_0(e')$ and $h_0(e) < h_0(e')$. For example, $\{0, 1\} < \{0, 2\}$ and $\{0, 1\} < \{2, 3\}$.

If the vertices of a graph is the set $V = \{0, 1, \ldots, n-1\}$ (or $V = \{1, 2, \ldots, n\}$) and if the edges of a graph are indexed $E = \{e_1, e_2, \ldots, e_m\}$ (e.g., using the default labeling) then we can associated to each orientation $\sigma$ a vector,

$$\vec{o}_\sigma = (o_1, o_2, \ldots, o_m) \in \{1, -1\}^m,$$

as follows: if $e_i = \{u, v\}$ and $h(e_i) = h_0(e_i)$ then define $o_i = 1$, but if $h(e_i) = t_0(e_i)$ then define $o_i = -1$. This $\vec{o}_\sigma$ is called the *orientation vector* associated to $\sigma$.

Recall, if $u$ and $v$ are two vertices in a graph $\Gamma = (V, E)$, then a $u$-$v$ walk $W$ is an alternating sequence of vertices and edges starting with $u$ and ending at $v$,

$$W : e_0 = (v_0, v_1), e_1 = (v_1, v_2), \ldots, e_{k-1} = (v_{k-1}, v_k), \tag{6}$$

where $v_0 = u$, $v_k = v$, and each $e_i \in E$.

Suppose $\Gamma = (V, E)$ is given an orientation $\sigma$ and that $\vec{o}$ is the associated orientation vector. Order the edges

$$E = \{e_1, e_2, \ldots, e_m\},$$

in some arbitrary but fixed way. Consider a path of length $k$ from $u$ to $v$ in $\Gamma$ $(u, v \in V)$,

$$P : u_0 = u, u_1, \ldots, u_{k-1}, u_k = v,$$

which we may regard as a subgraph $\Gamma' = (V', E')$ of $\Gamma$,

$$V' = \{u_0, u_1, \ldots, u_k\}, \quad E' = \{(u_0, u_1), (u_1, u_2), \ldots, (u_{k-1}, u_k)\} \subset E,$$

with a fixed orientation of the edges as in (6) so it is directed from $u = u_0$ to $v = u_k$. A *vector representation* of the subgraph $\Gamma'$ is an $m$-tuple

$$\vec{\Gamma'} = (a_1, a_2, \ldots, a_m) \in \{1, -1, 0\}^m, \tag{7}$$

where

$$a_i = a_i(\Gamma', \sigma) = \begin{cases} 1, & \text{if } e_i = (u_j, u_{j+1}) \in E', h(e_i) = u_{j+1} \\ -1, & \text{if } e_i = (u_j, u_{j+1}) \in E', h(e_i) = u_j \\ 0, & \text{if } e_i \notin E'. \end{cases}$$

In particular, this defines a mapping

$$\{\text{paths of } \Gamma = (V, E)\} \to \{1, -1, 0\}^m,$$
$$\Gamma' \mapsto \vec{\Gamma'}.$$

**Example 89.** Conside the house graph $\Gamma = (V, E)$ in Example 7. Label the edges as follows:

$$E = \{e_0 = (0, 1), e_1 = (0, 2), e_2 = (0, 4), e_3 = (1, 2), e_4 = (2, 3), e_5 = (3, 4)\},$$

oriented with the default orientation, so $\vec{o} = (1, 1, 1, 1, 1, 1)$.

Let $\Gamma'$ denote the cycle $0 \to 1 \to 2 \to 0$, oriented as indicated. The vector representation of this cycle is $\vec{\Gamma'} = (1, -1, 0, 1, 0, 0)$.

# 5 Incidence matrix

The incidence matrix is an algebraic way of describing the connection between the vertices and edges of a graph. There are two types of incidence matrices.

## 5.1 The unsigned incidence matrix

We shall describe the simplest kind of incidence matrix first.

**Definition 90.** The (unsigned) *incidence matrix* of a simple graph $\Gamma$ is an $n \times m$ matrix $B = B_\Gamma = (b_{ij})$, where $m$ and $n$ are the numbers of vertices and edges respectively, such that $b_{ij} = 1$ if the vertex $v_i$ is incident to the edge $e_j$, and $b_{ij} = 0$ otherwise.

**Example 91.** Sagemath *can easily compute the incidence matrix of a graph. For example, the incidence matrix of the cycle graph* $C_5$ *is*

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

*Here's the* Sagemath *computation.*

Sagemath

```
sage: Gamma = graphs.CycleGraph(5)
sage: Gamma.vertices()
[0, 1, 2, 3, 4]
sage: Gamma.edges(labels=None)
[(0, 1), (0, 4), (1, 2), (2, 3), (3, 4)]
sage: Gamma.incidence_matrix()

[1 1 0 0 0]
[1 0 1 0 0]
[0 0 1 1 0]
[0 0 0 1 1]
[0 1 0 0 1]
```

**Exercise 5.1.** *Compute the (unsigned) incidence matrix for the house graph.*

**Example 92.** *This is a continuation of Example 12.*

*The incidence matrix of a finite projective plane is a* $p \times q$ *matrix* $B$*, where* $p$ *and* $q$ *are the number of points and lines respectively, such that* $B_{i,j} = 1$ *if the point* $p_i$ *and line* $L_j$ *are incident, and* $0$ *otherwise.*

**Lemma 93.** *The incidence matrix of a finite projective plane is the incidence matrix of the associated graph (defined in Example 12).*

*Proof.* Exercise. □

**Example 94.** Sagemath *can easily compute the (unsigned) incidence matrix* $B$ *of the house graph:*

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

*Here's the* Sagemath *computation.*

Sagemath

```
sage: Gamma = graphs.HouseGraph()
sage: Gamma.edges()

[(0, 1, None),
 (0, 2, None),
 (1, 3, None),
 (2, 3, None),
 (2, 4, None),
 (3, 4, None)]
sage: Gamma.incidence_matrix()

[ 1  1  0  0  0  0]
[ 1  0  1  0  0  0]
[ 0  1  0  1  1  0]
[ 0  0  1  1  0  1]
[ 0  0  0  0  1  1]
```

**Lemma 95.** *If $B(\Gamma)$ is the (unsigned) incidence matrix of a graph $\Gamma$ and if $A(L(\Gamma))$ is the adjacency matrix of the line graph of $\Gamma$ then*

$$B(\Gamma)^T B(\Gamma) = 2I + A(L(\Gamma)).$$

*Proof.* The $(i, j)$th entry of $B(\Gamma)^T B(\Gamma)$ is the dot product of the $i$th column of $B(\Gamma)$ with the $j$th column. This inner product is non-zero if and only if the $i$th edge of $\Gamma$ is incident with the $j$th edge (i.e., they share a vertex). If $i = j$ then this inner product is 2 (since each edge has two vertices). Since the line graph swaps vertices with edges, the $i$th edge of $\Gamma$ is incident with the $j$th edge iff the $i$th vertex of $L(\Gamma)$ is adjacent with the $j$th vertex. Thus, if $i \neq j$ then this inner product is the $(i, j)$th entry of $A(L(\Gamma))$. □

**Lemma 96.** *If $B(\Gamma)$ is the (unsigned) incidence matrix of a graph $\Gamma$ and if $\Delta(\Gamma)$ is the diagonal matrix whose $i$th diagonal entry is the degree of the $i$th vertex of $\Gamma$, then*

$$B(\Gamma)B(\Gamma)^T = \Delta(\Gamma) + A(\Gamma).$$

*Proof.* The $(i, j)$th entry of $B(\Gamma)B(\Gamma)^T$ is the dot product of the $i$th row of $B(\Gamma)$ with the $j$th row. This inner product is non-zero if and only if the $i$th vertex of $\Gamma$ is adjacent with the $j$th vertex. If $i = j$ then this inner product is $d_i$, the degree of the $i$th vertex. If $i \neq j$ then this inner product is the $(i, j)$th entry of $A(\Gamma)$. □

## 5.2 The oriented case

In this section, we introduce the signed incidence matrix.

**Definition 97.** The (signed) *incidence matrix* of an oriented graph $\Gamma$, with orientation $\sigma$, is an $n \times m$ matrix $D = D_{\Gamma,\sigma} = (d_{ij})$, where $m$ and $n$ are the numbers of edges and vertices respectively, such that $d_{ij} = 1$ if the vertex $v_i$ is the head of edge $e_j$, $d_{ij} = -1$ if the vertex $v_i$ is the tail of edge $e_j$, and $d_{ij} = 0$ otherwise.

**Exercise 5.2.** *Compute the (signed) incidence matrix for the house graph, using the default orientation.*

**Lemma 98.** *Let $\Gamma$ be an oriented graph and $B$ its signed incidence matrix. The rank of $D$ is $n - c(\Gamma)$, where $n$ is the number of vertices of $\Gamma$ and $c(\Gamma)$ is the number of connected components.*

*Proof.* Suppose the matrix $D$ is $n \times m$ and $z \in \mathbb{R}^n$ is in the left-kernel of $B$: i.e., $zD = 0$. For each edge $(u, v) \in E$, $zD = 0$ implies $z_u = z_v$. Therefore, $z$ is constant on connected components. This implies that the left-kernel of $B$ has dimension equal to the number of connected components of the graph. The result now follows from the rank plus nullity theorem from matrix theory. □

**Lemma 99.** *If $B(\Gamma)$ is the (signed) incidence matrix of an oriented graph $\Gamma$ and if $\Delta(\Gamma)$ is the diagonal matrix whose diagonal entries is the degree sequence of $\Gamma$, and if $A(\Gamma)$ is the adjacency matrix of $\Gamma$ then*

$$D(\Gamma)D(\Gamma)^T = \Delta(\Gamma) - A(\Gamma).$$

*Proof.* Let $D = D(\Gamma)$. The $(i, j)$th entry of $DD^T$ is the dot product of the $i$th row of $D$ with the $j$th row. This inner product is non-zero only if the $i$th vertex of $\Gamma$ is adjacent with the $j$th vertex. If $i = j$ then this inner product is $d_i$. If $i \neq j$ then this dot product is $\pm 1$. However, the dot product of the $i$th row of $D$ with the $j$th row can never be $+1$ for the following reason: if the $i$th vertex of $\Gamma$ is adjacent with the $j$th vertex then they share an edge, say the $k$th edge. This means that the the dot product of the $i$th row of $D$ with the $j$th row is simply the product of the two non-zero entries if the $k$th column of $D$. Since $\Gamma$ is oriented, one of these two entries must be $+1$ (corresponding to the head vertex of the $k$th edge) and one must be $-1$ (the tail of the $k$th edge). □

**Example 100.** *Solution to Exercise 5.2 using* Sagemath *: a signed incidence matrix $D$ of the house graph:*

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

*Here's the* Sagemath *computation (using the module* `alg-graph-thry2.sage`*), using a pre-selected orientation and* Sagemath *'s ordering of the vertices (see Figure 20) and edges (see* Sagemath *output below).*

Sagemath

```
sage: Gamma = graphs.HouseGraph()
sage: Gamma.edges()

[(0, 1, None),
 (0, 2, None),
 (1, 3, None),
 (2, 3, None),
 (2, 4, None),
 (3, 4, None)]
sage: incidence_matrix(Gamma, 6*[1])

[ 1  1  0  0  0  0]
[-1  0  1  0  0  0]
[ 0 -1  0  1  1  0]
[ 0  0 -1 -1  0  1]
[ 0  0  0  0 -1 -1]
```

Let $\Gamma$ denote the (disjoint) union $C_3 \cup P_2$ of the cycle graph $C_3$ and the path graph $P_2$. Compute the rank and nullity of

(a) The adjacency matrix $A$ of $\Gamma$,

(b) The unsigned incidence matrix $B$ of $\Gamma$,

(c) The signed incidence matrix $D$ of $\Gamma$, for the default orientation.

## 5.3 Cycle space and cut space

Let $\Gamma = (V, E)$ be a graph.

If $F$ is a field such as $\mathbb{R}$ or $GF(q)$, or a ring such as $\mathbb{Z}$, let

$$C^0(\Gamma, F) = \{f : V \to F\}, \quad C^1(\Gamma, F) = \{f : E \to F\},$$

be the sets of $F$-valued functions defined on $V$ and $E$, respectively. These sets are sometimes called the *vertex space over* $F$ and the *edge space over* $F$. In case $F$ is a field then these vector spaces are inner product spaces with inner product

$$(f, g) = \sum_{x \in X} f(x)g(x), \qquad (X = V, \ \text{resp.} \ X = E), \tag{8}$$

and

$$\dim C^0(\Gamma, F) = |V|, \quad \dim C^1(\Gamma, F) = |E|.$$

Index the sets $V$ and $E$ in some arbitrary but fixed way and define, for $1 \le i \le |V|$ and $1 \le j \le |E|$,

$$f_i(v) = \begin{cases} 1, & v = v_i, \\ 0, & \text{otherwise}, \end{cases} \qquad g_j(e) = \begin{cases} 1, & e = e_j, \\ 0, & \text{otherwise}. \end{cases}$$

**Lemma 101.** *(a) $\mathcal{F} = \{f_i\} \subset C^0(\Gamma, F)$ is a basis. (b) $\mathcal{G} = \{g_j\} \subset C^1(\Gamma, F)$ is a basis.*

We call these the *standard bases* for $C^0(\Gamma, F)$ and $C^1(\Gamma, F)$, resp..

*Proof.* The proof is left as an exercise. □

In terms of linear transformations, define the *incidence map*,

$$D : C^1(\Gamma, F) \to C^0(\Gamma, F),$$

by

$$(Df)(v) = \sum_{h(e)=v} f(e) - \sum_{t(e)=v} f(e). \tag{9}$$

With respect to the standard bases $\mathcal{F}$ and $\mathcal{G}$, the matrix representing the linear transformation $D : C^1(\Gamma, F) \to C^0(\Gamma, F)$ is the incidence matrix.

**Remark 1.** *By "abuse of notation," we use $D$ to denote both the incidence map, as a linear transformation, and its matrix representation. If it is necessary to distinguish them, we use $D$ for the incidence map and $[D]$ for the matrix representation.*

**Exercise 5.3.** *Let $\Gamma = (V, E)$ denote a connected, graph, with orientation $\sigma$. (You may assume that $\sigma$ is the default orientation.) Compute the left kernel of the signed incidence matrix, $D = D_\Gamma$.*

*Use this, and the rank plus nullity theorem, to find the nullity (the dimension of the right kernel) of $D$. (Ans: $m - n + 1$.)*

If $F$ is a field and $\Gamma$ is a graph with orientation $\sigma$, then the kernel of the signed incidence map $D = D_{\Gamma,\sigma} : C^1(\Gamma, F) \to C^0(\Gamma, F)$ is the *cycle space*,

$$\mathcal{Z} = \mathcal{Z}_{\Gamma,\sigma} = \ker(D).$$

Via the standard basis, we may identify the cycle space with a subspace of $F^m$ using the right kernel of the $n \times m$ matrix representation of the map $D$.

Let $\Gamma = (V, E)$ be a connected graph.

A "cocycle" or "minimal cut" (or "bond," in the terminology of [GR04]) is a minimal set of edges whose removal from $\Gamma$ disconnects $\Gamma$. A more precise definition is below.

**Definition 102.** Fix a non-trivial partition

$$V = V_1 \cup V_2, \qquad V_i \neq \emptyset, \quad V_1 \cap V_2 = \emptyset,$$

of the vertices of $\Gamma$. If the set $H = H(V_1, V_2)$ of all edges $e = (v_1, v_2) \in E$ with $v_i \in V_i$ $(i = 1, 2)$ is non-empty, then $H$ is called a *cut* of $\Gamma$. A cut is minimum if the size of the cut is not larger than the size of any other cut. Such a cut is called a *minimum cut.* If a cut results in exactly two disjoint connected subgraphs of $\Gamma$ then the cut is called a *cut-set* or a *cocycle.*

**Remark 2.** *A word of* **caution:** *A minimum cut is a cocycle but not all cocycles are minimum cuts. Minimum cuts arise in the min-cut max-flow theorem. The min-cut max-flow theorem states that in a flow network, the maximum amount of flow passing from the source vertex to the sink vertex is equal to the total weight*[21] *of the edges in an minimum cut separating these vertices [La01].*

The following result[22] introduces fundamental cycles and cocycles.

[21] For an unweighted graph, the *weight* of a cut is the cardinality of $H = H(V_1, V_2)$, i.e., the number of edges removed in the cut.

[22] See Lemma 5.1 in Biggs [Bi93].

**Proposition 103.** *Let $T = (V, E_T)$ be a spanning tree of a connected graph $\Gamma = (V, E)$.*

*(a) For each edge $e \in E - E_T$ there is a unique cycle in $\Gamma$ containing only $e$ and some subset of $E_T$.*

*(b) For each edge $e \in E_T$ there is a unique cut in $\Gamma$ containing only $e$ and some subset of $E - E_T$.*

The edges in $E - E_T$ are called the *chords* of $T$. Write $C_e = cyc(e, T)$ for the cycle in (a) above. It will be called a *fundamental cycle*. Write $C_e^* = cut(e, T)$ for the cut in (b) above. It will be called a *fundamental cut* or *fundamental cocycle*.

*Proof.* (a): If $T$ is a spanning tree of $\Gamma$ then $T$ is maximal (in the sense of inclusion) as an acyclic subgraph. Therefore, adding an edge to $T$ must create a cycle. To show it is unique, we argue by contradiction. Fix $e \in E - E_T$ and assume that the graph $T \cup \{e\}$ contains two cycles, $C_1$ and $C_2$, $C_1 \neq C_2$. These must share at least the edge $e = (u, v)$. Consider the walk $W$ that starts at the vertex $u$ of $e$ and traverses $C_1$ until it gets to the vertex $v$ of $e$, then traverses $C_2$ until it returns back to $u$. This $W$ is a closed walk in $\Gamma$ which avoids $e$, and therefore is contained entirely in $T$. Since $W$ contains a closed path (see Lemma 34), this contradicts the assumption that $T$ is acyclic.

(b): It suffices to verify that the subgraph $cut(e, T)$ is a cut. Consider the decomposition of the vertices of $\Gamma$ associated to the cut of $T$ obtained by removing the edge $e \in E_T$. This corresponds to a decomposition of the vertices of $\Gamma$ since the vertices of $\Gamma$ and the vertices of $T$ are the same. Thus $e$ and $E - E_T$ is associated to a unique cut of $\Gamma$.

□

**Exercise 5.4.** *Let $\Gamma$ denote the diamond graph, with default orientation. Show that for every cycle $C$ of $\Gamma$, $D\vec{C} = \vec{0}$, where $\vec{C}$ denotes the oriented characteristic vector of $C$ and $D$ denotes the signed incidence matrix of $\Gamma$.*

**Example 104.** *Let $\Gamma = (V, E)$ denote the house graph, labeled as in Figure 20, so that*

$$E = \{(0, 1), (0, 2), (1, 2), (2, 3), (3, 4), (1, 4)\}.$$

*Consider the spanning tree $T = (V, E_T)$, where*

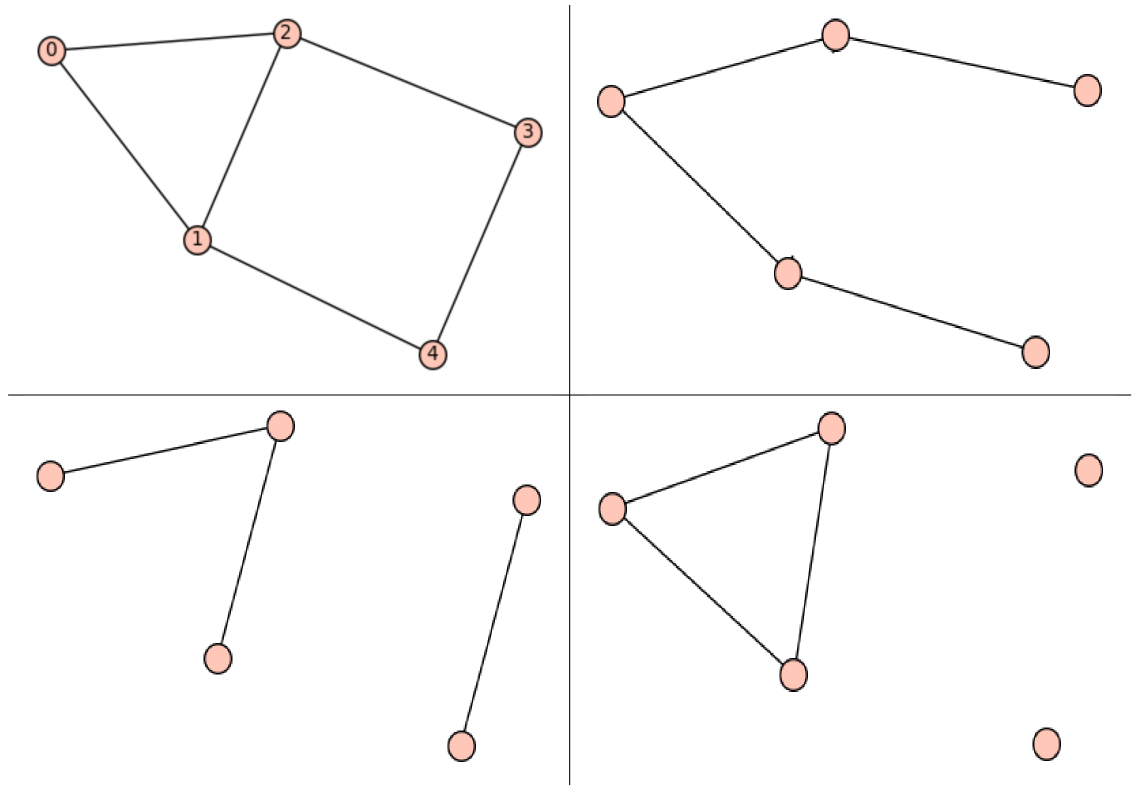


Figure 20: Left to right, top to bottom: The house graph $\Gamma$; a spanning tree $T$ of $\Gamma$; the cut $C_e^* = cut(e,T)$, $e = (0,2)$, of $\Gamma$; the cycle $C_e = cyc(e,T)$, $e = (1,4)$, of $\Gamma$.

$$E_T = E - \{(1,2),(3,4)\}.$$

*Picking* $e = (0,2) \in E_T$ *gives rise to the cut* $C_e^* = cut(e,T)$*. Picking* $e = (1,4) \in E_T$ *gives rise to the cycle* $C_e = cyc(e,T)$*.*

**Example 105.** *Let* $\Gamma$ *be the diamond graph, as depicted in Figure 26. The cocyles, or cut-sets, correspond to the partitions listed below.*

| *cocycle* | *weight* | *edges* |
|---|---|---|
| $C_0^* = \{0\} \cup \{1,2,3\}$ | 2 | *{(0,1),(0,2)}* |
| $C_1^* = \{1\} \cup \{0,2,3\}$ | 3 | *{(0,1),(1,2),(1,3)}* |
| $C_2^* = \{2\} \cup \{0,1,3\}$ | 3 | *{(0,2),(1,2),(2,3)}* |
| $C_3^* = \{3\} \cup \{0,1,2\}$ | 2 | *{(1,3),(2,3)}* |
| $C_4^* = \{0,1\} \cup \{2,3\}$ | 3 | *{(0,2),(1,2),(1,3)}* |
| $C_5^* = \{0,2\} \cup \{1,3\}$ | 3 | *{(0,1),(1,2),(1,3)}* |

*The minimum cuts are those of weight* 2.

**Exercise 5.5.** *List the cocycles of the house graph, depicted as in Figure 12.*

Suppose $\Gamma = (V, E)$ is given an orientation $\sigma$ and that $\vec{o}$ is the associated orientation vector. Order the edges

$$E = \{e_1, e_2, \ldots, e_m\},$$

in some arbitrary but fixed way. Consider a cut $\Gamma'$ of $\Gamma$ (associated to a non-trivial partition $V = V_1 \cup V_2$), which we may regard as a subgraph $\Gamma' = (V', E')$ of $\Gamma$. Consider an orientation on $\Gamma'$ such that for any edge $(u, v) \in E'$, $u \in V_1$ and $v \in V_2$. If we regard $V_1$ as the "negative" vertices and $V_2$ as the "positive" vertices, then each edge of an *oriented cut* has a negative tail and a positive head. A *(signed) vector representation* of the cut $\Gamma'$ is an $m$-tuple

$$\vec{H} = (b_1, b_2, \ldots, b_m) \in \{1, -1, 0\}^m, \tag{10}$$

where

$$b_i = b_i(\Gamma', \sigma) = \begin{cases} 1, & \text{if } e_i \in E', h(e_i) \in V_2 \\ -1, & \text{if } e_i \in E', h(e_i) \in V_1 \\ 0, & \text{if } e_i \notin E'. \end{cases}$$

This is also called the *(signed) characteristic vector* of $\Gamma'$. In particular, this defines a mapping

$$\{ \text{ cuts of } \Gamma = (V, E) \ \} \to \{1, -1, 0\}^m,$$
$$\Gamma' \mapsto \vec{\Gamma'}.$$

Each $v \in V$ determines an oriented cut $C^*(v)$ with $V_1 = V - \{v\}$ (the "− shore") and $V_2 = \{v\}$ (the "+ shore").

**Example 106.** *For the house graph depicted in Figure 21, we compute the signed characteristic vector of each $C(v)$.*

$$C_0^* : V_1 = \{1,2,3,4\}, V_2 = \{0\}, E_{C_0^*} = \{0,1\}, \vec{C_0^*} = (-1,-1,0,0,0,0),$$

$$C_1^* : V_1 = \{0,2,3,4\}, V_2 = \{1\}, E_{C_1^*} = \{0,2,3\}, \vec{C_1^*} = (1,0,-1,-1,0,0),$$
$$C_2^* : V_1 = \{0,1,3,4\}, V_2 = \{2\}, E_{C_2^*} = \{2,4\}, \vec{C_2^*} = (0,0,1,0,-1,0),$$
$$C_3^* : V_1 = \{0,1,2,4\}, V_2 = \{3\}, E_{C_3^*} = \{4,5\}, \vec{C_3^*} = (0,0,0,0,1,-1),$$
$$C_4^* : V_1 = \{0,1,2,3\}, V_2 = \{4\}, E_{C_4^*} = \{1,3,5\}, \vec{C_4^*} = (0,1,0,1,0,1).$$

**Lemma 107.** *The vth row of the oriented incidence matrix is the signed characteristic vector of the fundamental cocycle $C(v)$.*

*Proof.* This follows from the definitions. □

Also, recall the $F$-vector space spanned by the vector representations of all the cuts is the cocycle space of $\Gamma$,

$$\mathcal{Z}^*(\Gamma) = \mathcal{Z}^*(\Gamma, F).$$

This is the column space of the transpose of the incidence matrix of $\Gamma$ and may be regarded as a subspace of $C^1(\Gamma, F)$.

Let $\Gamma' = (V', E')$ be a cycle of the oriented graph $\Gamma$, having orientation $\sigma$. Let $V' = \{u_1, u_2, \dots, u_r\}$ denote a fixed sequence of vertices of this cycle obtained by traversing the cycle in some manner. Let $E = \{e_1, e_2, \dots, e_m\}$ denote a labeling of the edges of $\Gamma$. A *(signed) vector representation* of the cycle $\Gamma'$ is an $m$-tuple

$$\vec{H} = (b_1, b_2, \dots, b_m) \in \{1, -1, 0\}^m, \tag{11}$$

where

$$b_i = b_i(\Gamma', \sigma) = \begin{cases} 1, & \text{if } e_i \in E', e = (u_i, u_{i+1}), h(e_i) = u_{i+1}, \\ -1, & \text{if } e_i \in E', e = (u_i, u_{i+1}), t(e_i) = u_{i+1}, \\ 0, & \text{if } e_i \notin E'. \end{cases}$$

This is also called the *(signed) characteristic vector* of the cycle $\Gamma'$. In particular, this defines a mapping

$$\{ \text{ cycles of } \Gamma = (V, E) \ \} \to \{1, -1, 0\}^m,$$
$$\Gamma' \mapsto \vec{\Gamma'}.$$

**Example 108.** *Consider the house graph $\Gamma = (V, E)$, with edge labels depicted in Figure 21, so that*

$$E = \{e_1 = (0,1), e_2 = (0,4), e_2 = (1,2), e_3 = (1,4), e_4 = (2,3), e_4 = (3,4)\}.$$

*Let $\Gamma$ be given the default orientation $\sigma_0 = (h_0, t_0)$ (so, for example, $h_0(2,3) = 3$ and $t_0(0,4) = 0$).*

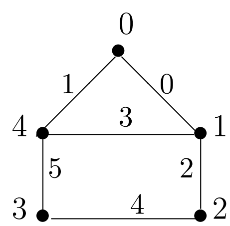


Figure 21: The house graph, with edge labels.

*If $C_1$ denotes the cycle $C_1 = (V_1, E_1)$, where*

$$V_1 = \{0, 1, 4\}, \qquad E_1 = \{(0,1), (0,4), (1,4)\},$$

*oriented clockwise, then*

$$\vec{C_1} = (1, -1, 0, 1, 0, 0).$$

*If $C_2$ denotes the cycle $C_2 = (V_2, E_2)$, where*

$$V_2 = \{1, 2, 3, 4\}, \qquad E_1 = \{(1,2), (1,4), (2,3), (3,4)\},$$

*oriented clockwise, then*

$$\vec{C_2} = (0, 0, 1, -1, 1, 1).$$

*If $C_3$ denotes the cycle $C_3 = (V_3, E_3)$, where*

$$V_3 = \{0, 1, 2, 3, 4\}, \qquad E_1 = \{(0,1), (0,4), (1,2), (2,3), (3,4)\},$$

*oriented clockwise, then*

$$\vec{C_3} = (1, -1, 1, 0, 1, 1).$$

*Note $\vec{C_1} + \vec{C_2} = \vec{C_3}$.*

*We have*

$$D = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

*and it's easy to verify that*

$$D\vec{C_i} = \mathbf{0}, \qquad i = 1, 2, 3.$$

**Proposition 109.** *If $\Gamma$ is an oriented graph with orientation $\sigma$ and if $D = D_{\sigma,\Gamma}$ is the oriented incidence matrix then, for any cycle $C$ as above*

$$D\vec{C} = \mathbf{0}.$$

*Proof.* If $C = (V_C, E_C)$ be a cycle of the oriented graph $\Gamma$, having orientation $\sigma$. Let $V_C = \{u_1, u_2, \dots, u_r\}$ denote a fixed sequence of vertices of this cycle obtained by traversing the cycle in some manner. Fix a $u_i \in V_C$. Let $e_1 = (u_{i-1}, u_i)$, $e_2 = (u_i, u_{i+1})$ be the edges in $C$ incident to $u_i$.

Let $D_i$ denote the $i$th row of $D$. There are four possibilities:

- $u_i = h(e_1)$ and $u_i = h(e_2)$: In this case, the entry of $D$ corresponding to the vertex $u_i$ and the edge $e_1$ is 1 and the entry of $D$ corresponding to the vertex $u_i$ and the edge $e_2$ is also 1. The entry of the characteristic vector $\vec{C}$ associated to $e_1$ is 1 and the entry associated to $e_2$ is $-1$. When we dot $D_i$ with the characteristic vector $\vec{C}$, the only non-zero terms will be these terms. Therefore, we find that

$$D_i \cdot \vec{C} = (1, 1) \cdot (1, -1) = 0.$$

- $u_i = h(e_1)$ and $u_i = t(e_2)$: In this case, the entry of $D$ corresponding to the vertex $u_i$ and the edge $e_1$ is 1 and the entry of $D$ corresponding to the vertex $u_i$ and the edge $e_2$ is $-1$. The entry of the characteristic vector $\vec{C}$ associated to $e_1$ is 1 and the entry associated to $e_2$ is also 1. When we dot $D_i$ with the characteristic vector $\vec{C}$, the only non-zero terms will be these terms. Therefore, we find that

$$D_i \cdot \vec{C} = (1, -1) \cdot (1, 1) = 0.$$

- $u_i = t(e_1)$ and $u_i = h(e_2)$: In this case, the entry of $D$ corresponding to the vertex $u_i$ and the edge $e_1$ is $-1$ and the entry of $D$ corresponding to the vertex $u_i$ and the edge $e_2$ is 1. The entry of the characteristic vector $\vec{C}$ associated to $e_1$ is $-1$ and the entry associated to $e_2$ is also $-1$. When we dot $D_i$ with the characteristic vector $\vec{C}$, the only non-zero terms will be these terms. Therefore, we find that

$$D_i \cdot \vec{C} = (-1, 1) \cdot (-1, -1) = 0.$$

- $u_i = t(e_1)$ and $u_i = t(e_2)$: In this case, the entry of $D$ corresponding to the vertex $u_i$ and the edge $e_1$ is $-1$ and the entry of $D$ corresponding to the vertex $u_i$ and the edge $e_2$ is also $-1$. The entry of the characteristic vector $\vec{C}$ associated to $e_1$ is $-1$ and the entry associated to $e_2$ is 1. When we dot $D_i$ with the characteristic vector $\vec{C}$, the only non-zero terms will be these terms. Therefore, we find that

$$D_i \cdot \vec{C} = (-1, -1) \cdot (-1, 1) = 0.$$

This proves $D\vec{C} = \mathbf{0}$ holds in each case.
□

**Proposition 110.** *The set of signed characteristic vectors of the collection fundamental cycles*

$$\{cyc(e, T) \mid e \in E - E_T\},$$

*defined as in way, is a set of linearly independent vectors in $\mathcal{Z}$. This is a basis of the cycle space $\mathcal{Z}$.*

*Proof.* Note that, by Proposition 109, the span of the vectors

$$\{cyc(\vec{e}, T) \mid e \in E - E_T\},$$

is contained in $\mathcal{Z}$. The first statement now follows from the proof of Proposition 103. For the second statement, recall that $\Gamma$ is assumed to be connected. Therefore, the rank of $D$ is $n - 1$. By the rank plus nullity theorem, this implies that the kernel of $D$, namely $\mathcal{Z}$, has dimension $m-n+1$. But this is the size of the set of independent vectors above, since $|E-E_T| = |E|-|E_T| = m - (n - 1)$. $\Box$

Let $\Gamma = (V, E)$, $V = \{1, 2, \ldots, n\}$, be a graph with orientation $\sigma$. From Lemma 107 and Proposition 109, it follows that the signed characteristic vectors of the fundamental cocyles are orthogonal to the signed characteristic vectors of the fundamental cycles.

The following result is a very pretty fact that relates cocycles and cycles in a non-trivial way. It's a standard result in algebraic graph theory textbooks. (See for example Theorem 8.3.1 in Godsil and Royle [GR04] or Proposition 4.7 in Biggs [Bi93].)

**Lemma 111.** Assume $\Gamma$ is connected. Under the standard inner product on $C^1(\Gamma, F)$, the cycle space is orthogonal to the cocycle space.

*Proof.* Let $n = |V|$ and $m = |E|$. Since the signed characteristic vectors of the fundamental cocyles are orthogonal to the signed characteristic vectors of the fundamental cycles, we have

$$\mathcal{Z}^*(\Gamma) \subset \mathcal{Z}(\Gamma)^{\perp}.$$

Therefore,

$$\dim \mathcal{Z}^*(\Gamma) \leq m - \dim \mathcal{Z}(\Gamma) = m - (m - n + 1) = n - 1.$$

Since $\Gamma$ os connected, there are exactly $n - 1$ signed characteristic vectors of the fundamental cocycles, and they span $\mathcal{Z}^*(\Gamma)$. Therefore, we have equality, as claimed.

$\Box$

In summary: if $\Gamma$ is connected,

- the kernel of $D$ is dimension $m - n + 1$ and the rank of $D$ is dimension $n - 1$,

- the cycle space $\mathcal{Z}(\Gamma)$ of an oriented graph $\sigma$ is
  - (as a subspace of $F^m$) spanned by the signed characteristic vectors of the fundamental cycles of $\Gamma$,
  - (as a subspace of $F^m$) the kernel of the signed incidence matrix, $D_{\Gamma,\sigma}$,
  - (as a subspace of $C^1(\Gamma, F)$) the kernel of the incidence map in (9).
- the cocycle space $\mathcal{Z}^*(\Gamma)$ of an oriented graph $\sigma$ is
  - (as a subspace of $F^m$) spanned by the $n-1$ signed characteristic vectors of the fundamental cocycles of $\Gamma$,
  - (as a subspace of $F^m$) $\mathcal{Z}^*(\Gamma)$ is orthogonal to $\mathcal{Z}(\Gamma)$ with respect to the Euclidean inner product,
  - (as a subspace of $C^1(\Gamma, F)$) $\mathcal{Z}^*(\Gamma)$ is orthogonal to $\mathcal{Z}(\Gamma)$ with respect to the inner product in (8) with $X = E$.

**Exercise 5.6.** *1. Compute the cycle space $Z$ of the complete graph $\Gamma$, depicted in Figure 22, by finding a basis of signed characteristic vectors of cycles.*

*2. Find the cocycle space $Z^*$ of the graph $\Gamma$, depicted in Figure 22, by finding a basis of signed characteristic vectors of cocycles.*

*3. Verify that $Z^* = Z^\perp$ in $\mathbb{R}^6$.*

*4. Verify that $Z = ker(D)$, where $D$ is the signed incidence matrix of $\Gamma$, with default orientation and default edge labeling.*

# 6 Laplacian matrix

Let's start with some motivation for the definition of the Laplacian.

If $\Gamma = (V, E)$ is a graph and $F$ is a ring, let $C^0(\Gamma, F)$ denote the $F$-module of all $F$-valued functions on the vertex set $V$ of $\Gamma$, and $C^1(\Gamma, F)$ is the $F$-module of all $F$-valued functions on the edge set $E$ of $\Gamma$. Let $S$ be a set and let

$$\delta_S : S \times S \to \{0, 1\},$$

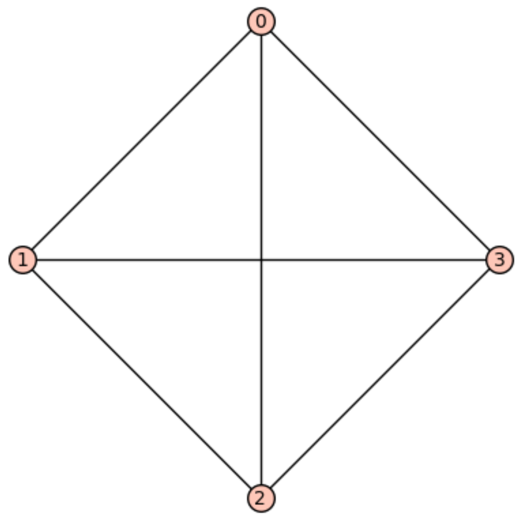


Figure 22: The complete graph $\Gamma = K_4$, with default orientation and default edge labeling.

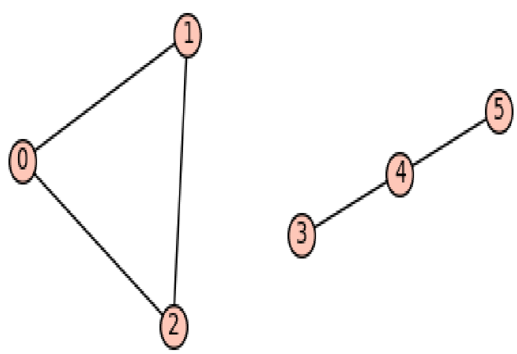


Figure 23: The disconnected graph $\Gamma$, with default orientation and default edge labeling.

denote the Kronecker delta function on $S$:

$$\delta_S(s,t) = \begin{cases} 1, & \text{if } s = t, \\ 0, & \text{if } s = t. \end{cases}$$

The set $\mathcal{B}_V = \{\delta_V(u,*) \mid u \in V\}$ forms a basis for $C^0(\Gamma, F)$, and the set $\mathcal{B}_E = \{\delta_E(e,*) \mid e \in E\}$ forms a basis for $C^1(\Gamma, F)$. We call these the *standard bases* for these $F$-modules.

Order the edges

$$E = \{e_1, e_2, \ldots, e_m\},$$

of $\Gamma$ in some arbitrary but fixed way. The *vector representation* (or *characteristic vector*) of a subgraph $\Gamma' = (V, E')$ of $\Gamma = (V, E)$, $E' \subset E$, is the binary $m$-tuple

$$\vec{\Gamma'} = (a_1, a_2, \ldots, a_m) \in \{0,1\}^m,$$

where

$$a_i = a_i(E') = \begin{cases} 1, & \text{if } e_i \in E', \\ 0, & \text{if } e_i \notin E'. \end{cases}$$

Any such vector representation is associated to a unique function in $C^1(\Gamma, F)$ in the obvious way.

If the graph $\Gamma$ is a large square lattice grid, and if $f \in C^0(\Gamma, F)$, then the usual definition of the Laplacian,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

corresponds to the discrete *Laplacian*[23] $Q$ on $f$

$$(Qf)(v) = \sum_{w:d(w,v)=1} [f(w) - f(v)] \tag{12}$$

where $\delta(w,v) = \delta_\Gamma(w,v)$ is the graph distance function (for $v, w \in V$). Indeed,

$$\frac{\partial^2 f}{\partial x^2} = \lim_{\epsilon \to 0} \frac{[f(x+\epsilon, y) - f(x,y)] + [f(x-\epsilon, y) - f(x,y)]}{\epsilon^2}.$$

and

$$\frac{\partial^2 f}{\partial y^2} = \lim_{\epsilon \to 0} \frac{[f(x, y+\epsilon) - f(x,y)] + [f(x, y-\epsilon) - f(x,y)]}{\epsilon^2}.$$

[23] This is called the vertex Laplacian, denoted $\Delta_0$, in the context of simplicial complexes. See §7.1 below.

so taking $\epsilon = 1$ given the desired discrete analog of $f_{xx} + f_{yy}$ on the grid graph. This operator only depends on "local" properties. That is, $(Qf)(v)$ depends only on the neighbors of $v$ in the graph $\Gamma$. It may come as a surprise to find out that $Q$ governs a number of "global properties" of $\Gamma$ as well, such as connectivity. We shall see these, and other fascinating properties of $Q$ below.

Let us label the vertices

$$V = \{0, 1, \ldots, n-1\},$$

and the edges $E = \{e_1, \ldots, e_m\}$. We sometimes write each edge $e_i$ connecting vertex $j$ with vertex $k$ as a pair $e_i = (j, k)$ if $j < k$.

In this case, we have bijections

$$\rho_0 : C^0(\Gamma, F) \to F^n, f \mapsto (f(0), f(1), \ldots, f(n-1)).$$

and

$$\rho_1 : C^1(\Gamma, F) \to F^m, g \mapsto (g(e_1), g(e_2), \ldots, g(e_m)).$$

Recall the (undirected, unweighted) adjacency matrix of $\Gamma$ is the $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = 1$ if vertex $i$ shares an edge with vertex $j$, and $a_{ij} = 0$ otherwise.

Given an orientation on $\Gamma$, there is a linear transformation

$$D : C^1(\Gamma, F) \to C^0(\Gamma, F)$$

given by

$$(Df)(v) = \sum_{h(e)=v} f(e) - \sum_{t(e)=v} f(e) \tag{13}$$

and a dual linear transformation

$$D^* : C^0(\Gamma, F) \to C^1(\Gamma, F).$$

The matrix of $D$ with respect to the standard basis is the (signed) incidence matrix.

**Example 112.** For the house graph $\Gamma = (V, E)$, we have $V = \{0, 1, 2, 3, 4\}$ and $E = \{(0, 1), (0, 2), (1, 3), (2, 3), (2, 4), (3, 4)\}$. A function $f : V \to F$ is simply an assignment of an element in $F$ to each vertex in $V$. We may regard such an assignment as a vector $(f(0), f(1), f(2), f(3), f(4)) \in F^5$. Similarly, a function $g : E \to F$ is simply an assignment of an element in $F$ to each edge in $E$. We may regard such an assignment as a vector $(g((0,1)), g((0,2)), \ldots, g((3,4))) \in F^6$.

Sagemath

```
sage: Gamma = graphs.HouseGraph()
sage: Gamma.vertices()
[0, 1, 2, 3, 4]
sage: Gamma.edges(labels=None)
[(0, 1), (0, 2), (1, 3), (2, 3), (2, 4), (3, 4)]
sage: Gamma.adjacency_matrix()
[0 1 1 0 0]
[1 0 0 1 0]
[1 0 0 1 1]
[0 1 1 0 1]
[0 0 1 1 0]
sage: load("/home/wdj/sagefiles/alg-graph-thry2.sage")
sage: e0 = [1]*6
sage: incidence_matrix(Gamma,e0)
[ 1  1  0  0  0  0]
[-1  0  1  0  0  0]
[ 0 -1  0  1  1  0]
[ 0  0 -1 -1  0  1]
[ 0  0  0  0 -1 -1]
```

The adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

and the incidence matrix (where all the edges listed are oriented lexicographically) is

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

There are natural bases for the spaces $C^0(\Gamma, F) = F[V]$ and $C^1(\Gamma, F) = F[E]$. We endow each of these vector spaces with an inner product that makes these bases orthonormal. Once we fix orderings of the vertices and edges of $\Gamma$ as above, the linear transformation $D : C^1(\Gamma, F) \to C^0(\Gamma, F)$ can be represented with respect to these bases by the incidence matrix, which we also denote by $B$. The matrix representation of the dual $D^* : C^0(\Gamma, F) \to C^1(\Gamma, F)$ of this linear transformation is given by the transpose of the corresponding incidence matrix.

The *vertex Laplacian* (or simply "the Laplacian") is the linear transformation $Q = Q_\Gamma : C^0(\Gamma, F) \to C^0(\Gamma, F)$ defined by

$$Q = DD^*, \tag{14}$$

where $D$ is the linear transformation of Equation (13) above, and $D^*$ is its dual.

This is the analog of Lemma 54.

**Lemma 113.** *If $\Gamma = (V, E)$ is disconnected having two connected components, $\Gamma_1$ and $\Gamma_2$ say, then $V$ can be indexed in such a way that $Q$ is a block diagonal matrix of the form*

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix},$$

*where $Q_i$ is the Laplacian matrix of $\Gamma_i$ ($i = 1, 2$). A similar decomposition holds in the case where $\Gamma$ has $k$ connected components.*

*Proof.* Exercise. □

## 6.1 The Laplacian spectrum

Since $Q$ is symmetric, the eigenvalues of $Q$ are real. The $V$ eigenvalues,

$$\lambda_0 \le \lambda_1 \le \cdots \le \lambda_{|V|-1},$$

is called the *(Laplacian) spectrum* of $\Gamma$.

**Lemma 114.** *If $Q$ is the Laplacian matrix of a an oriented graph $\Gamma$ with signed incidence matrix $D$ then*

$$ker(Q) = ker(D^T).$$

*Proof.* The containment

$$ker(D^T) \subset ker(Q)$$

follows from the definition $Q = DD^T$. Pick $v \in ker(Q)$, so $DD^T v = 0$, so $\langle DD^T v, v\rangle = 0$, where $\langle\ ,\ \rangle$ is the Euclidean inner product. We therefore have

$$0 = \langle DD^T v, v\rangle = \langle D^T v, D^T v\rangle = ||D^T v||^2,$$

so $D^T v = 0$. This implies $v \in ker(D^T)$. We have proven

$$ker(Q) \subset ker(D^T),$$

proving the desired equality. □

It follows now from the previous section, if $\Gamma$ is connected, the rank of $D^T$ is dimension $n-1$, so the kernel has dimension 1. Therefore, by the above lemma, if $\Gamma$ is connected, the kernel of $Q$ is spanned by the all 1's vector:

$$ker(Q) = Span\{(1, 1, \ldots, 1)\}.$$

**Example 115.** Let $n > 1$ be an integer. The *zero-divisor graph* of $n$, denoted $Z_n$, has as its vertices the set

$$V = \{k \mid 1 \leq k \leq n,\ gcd(k, n) > 1\}.$$

We say $(a, b) \in V \times V$ forms an edge of $Z_n$ if the product is 0, $ab \equiv 0 \pmod{n}$.

Consider the graph $\Gamma = Z_{10}$ depicted in Figure 24, whose edges are oriented in such a way that each of the edges

$$(2, 6), (3, 4), (3, 8), (4, 6), (4, 9), (6, 8), (6, 10), (8, 9)$$

has a positive orientation. The associated adjacency matrix is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

and the associated incidence matrix is

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The vertex Laplacian matrix is

$$Q = B \cdot {}^tB = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Its spectrum,

$$2.6131\ldots, 1.0823\ldots, 0, 0, 0, -1.0823\ldots, -2.6131\ldots,$$

is symmetric about 0.

For $n \leq 35$ (in the cases when the graph is non-trivial), the spectrum is symmetric about 0 in the cases

$$n = 6, 8, 9, 10, 12, 14, 15, 20, 21, 22, 26, 28, 33, 35,$$

and not symmetric about 0 in the cases

$$n = 16, 18, 25, 27, 30, 32.$$

Question: Is there a simple characterization of those $n$ for which the spectrum of $\Gamma = Z_n$ is symmetric about 0?

It is known that $\Gamma$ is a bipartite graph if and only if it's spectrum is symmetric to 0 (see Lemma 62). Therefore, we ask if there a simple characterization of those $n$ for which $Z_n$ is bipartite. It is known that if $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, where $\ell, p$ are distinct primes, then $\Gamma = Z_n$ is bipartite (see for example, [AL99]). However, this only covers some of the cases.

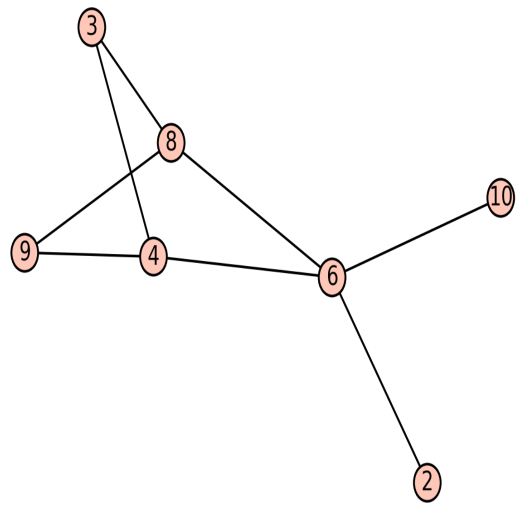


Figure 24: The graph $Z_{10}$.

There is a simple connection between the Laplacian and the adjacency matrix.

**Lemma 116.** *For an oriented graph $\Gamma$ with unsigned adjacency matrix $A$, there is a natural basis of $C^0(\Gamma, F)$ for which the matrix representation of the Laplacian is given by*

$$Q = \Delta - A,$$

*where $\Delta$ denotes the diagonal matrix of the degrees of the vertices of $V$:*

$$\Delta = \begin{pmatrix} d_0 & 0 & 0 & \dots & 0 \\ 0 & d_1 & 0 & \dots & 0 \\ 0 & 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-1} \end{pmatrix},$$

*where $d_i = \deg_\Gamma(i)$, for $i \in V$.*

*Proof.* For a proof, see Lemma 99. □

**Example 117.** Consider the $3 \times 3$ grid graph in Figure 25.

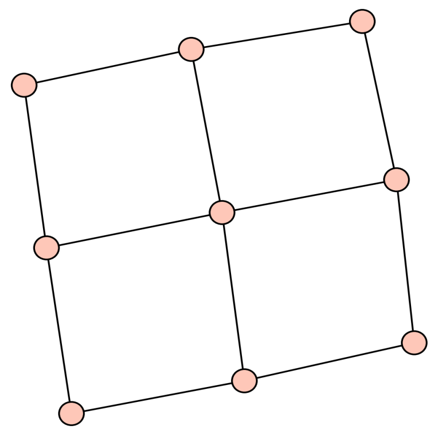

Figure 25: The $3 \times 3$ grid graph

Sagemath can be used to calculate the Laplacian matrix of this graph[24].

Sagemath

```
sage: Gamma = graphs.GridGraph([3,3])
   ## this is the 3x3 grid graph with 9 vertices
sage: B = incidence_matrix(Gamma,len(Gamma.edges())*[1])
sage: B
[ 1  1  0  0  0  0  0  0  0  0  0  0]
[-1  0  1  1  0  0  0  0  0  0  0  0]
[ 0  0 -1  0  1  0  0  0  0  0  0  0]
[ 0 -1  0  0  0  1  1  0  0  0  0  0]
[ 0  0  0 -1  0 -1  0  1  1  0  0  0]
[ 0  0  0  0 -1  0  0 -1  0  1  0  0]
[ 0  0  0  0  0  0 -1  0  0  0  1  0]
[ 0  0  0  0  0  0  0  0 -1  0 -1  1]
[ 0  0  0  0  0  0  0  0  0 -1  0 -1]
sage: B*transpose(B) == Gamma.laplacian_matrix()
True
sage: Gamma.laplacian_matrix()
[ 2 -1  0 -1  0  0  0  0  0]
[-1  3 -1  0 -1  0  0  0  0]
[ 0 -1  2  0  0 -1  0  0  0]
[-1  0  0  3 -1  0 -1  0  0]
[ 0 -1  0 -1  4 -1  0 -1  0]
[ 0  0 -1  0 -1  3  0  0 -1]
[ 0  0  0 -1  0  0  2 -1  0]
[ 0  0  0  0 -1  0 -1  3 -1]
[ 0  0  0  0  0 -1  0 -1  2]
```

The "4" in the center of the Laplacian matrix illustrates the fact that there

[24]The command `incidence_matrix(Gamma,e0)` requires an additional Python function.

are four edges emanating from the central vertex of the $3 \times 3$ grid graph in Figure 25.

The following proposition describes the kernel of the Laplacian matrix of a connected graph.

**Proposition 118.** *If $\Gamma$ is a connected graph, the kernel of the Laplacian matrix $Q$ consists of all multiples of the all 1's vector $\mathbf{1} = (1, 1, \ldots, 1)$, i.e., $\mathbf{1}$ is an eigenvector of $Q$ corresponding to the eigenvalue 0, and the eigenspace of 0 is 1-dimensional.*

*Proof.* Each row of $B^t$ contains 1 once and $-1$ once and all other entries of the row are 0. Thus $B^t\mathbf{1} = \mathbf{0}$, the zero vector. Furthermore, if $x$ is a vector in the kernel of $BB^t$, then $x^tBB^tx = \mathbf{0}$, so $B^tx = \mathbf{0}$. But if $x$ is in the kernel of $B^t$, then $x$ takes the same value on the head and tail vertices of each edge. Since $\Gamma$ is assumed to be connected, $x$ must take the same value on all vertices of $\Gamma$. $\square$

**Corollary 119.** *If $\Gamma$ is a connected graph, the rank of the Laplacian matrix $Q$ is $n - 1$, where $n$ is the number of vertices of $\Gamma$.*

**Exercise 6.1.** *If $Q$ denotes the Laplacian of a $k$-regular graph $\Gamma$, prove*

$$\det(xI - Q) = \chi_\Gamma(k - x).$$

**Exercise 6.2.** *If $Q$ denotes the Laplacian of the graph $\Gamma$, depicted in Figure 23:*

*(a) Find the eigenvalues and eigenvectors of $Q$.*

*(b) Find $ker(D^T)$, where $D$ denotes the signed incidence matrix of $\Gamma$, with default orientation and default edge labeling.*

# 7 Hodge decomposition for graphs

We follow the excellent presentation in Lim [Li15].

Consider a simple graph, $\Gamma = (V, E)$ having $m$ vertices and $n$ edges. Let $C_0 = \mathbb{R}[V]$, $C_1 = \mathbb{R}[E]$, and $C_2 = \mathbb{R}[T]$, where $T$ denotes the set of triangles of $\Gamma$. If $[i, j] \in E$ and $[i, j, k] \in T$, for $i, j, k \in V$, then define $\partial_1([i, j]) = [j] - [i]$ and $\partial_2([i, j, k]) = [j, k] - [i, k] + [i, j]$. These define maps $\partial_2 = \mathsf{curl}^* : C_2 \to C_1$ and $\partial_1 = \mathsf{grad}^* : C_1 \to C_0$ , which we may regard as the (analog of the ) dual of the curl and the gradient. For graphs[25], the 2-*dimensional combinatorial Laplacian*[26] is given by

$$\Delta = \partial_2 \partial_2^* + \partial_1^* \partial_1 : C_1 \to C_1,$$

where the $^*$ denotes the adjoint (or transpose). The *up combinatorial Laplacian* is given by

$$\Delta^{up} = \partial_2 \partial_2^*,$$

and the *down combinatorial Laplacian* is given by

$$\Delta^{down} = \partial_1^* \partial_1.$$

The *vertex Laplacian* is given by

$$\Delta_0 = \partial_1 \partial_1^* : C_0 \to C_0.$$

The *Hodge decomposition* is given by

$$im(\partial_2) \oplus ker(\partial_2^*) = ker(\partial_1) \oplus im(\partial_1^*)$$

$$= im(\partial_2) \oplus ker(\Delta) \oplus im(\partial_1^*) = C_1 = \mathbb{R}^n. \tag{15}$$

(For a proof, see Friedman [F97], Proposition 2.1, or §4.1 in Goldberg [Go02].)

## 7.1 Abstract simplicial complexes

One generalization of a graph is an abstract simplicial complex, which in some sense may be regarded as a higher-dimensional analog of a graph.

We start with an example.

**Example 120.** Consider the diamond graph $\Gamma = (V, E)$ in Figure 26.

Let $I_k$ denote the $k$-cliques[27] of $\Gamma$. From Definition 16, recall a clique of $\Gamma$ is a complete subgraph[28].

$$\mathcal{I}_0 = \{[0], [1], [2], [3]\} = V,$$

[25] The general definition for a simplicial complex is given in (28) below.

[26] Lim et al sometimes calls this a *Hodge Laplacian*.

[27] These are the $k-1$-faces, or faces of dimension $k-1$, in the associated clique complex.

[28] Since the subgraph obtained by removing a vertex from a clique is still a clique, the clique complex satisfies (I2) below.

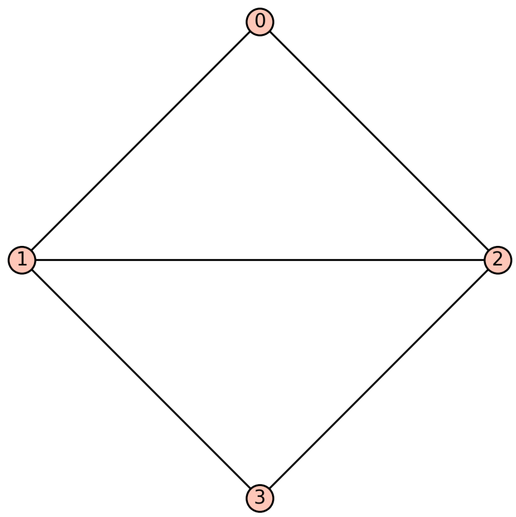


Figure 26: The diamond graph.

$$\mathcal{I}_1 = \{[0,1],[0,2],[1,2],[1,3],[2,3]\} = E,$$
$$\mathcal{I}_2 = \{[0,1,2],[1,2,3]\},$$

so $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1 \cup \mathcal{I}_2$ forms an abstract simplicial complex (whose definition is recalled in Definition 121 below).

Sagemath

```
sage: Gamma = graphs.DiamondGraph()
sage: S = Gamma.clique_complex()
sage: S
Simplicial complex with vertex set (0, 1, 2, 3) and facets {(0, 1, 2), (1, 2, 3)}
```

The following definition suggests that each of these sets $\mathcal{I}$ has some interesting extra structure.

**Definition 121.** Let $E$ be a finite set. Consider a set of subsets, $\mathcal{I} \subset 2^E$, satisfying

I1: $\emptyset \in \mathcal{I}$,

I2: $I_1 \in \mathcal{I}$ and $I_2 \subset I_1$ implies $I_2 \in \mathcal{I}$.

In this case, we call $\mathcal{I}$ an *abstract simplicial complex* on $E$.

The elements of $\mathcal{I}$ are *simplicies* or faces, and the *facets* are the maximal (w.r.t. inclusion) faces.

The *dimension* of a simplex is one less than its cardinality: $\dim(I) = |I| - 1$. If $I \in \mathcal{I}$ is a simplex with dimension $d$ then we call $I$ a $d$-simplex or a $d$-face. The *dimension* of the complex, $\dim(\mathcal{I})$, is defined as the largest dimension of any of its simplicies.

In general, define the spaces of $k$-*chains* ($k = 0, 1, 2, \dots$) by

$$C_0(\mathcal{I}) = \mathsf{Span}_{\mathbb{R}}(I_0), \quad C_1(\mathcal{I}) = \mathsf{Span}_{\mathbb{R}}(I_1), \quad C_2(\mathcal{I}) = \mathsf{Span}_{\mathbb{R}}(I_2), \ \dots .$$

Give $C_k$ the inner product such that the basis $I_k$ is orthonormal (o.n.). In other words, if $I_k = \{F_1, F_2, \dots, F_m\}$ (where $F_i$ is a $k$-simplex of $\mathcal{I}$) then we identify $F_1$ with $e_1 \in \mathbb{R}^m$, $F_2$ with $e_2 \in \mathbb{R}^m$, and so on (where $e_i$ is the $i$th standard basis vector of $\mathbb{R}^m$).

**Example 122.** Let $n > 1$ be an integer. The *divisor simplicial complex* of $n$, denoted $D_n$, has as its vertices the set of positive integers less than or equal to $n$. If $S \subset \{1, 2, \dots, n\}$ consists of $k + 1$ integers, we form a $k$-simplex from $S$ if every element of $S$ except the maximal element, $\max(S)$, divides the next largest element.

This is a generalization of the well-known divisor graph.

When $n = 6$, the vertices (or 0-simplices) are

$$I_0 = \{1, 2, 3, 4, 5, 6\},$$

the 1-faces are

$$I_1 = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 4), (2, 6), (3, 6)\},$$

and the 2-faces are

$$I_2 = \{(1, 2, 4), (1, 2, 6), (1, 3, 6)\}.$$

**Example 123.** *In the notation of Example 120, the spaces of $k$-chains ($k = 0, 1, 2$) are given by*

$$C_0 = \mathsf{Span}_{\mathbb{R}}(\mathcal{I}_0) \cong \mathbb{R}^4, \quad C_1 = \mathsf{Span}_{\mathbb{R}}(\mathcal{I}_1) \cong \mathbb{R}^5, \quad C_2 = \mathsf{Span}_{\mathbb{R}}(\mathcal{I}_2) \cong \mathbb{R}^2.$$

Let $X$ denote an $\ell$-dimensional abstract simplicial complex, and fix an ordering of $\mathcal{I}_0$ and hence a lexicographical ordering of $\mathcal{I}_k$, $k = 0, 1, 2, \ldots, \ell$. By virtue of this ordering, we may regard $\mathcal{I}_1$ as a subset of $\mathcal{I}_0^2$. Let

$$\phi : \mathcal{I}_0^2 \to \mathbb{R}$$

be a skew-symmetric function, $\phi(x, y) = -\phi(y, x)$, such that $\phi(x, y) = 0$ if $(x, y) \notin \mathcal{I}_1$. The set of all such skew-symmetric functions forms an inner product space $L_1$ over $\mathbb{R}$ with inner product

$$\langle \phi_1, \phi_2 \rangle = \sum_{x,y \in \mathcal{I}_0,\, x<y} \phi_1(x, y)\phi_2(x, y).$$

An example of such a function is given as follows. Let $f : \mathcal{I}_0 \to \mathbb{R}$ be arbitrary and define

$$\mathsf{grad}\ f(x, y) = f(y) - f(x),$$

if $(x, y) \in I_1$, and *grad* $f(x, y) = 0$ otherwise. This is called the *gradient* function on $X$.

We define the *divergence* of $\phi \in L_1$ by

$$\mathsf{div}\, \phi(x) = \sum_{y \in \mathcal{I}_0} \phi(x, y).$$

**Exercise 7.1.** *Prove the vertex Laplacian satisfies*

$$\Delta_0 = \mathsf{div} \circ \mathsf{grad}\ .$$

For each face, $F \in \mathcal{I}_2$, we may regard $F$ as being in $\mathcal{I}_0^3$, say $F = F_{x,y,z} = (x, y, z)$, for $x, y, z \in \mathcal{I}_0$ (wlog, we assume $x < y < z$). If $F = F(x, y, z)$, for $x, y, z \in E$, we define the *curl* of $\phi \in L_1$ by

$$\begin{aligned} \mathsf{curl}_X \phi(F) = \mathsf{curl}_X \phi(x, y, z) &= \phi(x, y) + \phi(y, z) + \phi(z, x) \\ &= \phi(x, y) + \phi(y, z) - \phi(x, z). \end{aligned}$$

**Example 124.** Consider again the diamond graph (in Examples 120 and 123), $\Gamma = (V, E)$ in Figure 26. Let $X$ denote the clique complex of $\Gamma$.

Let $\partial_2$ denote the (adjoint of the $\mathsf{curl}$ ) boundary operator (see (17) below):

$$\partial_2[0,1,2] = [1,2] - [0,2] + [0,1],$$
$$\partial_2[1,2,3] = [2,3] - [1,3] + [1,2].$$

With respect to the above o.n. bases, the matrix of $\partial_2 : C_2 \to C_1$ is the $5 \times 2$ matrix

$$\partial_2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

Note that

$$\partial_2 \partial_2^* = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 2 & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

Let $\partial_1$ denote the (dual of the $\mathsf{grad}$ ) boundary operator:

$$\partial_1[0,1] = [1] - [0],$$
$$\partial_1[0,2] = [2] - [0],$$

and so on. With respect to the above o.n. bases, the matrix of $\partial_1 : C_1 \to C_0$ is the $4 \times 5$ matrix

$$\partial_1 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad (16)$$

Note that

$$\partial_1^*\partial_1 = \begin{pmatrix} 2 & 1 & -1 & -1 & 0 \\ 1 & 2 & 1 & 0 & -1 \\ -1 & 1 & 2 & 1 & -1 \\ -1 & 0 & 1 & 2 & 1 \\ 0 & -1 & -1 & 1 & 2 \end{pmatrix},$$

so the 1-dimensional combiatorial Laplacian of $X$ is

$$\Delta = \partial_1^*\partial_1 + \partial_2\partial_2^* == \begin{pmatrix} 3 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \\ 0 & 0 & 4 & 0 & 0 \\ -1 & 0 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 & 3 \end{pmatrix}.$$

This matrix has determinant 256, so it non-singular. The Hodge decomposition in this case says

$$C_1(X) = im(\partial_2) \oplus ker(\Delta) \oplus im(\partial_1^*)$$

The vertex Laplacian is

$$\Delta_0 = \partial_1\partial_1^* = -\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}.$$

In this case, $ker(\Delta) = \{0\}$,

$$ker(\partial_1) = im(\partial_2) = Span[(1,-1,0,1,-1),(0,0,1,-1,1)],$$

and

$$ker(\partial_2^*) = im(\partial_1^*) = Span[(1,0,-1,0,1),(0,1,1,0,-1),(0,0,0,1,1)],$$

so the Hodge decomposition can be easily verified.

Sagemath

```
sage: L2 = [[1,0],[-1,0],[1,1],[0,-1],[0,1]]
sage: D2 = matrix(ZZ,L2)
sage: L1 = [[1,1,0,0,0],[-1,0,1,1,0],[0,-1,-1,0,1],[0,0,0,-1,-1]]
sage: D1 = matrix(ZZ,L1)
sage: (D2.transpose()).right_kernel()
Free module of degree 5 and rank 3 over Integer Ring
Echelon basis matrix:
```

```
[ 1  0 -1  0  1]
[ 0  1  1  0 -1]
[ 0  0  0  1  1]
sage: D1.right_kernel()
Free module of degree 5 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1 -1  0  1 -1]
[ 0  0  1 -1  1]
sage: (D2.transpose()).image()  ### ``right_image'' of D3
Free module of degree 5 and rank 2 over Integer Ring
Echelon basis matrix:
[ 1 -1  0  1 -1]
[ 0  0  1 -1  1]
sage: D1.image()          ### ``right_image'' of D2^t
Free module of degree 5 and rank 3 over Integer Ring
Echelon basis matrix:
[ 1  0 -1  0  1]
[ 0  1  1  0 -1]
[ 0  0  0  1  1]
```

## 7.2 The Björner complex and the Riemann hypothesis

This example comes from Björner [Bj11].

Let $n > 1$ be an integer and let $\Delta_n$ denote the simplicial complex of squarefree[29] integers less that or equal to $n$, ordered by divisibility. In other words, the 0-faces are

$$I_0 = \{m \mid 1 \leq m \leq n,\ m \text{ square} - \text{free}\},$$

the 1-faces are

$$I_1 = \{(a, b) \mid a, b \in I_0, a|b\},$$

and the 2-faces are

$$I_2 = \{(a, b, c) \mid a, b, c \in I_0, a|b \text{ and } b|c\}.$$

Recall, the facets are the maximal faces.

In the case $n = 10$, the vertices (or 0-faces) are

$$I_0 = \{1, 2, 3, 5, 6, 7, 10\},$$

the 1-faces are

[29] A number is said to be *squarefree* if its prime decomposition contains no repeated factors. The number 1 is, by convention, squarefree.

$$I_1 = \{(1,2),(1,3),(1,5),(1,6),(1,7),(1,10),(2,6),(2,10),(3,6),(5,10)\},$$

the 2-faces are

$$I_2 = \{(1,2,6),(1,2,10),(1,3,6),(1,5,10)\},$$

and the facets are

$$I_2 \cup \{(1,7)\}.$$

Therefore, this simplicial complex is not pure.

Define the spaces of $k$-chains ($k = 0,1,2$) by

$$C_0 = \mathsf{Span}\,(\mathcal{I}_0) \cong \mathbb{R}^7, \quad C_1 = \mathsf{Span}\,(\mathcal{I}_1) \cong \mathbb{R}^9, \quad C_2 = \mathsf{Span}\,(\mathcal{I}_2) \cong \mathbb{R}^3.$$

Give $C_k$ the inner product such that the basis is orthonormal (o.n.).

Let $\partial_2 : C_2 \to C_1$ denote the (dual of the $\mathsf{curl}$) boundary operator:

$$\partial_2[1,2,6] = [2,6] - [1,6] + [1,2],$$

and so on. The matrix representation of $\partial_2$ is

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\partial_1 : C_1 \to C_0$ denote the boundary operator:

$$\partial_1[1,2] = [2] - [1],$$

and so on. The matrix representation of $\partial_1$ is

$$\partial_1 = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The kernel of $\partial_1$ is

$$ker(\partial_1) = \mathsf{Span}_{\mathbb{Z}}[(1,0,0,0,0,-1,0,1,0,0),(0,1,0,0,0,-1,-1,1,1,0),$$
$$(0,0,1,0,0,-1,0,0,0,1),(0,0,0,1,0,-1,-1,1,0,0)].$$

The image of $\partial_2$ is

$$im(\partial_2) = \mathsf{Span}_{\mathbb{Z}}[(1,0,0,0,0,-1,0,1,0,0),(0,1,0,0,0,-1,-1,1,1,0),$$
$$(0,0,1,0,0,-1,0,0,0,1),(0,0,0,1,0,-1,-1,1,0,0)].$$

and the kernel of $\partial_2^*$ is

$$ker(\partial_2^*) = \mathsf{Span}_{\mathbb{Z}}\{(1,0,0,0,0,0,-1,-1,0,0),(0,1,0,0,0,0,0,0,-1,0),$$
$$(0,0,1,0,0,0,0,0,0,-1),(0,0,0,1,0,0,1,0,1,0),$$
$$(0,0,0,0,1,0,0,0,0,0),(0,0,0,0,0,1,0,1,0,1)\}.$$

The 0th homology group (29) is

$$H_0(\Delta_{10},\mathbb{Z}) = ker(\partial_0^*)/im(\partial_1^*) = \mathbb{Z}^4,$$

since $ker(\partial_0^*) = \mathbb{Z}^{10}$, and $im(\partial_1^*) \cong \mathbb{Z}^6$ is a rank 6 submodule of $\mathbb{Z}^{10}$. The 1st homology group (29) is

$$H_1(\Delta_{10},\mathbb{Z}) = ker(\partial_1^*)/im(\partial_2^*) = 0$$

since $ker(\partial_1^*) = im(\partial_2^*) = \mathbb{Z}^4$. The 2nd homology group (29) is

$$H_2(\Delta_{10}, \mathbb{Z}) = ker(\partial_2^*)/im(\partial_3^*) \cong \mathbb{Z}^6,$$

since $\partial_3 = 0$ and $\partial_2^*$ has rank 4.

Note that this is consistent with what Sagemath predicts:

Sagemath

```
sage: S = SimplicialComplex(maximal_faces=[(1, 3, 6), (1, 2, 6),\
    (1, 2, 10), (1, 5,10),(1, 7)])
sage: S
Simplicial complex with vertex set (1, 2, 3, 5, 6, 7, 10) and 5 facets
sage: S.homology(base_ring=ZZ)
{0: 0, 1: 0, 2: 0}
sage: S.n_chains(1)
Free module generated by {(3, 6), (1, 2), (2, 6), (5, 10), (1, 10), (1, 7),
  (1, 6), (2, 10), (1, 3), (1, 5)} over Integer Ring
sage: S.n_chains(2)
Free module generated by {(1, 3, 6), (1, 2, 6), (1, 2, 10),
  (1, 5, 10)} over Integer Ring
sage: S.f_vector()
[1, 7, 10, 4]
sage: S.n_cells(1)
[(3, 6), (1, 2), (2, 6), (5, 10), (1, 10),
 (1, 7), (1, 6), (2, 10), (1, 3), (1, 5)]
sage: S.n_cells(2)
[(1, 3, 6), (1, 2, 6), (1, 2, 10), (1, 5, 10)]
sage: S.h_vector()
[1, 4, -1, 0]
sage: S.is_acyclic()
True
sage: S.is_cohen_macaulay()
False
```

Sagemath

```
sage: V1 = ZZ^7
sage: V2 = image(D1*D1.transpose())
sage: V1/V2
Finitely generated module V/W over Integer Ring with invariants (55, 0)
sage: L2 = [[1, 1, 0, 0], [0, 0, 1, 0],
....:  [0, 0, 0, 1], [-1, 0, -1, 0],
....:  [0, 0, 0, 0], [0, -1, 0, -1],
....:  [1, 0, 0, 0], [0, 1, 0, 0],
....:  [0, 0, 1, 0], [0, 0, 0, 1]]
sage: D2 = matrix(ZZ,L2)
sage: L1 = [[-1,-1,-1,-1,-1,-1,0,0,0,0],\
....: [1,0,0,0,0,0,-1,-1,0,0],\
....: [0,1,0,0,0,0,0,0,-1,0],\
....: [0,0,1,0,0,0,0,0,0,-1],\
....: [0,0,0,1,0,0,1,0,1,0],\
....: [0,0,0,0,1,0,0,0,0,0],\
....: [0,0,0,0,0,1,0,1,0,1]]
sage: D1 = matrix(ZZ,L1)
```

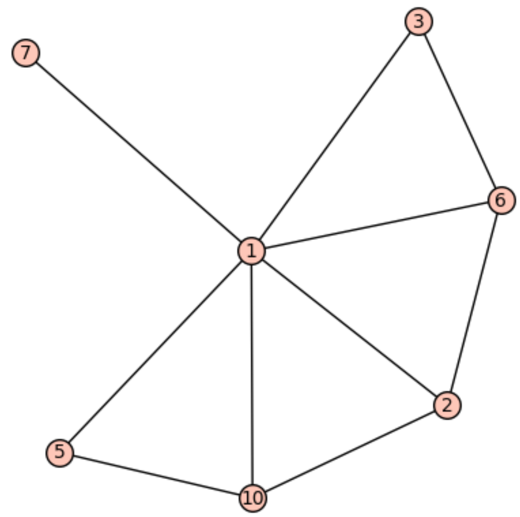


Figure 27: The graph associated with the simplicial complex $\Delta_{10}$.

```
sage: V1 = D1.right_kernel()
sage: V2 = image(D2*D2.transpose())
sage: V1/V2
Finitely generated module V/W over Integer Ring with invariants ()
```

It turns out that a certain conjectural estimate on the Euler characteristic of $\Delta_n$, as $n \to \infty$, is equivalement to the Riemann hypothesis. See Björner [Bj11] and Noguchi [No17] for details.

## 7.3 Homology groups

Let $\mathcal{I}$ denote an abstract simplicial complex.

We define the $d$-th *boundary operator*

$$\partial_d : C_d(\mathcal{I}) \to C_{d-1}(\mathcal{I})$$

by

$$\partial_d([i_0, \ldots, i_d]) = \sum_{k=0}^{d} (-1)^k [i_0, \ldots, \hat{i}_k, \ldots, i_d], \tag{17}$$

where the hat means to omit that vertex from the list. We define $\partial_0 = 0$, by convention.

**Lemma 125.** $\partial_d \circ \partial_{d+1} = 0$.

*Proof.* ... □

The vector space of *d-cochains* is defined to be the space of "alternating" functions on the set of $d$-simplicies which preserve their orientation:

$$C^d = C^d(\mathcal{I}) = \{f : \{I \in \mathcal{I} \mid \dim(I) = d\} \to \mathbb{R} \mid f(\sigma \circ I) = \mathsf{sgn}\,(\sigma) f(I)\}.$$

Since each linear functional in $C_d(\mathcal{I})^*$ is determined by its value on the set of $d$-faces, we may identify it with the space of $d$-cochains. This allows us to identify

$$\partial_d^* = \delta_{d-1}.$$

We define the $d+1$-st *coboundary operator*

$$\delta_d : C^d(\mathcal{I}) \to C^{d+1}(\mathcal{I})$$

by

$$\delta_d f(i_0, \ldots, i_d) = \sum_{k=0}^{d} (-1)^k f(i_0, \ldots, \hat{i}_k, \ldots, i_d), \tag{18}$$

where the hat means to omit that vertex from the list.

**Lemma 126.** $\delta_d \circ \delta_{d-1} = 0$.

*Proof.* For any $f \in C^{k-1}(\mathcal{I})$, we have[30]

[30] The proof below is from a class project by Zoe Green, "Introduction to Hodge Laplacians of Graphs." I thank her for allowing me to use her LaTeX source.

$$(\delta_k \delta_{k-1} f)(i_0, \ldots, i_{k+1}) = \sum_{j=0}^{k+1} (-1)^j \delta_{k-1} f(i_0, \ldots, \hat{i}_j, \ldots, i_{k+1}) \tag{19}$$

$$\textcircled{1} = \sum_{j=0}^{k+1} (-1)^j \Big[ \sum_{\ell=0}^{j-1} (-1)^\ell f(i_0, \ldots, \hat{i}_\ell, \ldots, \hat{i}_j, \ldots, i_{k+1}) \tag{20}$$

$$+ \sum_{\ell=j+1}^{k+1} (-1)^{\ell-1} f(i_0, \ldots, \hat{i}_j, \ldots, \hat{i}_\ell, \ldots, i_{k+1}) \Big] \tag{21}$$

$$= \sum_{j=0}^{k+1} \sum_{j<\ell} (-1)^j (-1)^\ell f(i_0, \ldots, \hat{i}_j, \ldots, \hat{i}_\ell, \ldots, i_{k+1}) \tag{22}$$

$$+ \sum_{j>\ell} (-1)^j (-1)^{\ell-1} f(i_0, \ldots, \hat{i}_\ell, \ldots, \hat{i}_j, \ldots, i_{k+1}) \tag{23}$$

$$\textcircled{2} = \sum_{j=0}^{k+1} [\sum_{j<\ell} (-1)^{j+\ell} f(i_0, \ldots, \hat{i}_j, \ldots, \hat{i}_\ell, \ldots, i_{k+1}) \tag{24}$$

$$+ \sum_{\ell>j} (-1)^{j+\ell-1} f(i_0, \ldots, \hat{i}_j, \ldots, \hat{i}_\ell, \ldots, i_{k+1})] \tag{25}$$

$$= \sum_{j=0}^{k+1} [\sum_{j<\ell} (-1)^{j+\ell} f(i_0, \ldots, \hat{i}_j, \ldots, \hat{i}_\ell, \ldots, i_{k+1}) \tag{26}$$

$$- \sum_{j<\ell} (-1)^{j+\ell} f(i_0, \ldots, \hat{i}_j, \ldots, \hat{i}_\ell, \ldots, i_{k+1})] = 0. \tag{27}$$

The power of $-1$ in the third sum in (1) is $\ell - 1$ because an argument preceeding $\hat{i}_\ell$ is omitted so $\hat{i}_l$ is the $(\ell-1)$th argument (which is also omitted). (2) follows from swapping labels $j$ and $l$ in the second sum.
□

The $d$-dimensional *combinatorial Laplacian* is given by

$$\Delta_d = \partial_{d+1}\delta_d + \delta_{d-1}\partial_d = \partial_{d+1}\partial_{d+1}^* + \partial_d^*\partial_d : C_d \to C_d. \tag{28}$$

The *up combinatorial Laplacian*[31] is given by

$$\Delta_d^{up} = \partial_{d+1}\partial_{d+1}^*,$$

[31]Some call this the *up-down combinatorial Laplacian*.

and the *down combinatorial Laplacian*[32] is given by

$$\Delta_d^{down} = \partial_d^* \partial_d.$$

Define the homology of $\mathcal{I}$ by

$$H_r(\mathcal{I}, \mathbb{Z}) = ker(\partial_r^*)/im(\partial_{r+1}^*), \tag{29}$$

for $1 \leq r \leq \dim(\mathcal{I}) - 1$.

**Example 127.** For the diamond graph (in Examples 120 and 123), the 0th homology group (29) is

$$H_0(M, \mathbb{Z}) = ker(\partial_0^*)/im(\partial_1^*) = \mathbb{Z}^2,$$

since $ker(\partial_0^*) = \mathbb{Z}^5$ and $m(\partial_1^*) \cong \mathbb{Z}^3$ is a rank 3 submodule of $\mathbb{Z}^5$. The 1st homology group (29) is

$$H_1(M, \mathbb{Z}) = ker(\partial_1^*)/im(\partial_2^*) = 0,$$

since $ker(\partial_1^*) = im(\partial_2^*)$, as a rank 3 submodule of $\mathbb{Z}^5$. The 2nd homology group (29) is

$$H_2(M, \mathbb{Z}) = ker(\partial_2^*)/im(\partial_3^*) = 0,$$

since $\partial_3 = 0$ and $\partial_2^*$ has full rank.

# 8 Comparison graphs

We follow Sizemore [Si13] or §in Jiang et al [JLYY10].

## 8.1 Comparison matrices

We are given $n$ items to rank (for example, $n$ sports teams) and $r$ rankings (for example, determined by voters, or by sports matches). Denote the items to rank by

$$V = \{1, 2, \ldots, n\},$$

[32]Some call this the *down-up combinatorial Laplacian.*

and the voters by

$$B = \{\beta_1, \beta_2, \ldots, \beta_r\}.$$

The comparisons must be given pairwise. More precisely, for each voter $\beta$, we have a *decision matrix* $W^\beta = (W_{ij}^\beta)_{i.j \in V}$, defined by

$$W_{i,j}^\beta = \begin{cases} 1, & \text{if } \beta \text{ decided between } i, j, \\ 0, & \text{if } \beta \text{ didn't compare } i, j. \end{cases}$$

This is a symmetric matrix which has a 0 in the $(i, j)$ position if and only if the voter made no comparison between $i$ and $j$. If we regard this matrix $W^\beta$ as the adjacency matrix of a graph then the associated graph is the *decision graph* of $\beta$.

**Example 128.** If six baseball teams play each other in league, then, at the end of the season, the comparison graph is the complete graph $K_6$. For instance, see Figure 16.

If $\beta$ makes comparisons between all the different pairs available then the associated decision graph is the complete graph, $K_n$. The

The *comparison matrix*, $Y^\beta = (Y_{ij}^\beta)_{i.j \in V}$, is defined by

$$Y_{i,j}^\beta = \begin{cases} \text{degree of preference of } i \text{ over } j, & \text{if } W_{ij}^\beta = 1, \\ 0, & \text{if } W_{ij}^\beta = 0. \end{cases}$$

This is a skew-symmetric matrix whose $(i, j)th$ entry $Y_{ij}^\beta > 0$ if and only if $\beta$ prefers $i$ to $j$.

**Example 129.** If six baseball teams play each other in league, then, at the end of the season, a comparison digraph associated to $Y^\beta$ is depicted in Figure 16.

The *aggregate decision matrix* $W = (W_{ij})_{i.j \in V}$, defined by

$$W_{ij} = \sum_{\beta \in B} W_{ij}^\beta.$$

This matrix $W$ is symmetric. The *aggregate comparison matrix* $Y = (Y_{ij})_{i.j \in V}$, defined by

$$Y_{ij} = \frac{1}{W_{ij}} \sum_{\beta \in B} Y_{ij}^\beta.$$

This matrix $Y$ is skew-symmetric. Note $Y_{ij} > 0$ if and only if the average voter prefers $i$ to $j$.

*Assume*, for each distinct $i, j \in V$ that some voter $\beta$ has compared $i$ and $j$, i.e., $W_{ij} \neq 0$.

*Assume*, for each distinct $i, j \in V$ that the average voter prefers either $i$ or $j$, i.e., $Y_{ij} \neq 0$.

Associated to $W$ is the *comparison graph* $\Gamma = (V, E)$, where

$$E = \{\{i, j\} \mid W_{ij} \neq 0\}.$$

From $Y$ we obtain an orientation on $\Gamma$: for an edge $e = \{i, j\} \in E$, we call $i$ the head (and $j$ the tail) if and only if $Y_{ij} > 0$. Otherwise, we call $j$ the head (and $i$ the tail).

## 8.2 HodgeRank

Roughly speaking, in the Hodge decomposition (15), the component $im(\partial_1^*)$ corresponds to the ranking associated with the HodgeRank method, $im(\partial_2)$ measures the local inconsistancy, and $ker(\Delta)$ measures the global inconsistancy.

How do we project onto the image of the "gradient", $\partial_1^*$? Since the columns of $\partial_1^*$ and not linearly independent, if we remove a dependent row from (16) we won't affect the image of its transpose. For example,

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

is such a matrix. Note $B$ does have linearly independent columns. The orthogonal projection $P_W : \mathbb{R}^n \to \mathbb{R}^n$ whose image is $W = im(\partial_1^*)$ is given by

$$P_W = B(B^t B)^{-1} B^t = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} & -\frac{1}{4} & -\frac{1}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{5}{8} & \frac{1}{4} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{8} & \frac{1}{8} & \frac{1}{4} & \frac{5}{8} & \frac{3}{8} \\ \frac{1}{8} & -\frac{1}{8} & -\frac{1}{4} & \frac{3}{8} & \frac{5}{8} \end{pmatrix}. \tag{30}$$

Now, we apply this to a hypothetical ranking, one that does not require all pairwise comparisons. Let the vertices of $\Gamma$, $V = \{0, 1, 2, 3\}$, denote sport teams and you have watched matches among the pairs listed in the edge set,

$$E = \{[0, 1], [0, 2], [1, 2], [1, 3], [2, 3]\}.$$

Suppose 0 beat 1, 2 beat 0, 1 beat 2, 1 beat 3 and 3 beat 2. We record that as the vector $\mathbf{r} = (-1, 1, -1, -1, 1)$. Using (30), the "gradient component" of $\mathbf{r} \in C_1$ is

$$P_W \mathbf{r} = (1/4, -1/4, -1/2, -1/4, 1/4) = \mathsf{grad}\,(s)$$
$$= (s(1) - s(0), s(2) - s(0), s(2) - s(1), s(3) - s(1), s(3) - s(2)),$$

where the "score" is $s = (0, 1/4, -1/4, 0)$. Therefore, 1 is best, 0 and 3 are tied for 2nd place, and 2 is last.

## 8.3 HodgeRank example

Our specific example is a graph $\Gamma$ whose vertices $V = \{0, 1, 2, 3, 4, 5\}$ correspond to the six teams

Army = 0, Bucknell = 1, Holy Cross = 2, Lafayette = 3, Lehigh = 4, Navy = 5,

and whose edge set,

$$E = \{[0, 1], [0, 2], [1, 2], \ldots, [4, 5]\}$$

corresponds to their matches.

In general, let $\partial_2 = \mathsf{curl}^* : C_2 \to C_1$ and $\partial_1 = \mathsf{grad}^* : C_1 \to C_0$ denote the adjoint of the curl and the gradient, resp., for a graph, $\Gamma = (V, E)$ having $n$ edges. Here the $C_i$ $(i = 0, 1, 2)$ denote the space of chains

$$C_0 = \mathbb{R}[V], \quad C_1 = \mathbb{R}[E], \quad C_2 = \mathbb{R}[T],$$

where $T$ denotes the set of triangles of $\Gamma$. A win-loss vector representing the result of the mathces between the teams will be regarded as an element of $C_1$.

For graphs, the *Hodge Laplacian* is given by

$$\Delta = \partial_2 \partial_2^* + \partial_1^* \partial_1 : C_1 \to C_1,$$

where the $^*$ denotes the adjoint (or transpose). The *combinatorial Laplacian*[33] is given by

$$\Delta_0 = \partial_1 \partial_1^* : C_0 \to C_0.$$

The *Hodge decomposition* is given by

$$\begin{aligned} im(\partial_2) \oplus ker(\partial_2^*) &= ker(\partial_1) \oplus im(\partial_1^*) \\ = im(\partial_2) \oplus ker(\Delta) \oplus im(\partial_1^*) &= C_1 \cong \mathbb{R}^n. \end{aligned}$$

For the HodgeRank method, we need to project a win-loss vector onto the image of the gradient.

In this ranking method, we record their (sum total) win-loss record (a 1 for a win, $-1$ for a loss) in the $15 \times 6$ matrix $M$ given by

$$M = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}. \tag{31}$$

The matrix $M$ does not have linearly independent columns, so let $M_0$ denote the submatrix of the first 5 columns. The orthogonal projection $P_W$ :

[33]This differs from the vertex Laplacian of the graph by a sign, so it not quite the usual Laplacian.

$\mathbb{R}^{15} \to \mathbb{R}^{15}$ whose image is $W = im(M)$ is given by $P_W = M_0(M_0^t M_0)^{-1} M_0^t =$

$$\left(\begin{array}{ccccccccccccccc}
\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 0 & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\
\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{6} & -\frac{1}{6} & 0 \\
\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & \frac{1}{6} & 0 & -\frac{1}{6} \\
\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\
-\frac{1}{6} & -\frac{1}{6} & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & 0 & 0 & 0 \\
\frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\
\frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & 0 & \frac{1}{6} & 0 & -\frac{1}{6} & 0 & \frac{1}{6} \\
-\frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & -\frac{1}{6} & -\frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & -\frac{1}{6} & 0 & -\frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & 0 & \frac{1}{6} \\
0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 & 0 & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} & \frac{1}{6} \\
0 & 0 & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 & \frac{1}{6} & -\frac{1}{6} & 0 & \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\
0 & 0 & -\frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\
0 & 0 & 0 & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{3}
\end{array}\right). \qquad (32)$$

If we use the win-loss vector with game-scores,

$$\mathbf{r} = (2, 1, 10, 2, 11, 3, 2, 3, 14, 4, 14, 10, 11, 22, 6) \in C_1,$$

Using (32), the "gradient component" of $\mathbf{r}$ is

$$P_W \mathbf{r} =$$

$$(7/3, -17/3, -5/6, 7/3, 29/2, 10/3, 19/6, 0, 73/6, 13/2, 10/3, 53/6, 19/6, 46/3, 73/6)$$

$$= \mathsf{grad}\,(s) = (s(1) - s(0), s(2) - s(0), s(2) - s(1), \ldots, s(5) - s(4)),$$

where the "ranking-score" is $s = (-29/2, -73/6, -53/6, -46/3, -73/6, 0)$. Therefore,

$$\text{Lafayette} < \text{Army} < \text{Bucknell} < \text{Lehigh} < \text{Holy Cross} < \text{Navy}.$$

# 9 Sagemath graph constructions

Sage graphs can be created from a wide range of inputs. An example or 2 are given below. See also the Sagemath , reference manual, available online (google "sagemath graph theory" or visit `http://doc.sagemath.org/html/en/reference/graphs/sage/graphs/graph.html#graph-format`).

A graph can be defined most simply by specifying its edges:

Sagemath

```
sage: E = [(0,1),(0,2),(0,4),(1,2),(2,3),(3,4)]
sage: Gamma = Graph(E)
sage: Gamma.show()
```

The last line outputs the graph depicted in Figure 28. By the way, this is the command line input. For the graphical *sagecell* input, it looks like Figure 29. Incidently, you can also input this graph into Sagemath using a more detailed version (which I prefer) where you input both the vertices and the edges:

Sagemath

```
sage: V = [0,1,2,3,4]
sage: E = [(0,1),(0,2),(0,4),(1,2),(2,3),(3,4)]
sage: Gamma = Graph([V,E])
```

Same graph as before, just a different way of entering it into Sagemath .

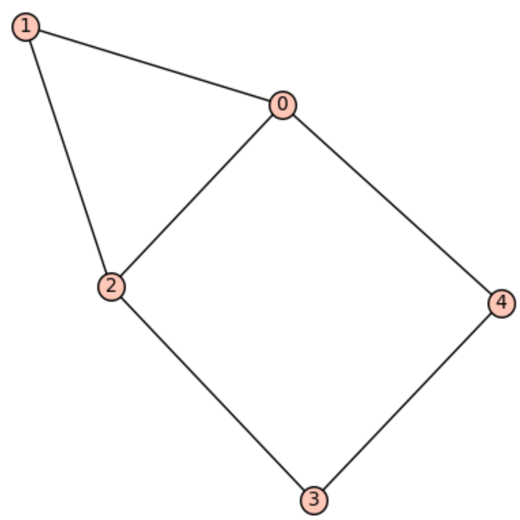


Figure 28: The house graph.

To do a computation on `Gamma` in Sagemath , for example the diameter, you enter `Gamma.diameter()`. Try it! (You should get 2.) For more details on a command, enter `Gamma.diameter??`. This shows you the explanation of the command, some examples, and the Python code used to write the command. For a list of the commands you can use, see the Sagemath reference manual or enter `Gamma.[tab]`. Here `[tab]` means to hit the tab key. Once you enter that, you should get an alphabetical list of the commands you can use. It's

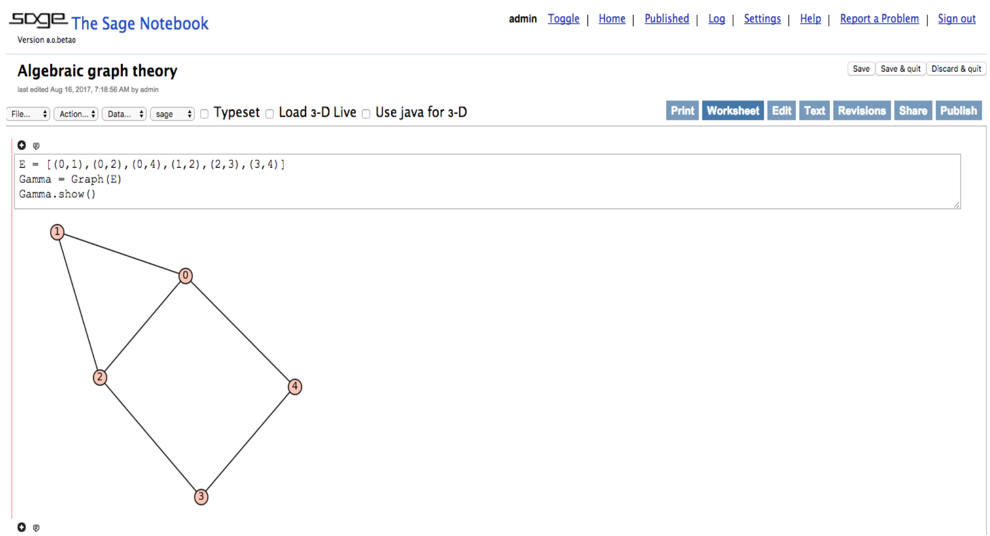


Figure 29: The Sagemath Notebook input for the house graph.

long, so the reference manual (or simply google what you want to do) might be easier.

There are also lots of "named" graphs which are pre-entered into Sagemath . For example, the Sagemath command below produces more-or-less the same "house graph" as above.

Sagemath

```
sage: Gamma = graphs.HouseGraph()
```

For other named graphs, enter `graphs.[tab]`.

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