The wave equation in one dimension

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The theory of the vibrating string touches on musical theory and the theory of oscillating waves, so has likely been a concern of scholars since ancient times. Nevertheless, it wasn't until the late 1700s that mathematical progress was made. Though the problem of describing mathematically a vibrating string requires no calculus, the solution does. With the advent of calculus, Jean le Rond dAlembert, Daniel Bernoulli, Leonard Euler, Joseph-Louis Lagrange were able to arrive at solutions to the one-dimensional wave equation in the eighteenth-century. Daniel Bernoulli's solution dealt with an infinite series of sines and cosines (derived from what we now call a "Fourier series", though it predates it), his contemporaries did not believe that he was correct. Bernoullis technique would be later used by Joseph Fourier when he solved the thermodynamic heat equation in 1807. It is Bernoulli's idea which we discuss here as well. Euler was wrong: Bernoulli's method was basically correct after all.

Now, d'Alembert was mentioned in the lecture on the transport equation and it is worthwhile very briefly discussing what his basic idea was. The theorem of dAlembert on the solution to the wave equation is stated roughly as follows: The partial differential equation:

$$\frac{\partial^2 w}{\partial t^2} = c^2 \cdot \frac{\partial^2 w}{\partial x^2}$$

is satisfied by any function of the form $w = w(x,t) = g(x+ct) + h(x-ct)$, where g and h are "arbitrary" functions. (This is called "the dAlembert solution".) Geometrically speaking, the idea of the proof is to observe that $\frac{\partial w}{\partial t} \pm c\frac{\partial w}{\partial x}$ is a constant times the directional derivative $D_{\vec{v_\pm}} w(x,t)$, where $\vec{v_\pm}$ is a unit vector in the direction $\langle \pm c, 1 \rangle$. Therefore, you integrate

$$D_{\vec{v_-}} D_{\vec{v_+}} w(x,t) = (\text{const.})\frac{\partial^2 w}{\partial t^2} - c^2 \cdot \frac{\partial^2 w}{\partial x^2} = 0$$

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twice, once in the $\vec{v_+}$ direction, once in the $\vec{v_-}$, to get the solution. Easier said than done, but still, that's the idea.

The wave equation with zero ends boundary conditions models the motion of a (perfectly elastic) guitar string of length L:

$$\begin{cases} c^2 \frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2} \\ w(0,t) = w(L,t) = 0. \end{cases}$$

Here $w(x,t)$ denotes the displacement from rest of a point x on the string at time t. The initial displacement $f(x)$ and initial velocity $g(x)$ at specified by the equations

$$w(x,0) = f(x), \qquad w_t(x,0) = g(x).$$

Method:

- Find the sine series of $f(x)$ and $g(x)$:

$$f(x) \sim \sum_{n=1}^{\infty} b_n(f) \sin(\frac{n\pi x}{L}), \qquad g(x) \sim \sum_{n=1}^{\infty} b_n(g) \sin(\frac{n\pi x}{L}).$$

- The solution is

$$w(x,t) = \sum_{n=1}^{\infty} (b_n(f) \cos(c\frac{n\pi t}{L}) + \frac{Lb_n(g)}{cn\pi} \sin(c\frac{n\pi t}{L})) \sin(\frac{n\pi x}{L}).$$

Example: Let

$$f(x) = \begin{cases} -1, & 0 \le t \le \pi/2, \\ 2, & \pi/2 < t < \pi, \end{cases}$$

and let $g(x) = 0$. Then $L = \pi$, $b_n(g) = 0$, and

$$b_n(f) = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx = -2 \frac{2 \cos(n\pi) - 3 \cos(1/2 \, n\pi) + 1}{n}.$$

Thus

$$f(x) \sim b_1(f) \sin(x) + b_2(f) \sin(2x) + ... = \frac{2}{\pi} \sin(x) - \frac{6}{\pi} \sin(2x) + \frac{2}{3\pi} \sin(3x) + ...$$

The function $f(x)$, and some of the partial sums of its sine series, looks like
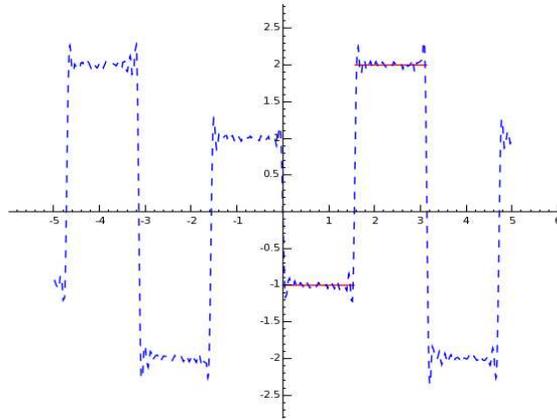
This was computed using the following SAGE commands:

Figure 1: Using 50 terms of the sine series of $f(x)$.

SAGE

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sage: x = var("x")
sage: f1 = lambda x: -1
sage: f2 = lambda x: 2
sage: f = Piecewise([[(0,pi/2),f1],[(pi/2,pi),f2]])
sage: P1 = f.plot(rgbcolor=(1,0,0))
sage: b50 = [f.sine_series_coefficient(n,pi) for n in range(1,50)]
sage: ss50 = sum([b50[i-1]*sin(i*x) for i in range(1,50)])
sage: b50[0:5]
[2/pi, -6/pi, 2/(3*pi), 0, 2/(5*pi)]
sage: P2 = ss50.plot(-5,5,linestyle="--")
sage: (P1+P2).show()
```

As you can see, taking more and more terms gives functions which better and better approximate $f(x)$.

The solution to the wave equation, therefore, is

$$w(x,t) = \sum_{n=1}^{\infty} (b_n(f) \cos(c\frac{n\pi t}{L}) + \frac{Lb_n(g)}{cn\pi} \sin(c\frac{n\pi t}{L})) \sin(\frac{n\pi x}{L}).$$

Taking only the first 50 terms of this series, the graph of the solution at $t = 0$, $t = 0.1$, $t = 1/5$, $t = 1/4$, looks approximately like:
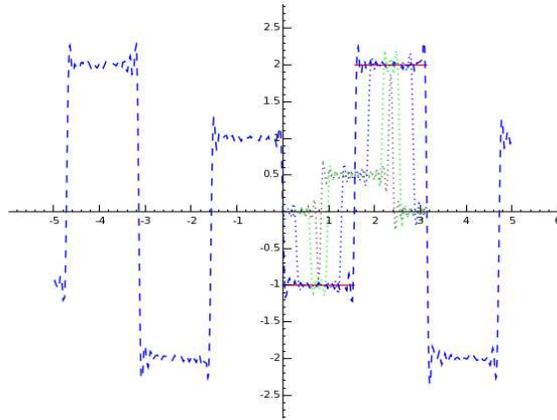
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Figure 2: Wave equation with $c = 3$.

This was produced using the SAGE commands:

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——————————————————— SAGE ———————————————————

sage: t = var("t")
sage: w50t1 = sum([b50[i-1]*sin(i*x)*cos(3*i*(1/10)) for i in range(1,50)])
sage: P3 = w50t1.plot(0,pi,linestyle=":")
sage: w50t2 = sum([b50[i-1]*sin(i*x)*cos(3*i*(1/5)) for i in range(1,50)])
sage: P4 = w50t2.plot(0,pi,linestyle=":",rgbcolor=(0,1,0))
sage: w50t3 = sum([b50[i-1]*sin(i*x)*cos(3*i*(1/4)) for i in range(1,50)])
sage: P5 = w50t3.plot(0,pi,linestyle=":",rgbcolor=(1/3,1/3,1/3))
sage: (P1+P2+P3+P4+P5).show()
```

Of course, taking terms would give a better approximation to $w(x, t)$. Taking the first 100 terms of this series (but with different times):
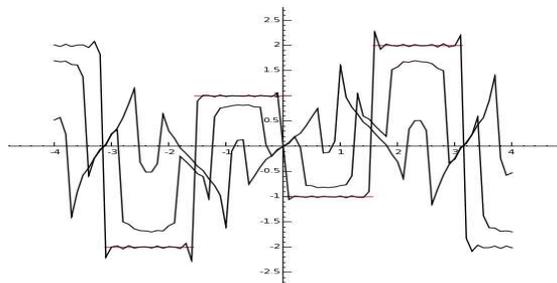

Figure 3: Wave equation with $c = 3$.

Exercise: Solve the wave equation

$$\begin{cases} 2\frac{\partial^2 w(x,t)}{\partial x^2} = \frac{\partial^2 w(x,t)}{\partial t^2} \\ w(0,t) = w(3,t) = 0 \\ w(x,0) = x \\ w_t(x,0) = 0, \end{cases}$$

using SAGE to plot approximations as above.

References

[B] Wikipedia entry for Daniel Bernoulli:
http://en.wikipedia.org/wiki/Daniel_Bernoulli

[M] Wikipedia entry for the physics of music:
http://en.wikipedia.org/wiki/Physics_of_music

[P] Howard L. Penn, "Computer Graphics for the Vibrating String," The College Mathematics Journal, Vol. 17, No. 1 (Jan., 1986), pp. 79-89

[W] Wikipedia entry for the wave equation:
http://en.wikipedia.org/wiki/Wave_equation