Integral Calculus and Sage

(preliminary version)

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Preface

Who cares about calculus? Well, calculus is used in virtually every branch of science where a problem can be mathematically modeled and an optimal solution is desired. In applications, quite often you "know" or strongly suspect something based on your understanding of the general principles in that field. Calculus allows you to quantify your expectations.

For example, you know if you drop a car from a helicopter it will fall to the ground in a short length of time, possibly causing a lot of damage to the vehicle. Now suppose you are directing a truck commercial for television. You really want to know *exactly* how much damage is caused by dropping the car from a specific height so you can prepare precise instructions for the chopper pilot and cameramen. This quantitative precision is where mathematics, namely calculus, can help.

Image a football quarterback throwing a football to a receiver during game. Now suppose you want to design a program for a computer game that allows players to play football on their computer or game console (or cell-phone). You need to know exact equations for the football's trajectory in its path to the receiver. Again, this is where calculus plays a role.

Hopefully this book will help provide a bit of foundation for you to go on to design the best video game or commercial or whatever your future brings to you!

Acknowledgements: We thank Minh Van Nguyen for his careful proofreading and many excellent suggestions for revisions.

Chapter 1

The Integral

Differential calculus starts with a “simple” geometric problem: given a curve and a line tangent to that curve, find the slope of the line. The subject then develops techniques used in solving this problem into a combination of theory about derivatives and their properties, techniques for calculating derivatives, and applications of derivatives. This book begins the development of integral calculus and starts with the “simple” geometric idea of area. This idea will be developed into another combination of theory, techniques, and applications. The integral will be introduced in two (completely different) ways: as a limit of “Riemann sums” and as an “inverse” of differentiation (also known as “antiderivative”). Conceptually, the limit approach to the integral is geometric or numerical, while the antiderivative approach is somewhat more algebraic.

One of the most important results in mathematics, The Fundamental Theorem of Calculus, appears in this chapter. It connects these two notions of the integral and also provides a relationship between differential and integral calculus. Historically, this theorem marked the beginning of modern mathematics, and it provided important tools for the growth and development of the sciences. The chapter begins with a look at area, some geometric properties of areas, and some applications. First, we will see ways of approximating the areas of regions such as tree leaves that are bounded by curved edges, and the areas of regions bounded by graphs of functions. Then we will find ways to calculate exactly the areas of some of these regions. Finally, we will explore more of the rich variety of uses of “areas”. The primary purpose of this introductory section is to help develop your intuition about areas and your ability to reason using geometric arguments about area. This type of reasoning will appear often in the rest of this book and is very helpful for applying the ideas of calculus.

1.1 Area

We know from previous experience how to compute the areas of simple geometrical shapes like triangles, circles and rectangles. Formulas for these have been known since the days of the ancient Greeks. But how do you find the area under a "more complicated" curve such as $y = x^2$ where $-1 < x < 1$? First, let's graph the function using Sage as follows[1]:

```
sage: a = -1; b = 1
sage: f = lambda x: x^2
sage: Lb = [[b,f(b)], [b,0], [a,0], [a,f(a)]]
sage: Lf = [[i/20, f(i/20)] for i in xrange(20*a, 20*b+1)]
sage: P = polygon(Lf+Lb, rgbcolor=(0.2,0.8,0))
sage: Q = plot(f(x), x, a-0.5, b+0.5)
sage: show(P+Q)
```

This results in the plot shown in Figure 1.1:

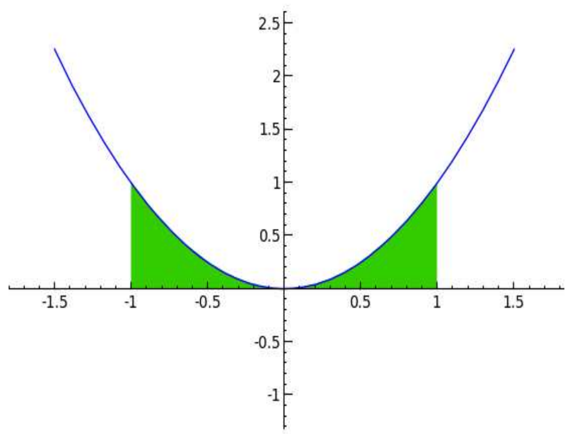

Figure 1.1: Using Sage to plot $y = x^2$ for $-1 \leq x \leq 1$.

The rough, general idea introduced in this section is the following. To compute the area of the "complicated" shaded region in Figure 1.1, we break it up into *lots* of "simpler" subregions, whose areas are easy to compute, then add up the areas of the subregions to get the total area. We shall return to this example later.

The basic shape we will use is the rectangle, whose area is given by the formula

$$\text{area} = \text{base} \times \text{height}$$

If the units for each side of the rectangle are "meters," then the area will have the units ("meters")×("meters") = "square meters" $= m^2$. Other area formulas needed for this section are: the area formula for triangles

$$\text{area} = bh/2 \tag{1.1}$$

[1] Feel free to try this yourself, changing a, b and x^2 to something else if you like.

where b is the length of the base and h is the height; and the corresponding formula for circles

$$\text{area} = \pi r^2$$

with r being the radius. We also assume the following familiar properties of area:

- Addition Property: The total area of a region is the sum of the areas of the non-overlapping pieces which comprise the region (see Figure 1.2).
- Inclusion Property: If region B is on or inside region A, then the area of region B is less than or equal to the area of region A (see Figure 1.3).
- Location-Independence Property: The area of a region does not depend on its location (see Figure 1.4).

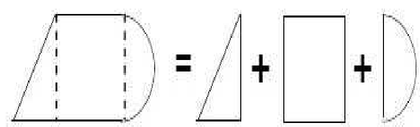

Figure 1.2: Adding areas of non-overlapping regions.

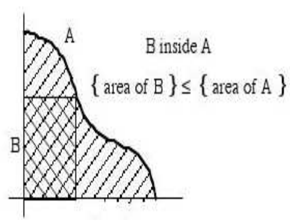

Figure 1.3: Estimating areas using rectangles.

Example 1.1.1. *Determine the area of the region in Figure 1.5(a).*

Solution. The region can easily be broken into two rectangles, Figure 1.5(b), with areas 35 square inches and 3 square inches respectively. So the total area of the original region is 38 square inches. □

Using the above three properties of area, we can get information about areas that are difficult to calculate exactly. For instance, let A be the region bounded by the graph of $f(x) = 1/x$, the x-axis, and vertical lines at $x = 1$ and $x = 3$ (refer to Figure 1.6). Since the two rectangles in Figure 1.6 are inside the region A and do not overlap each other, the area of the rectangles is $1/2 + 1/3 = 5/6$, which is less than the area of region A.

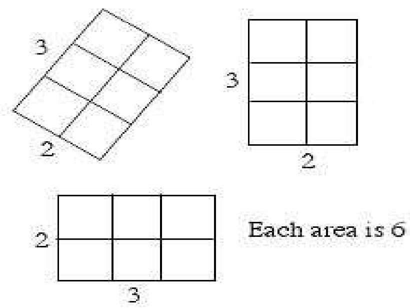

Figure 1.4: Independence of area under translations and rotations.

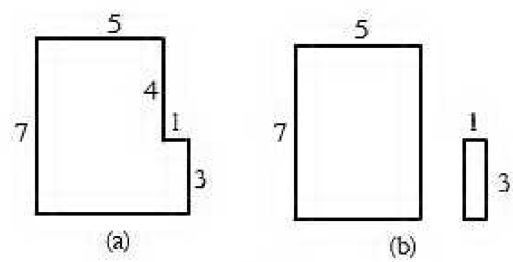

Figure 1.5: Calculating areas using the addition property.

Practice 1.1.1. *Build two rectangles, each with base 1 unit, outside the shaded region in Figure 1.6. Then use their areas to make a valid statement about the area of region A.*
(Ans: Outside rectangular area = 1.5*.)*

Practice 1.1.2. *Now use both inside and outside rectangles with base 1/2 unit. What can be said about the area of region A in Figure 1.6?*
(Ans: The area of the region is between 0.95 *and* 1.2*.)*

Example 1.1.2. *In Figure 1.7, there are 32 dark squares, each having a side length of 1 centimeter, and 31 lighter squares of the same side length. We can be sure that the area of the leaf is smaller than what number?*

Solution. The area of the leaf is smaller than $32 + 31 = 63$ cm^2. □

Practice 1.1.3. *We can be sure that the area of the leaf is at least how large?*

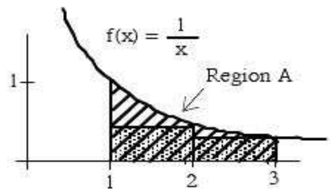

Figure 1.6: The area under of $y = 1/x$, where $1 \leq x \leq 3$.

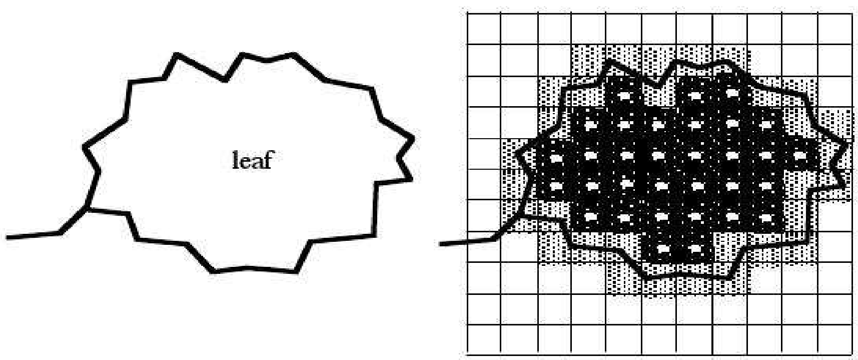

Figure 1.7: Approximating the area of a "leaf".

Functions can be defined in terms of areas. For the constant function $f(t) = 2$, define $A(x)$ to be the area of the rectangular region bounded by the graph of f, the t-axis, and the vertical lines at $t = 1$ and $t = x$ (see Figure 1.8(a)). Then $A(2)$ is the area of the shaded region in Figure 1.8(b), where $A(2) = 2$. Similarly, $A(3) = 4$ and $A(4) = 6$. In general, we have

$$A(x) = \text{base} \times \text{height} = (x - 1) \times 2 = 2x - 2$$

for any $x \geq 1$. Figure 1.8(c) shows the graph of $y = A(x)$, whose derivative is $A'(x) = 2$ for every value of $x \geq 1$.

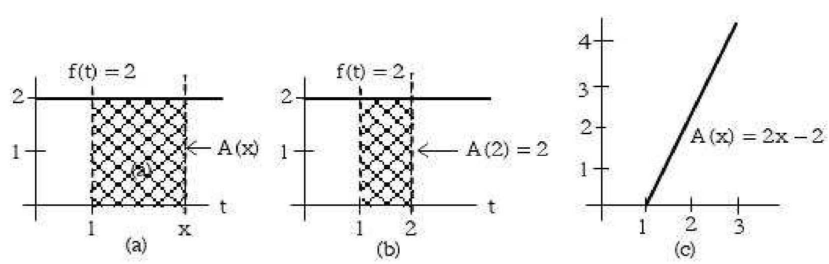

Figure 1.8: The area as a function.

Sometimes it is useful to move regions around. Consider the parallelogram in Figure 1.9 and notice the triangle on its left end. Moving this triangular region from the left side of the parallelogram to fill the region on the right side, we end up with a rectangle. So we have reduced the problem of finding the area of a parallelogram to a simpler, equivalent problem about finding the area of a rectangle.

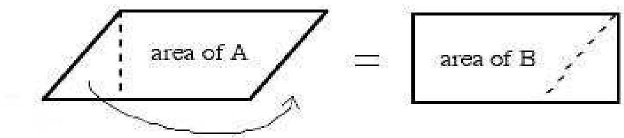

Figure 1.9: The area of a parallelogram.

For complicated regions such as the shaded regions in Figure 1.10(a), we can use a similar technique. At first glance, it is difficult to estimate the total area of the shaded regions in Figure 1.10(a). However, if we slide all of them into a single column as shown in Figure 1.10(b), then it is easy to determine that the shaded area is less than the area of the enclosing rectangle: that is, area of rectangle = base $\times$ height = $1 \times 2 = 2$.

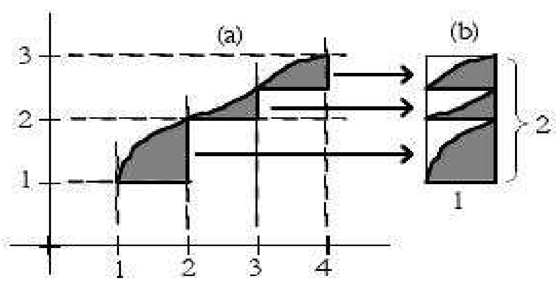

Figure 1.10: Estimating the area of an irregular region.

1.2 Some applications of area

One reason "areas" are so useful is that they can represent quantities other than simple geometric shapes. For example, if the units of the base of a rectangle are "hours" and the units of the height are "miles/hour", then the units of the "area" of the rectangle are (hours) $\times$ (miles/hour) = miles, a measure of distance. Similarly, if the base units are centimeters and the height units are grams, then the "area" units are gram $\times$ centimeters, a measure of work.

Example 1.2.1. *Distance as an "area": In Figure 1.11, $f(t)$ is the velocity of a car in "miles per hour" and t is the time in "hours." Then the shaded "area"* *is (base) × (height) = (3 hours) × (20 miles/hour) = 60 miles. In other words, the car traveled 60 miles in the 3 hours from 1 o'clock until 4 o'clock.*

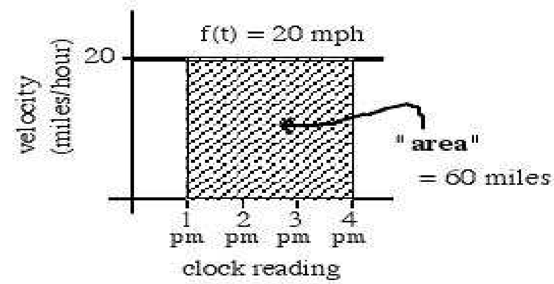

Figure 1.11: Calculating distance using the "area" concept.

Example 1.2.1 shows that we can use the familiar idea of "area" to calculate a quantity (in this case, miles) that at first seems totally unrelated to "area". Theorem 1.2.1 provides a general statement of the idea illustrated in Example 1.2.1.

Theorem 1.2.1. *("Area" as Distance) Let $f(t)$ be the (positive) forward velocity of an object at time t. Then the "area" bounded by the graph of f, the t-axis, and the vertical lines at times $t = a$ and $t = b$, will be the distance that the object has moved forward between times a and b.*

This "area as distance" fact can make some difficult distance problems much easier to handle.

Example 1.2.2. *A car starts from rest (velocity = 0) and steadily speeds up so that 20 seconds later its speed is 88 feet per second (60 miles per hour). How far did the car travel during those 20 seconds?*

Solution. If "steadily speeds up" means that the velocity increases linearly, then we can apply the idea of "area as distance". Figure 1.12 shows the graph of velocity versus time. The "area" of the triangular region in the figure represents the distance traveled. Using formula (1.1) for the area of a triangle, we have

$$\begin{aligned} \text{distance} &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times (20 \text{ seconds}) \times (88 \text{ feet/second}) \\ &= 880 \text{ feet} \end{aligned}$$

In 20 seconds the car covered a total distance of 880 feet. □

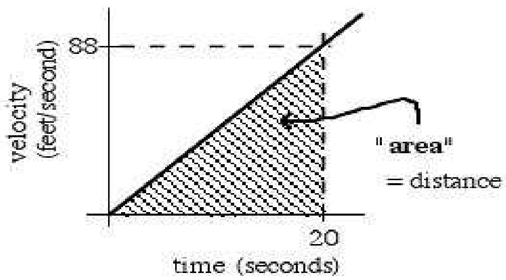

Figure 1.12: Using "area" to calculate the distance a car travels.

Practice 1.2.1. *A train traveling at 45 miles per hour (66 feet/second) takes 60 seconds to come to a complete stop. If the train slowed down at a steady rate (the velocity decreased linearly), how many feet did the train travel while coming to a stop?*

Practice 1.2.2. *You and a friend start off at noon and walk in the same direction along the same path at the rates shown in Figure 1.13.*

- *Who is walking faster at 2 pm? Who is ahead at 2 pm?*
- *Who is walking faster at 3 pm? Who is ahead at 3 pm?*
- *When will you and your friend be together? (Answer in words.)*

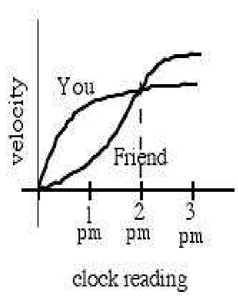

Figure 1.13: Illustration for Practice 1.2.2.

1.2.1 Total accumulation as "area"

In previous examples, the function represented a rate of travel (miles per hour), and the area represented the total distance traveled. For functions representing other rates, the area still represents the total amount of something. Some familiar rates include the production of a factory (bicycles per day), the flow of water in a river (gallons per minute), traffic over a bridge (cars per minute), or the spread of a disease (newly sick people per week).

Theorem 1.2.2. *("Area" as Total Accumulation) Let $f(t)$ be a positive rate (in units per time interval) at time t. Then the "area" between the graph of f, the t-axis, and the vertical lines at times $t = a$ and $t = b$, will be the total units which accumulate between times a and b.*

Practice 1.2.3. *Figure 1.14 shows the number of telephone calls made per hour on a Tuesday. Approximately how many calls were made between 9 am and 11 am?*

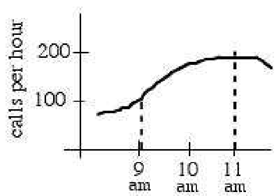

Figure 1.14: Illustration for Practice 1.2.3.

1.2.2 Problems

1. (a) Calculate the sum of the rectangular areas in Figure 1.15(a).
   (b) From part (a), what can we say about the area of the shaded region in Figure 1.15(b)?

2. (a) Calculate the sum of the areas of the shaded regions in Figure 1.15(c).
   (b) From part (a), what can we say about the area of the shaded region in Figure 1.15(b)?

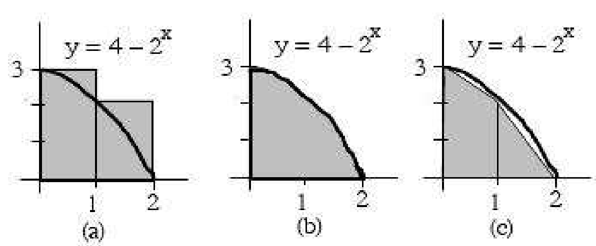

Figure 1.15: Estimating areas.

3. Consider the function $f(x)$ whose graph is shown in Figure 1.16. Let $A(x)$ represent the area bounded by the graph of $f(x)$, the horizontal axis, and vertical lines at $t = 0$ and $t = x$. Evaluate $A(x)$ for $x = 1$, 2, 3, 4, and 5.

4. Police chase: A speeder traveling 45 miles per hour (in a 25 mph zone) passes a stopped police car which immediately takes off after the speeder.

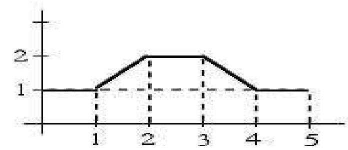

Figure 1.16: Computing areas.

The police car speeds up steadily to 60 miles/hour in 20 seconds and then travels at a steady 60 miles/hour. Figure 1.17 shows the velocity versus time graph for the police car and the speeder. How long and how far before the police car catches the speeder who continued traveling at 45 miles/hour?

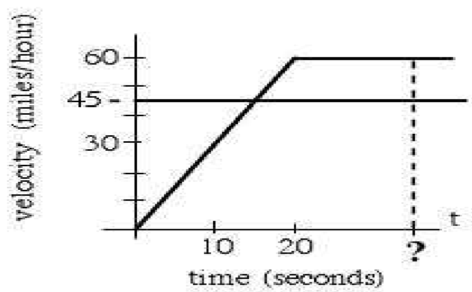

Figure 1.17: Computing areas to analyze the police chase problem.

5. What are the units for the "area" of a rectangle with the given base and height units?

Base units	Height units	"Area" units
miles per second	seconds	
hours	dollars per hour	
square feet	feet	
kilowatts	hours	
houses	people per house	
meals	meals	

1.3 Sigma notation and Riemann sums

So far, we have calculated the area of a region in terms of simpler regions. We cut the region into simple shapes, calculate the area of each simple shape, and then add these smaller areas together to get the area of the whole region. We will continue using that approach, but it is useful to have a notation for adding many values together. This is where the sigma notation (denoted with the capital Greek letter Σ) comes in handy.

Consider the function $f(x) = 3x + 1$ and for convenience we restrict x to take only integer values. Now consider the values of $f(x)$ where $x = 2, 3, 4, 5$ as shown in Table 1.1. The sigma notation allows us to express the sum $S = f(2) + f(3) + f(4) + f(5)$ using a convenient shorthand, namely

$$S = \sum_{k=2}^{5} 3k + 1 = f(2) + f(3) + f(4) + f(5) \tag{1.2}$$

x	$f(x) = 3x + 1$
2	$f(2) = 7$
3	$f(3) = 10$
4	$f(4) = 13$
5	$f(5) = 16$

Table 1.1: Values of $f(x) = 3x + 1$ for $x = 2, 3, 4, 5$.

Figure 1.18 shows the various parts of this notation. The function to the right of the sigma Σ is called the **summand**. Below and above the sigma are numbers called the **lower** and **upper limits** of the summation, respectively. Notice that instead of using the symbol x, in equation (1.2) and Figure 1.18 we replaced x with the variable k. The variable (typically i, j, or k) used in the summation is called the **counter** or **index variable**. Table 1.2 provides various ways of reading the Σ notation. Using Sage, we can obtain the value of the sum in Figure 1.18 as follows:

```
sage: maxima("sum(3*k + 1, k, 2, 5)")
46
```

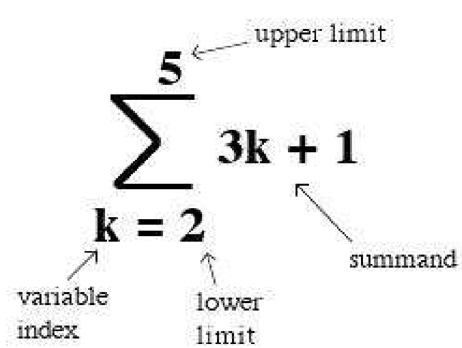

Figure 1.18: The summation or sigma notation.

Summation notation	A way to read the sigma notation	Sigma notation
$1^2+2^2+3^2+4^2+5^2$	the sum of k squared for k equals 1 to k equals 5	$\sum_{k=1}^{5} k^2$
$\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$	the sum of 1 over k for k equals 3 to k equals 7	$\sum_{k=3}^{7} k^{-1}$
$2^0+2^1+2^2+2^3+2^4+2^5$	the sum of 2 to the j-th power for j equals 0 to j equals 5	$\sum_{j=0}^{5} 2^j$
$a_2+a_3+a_4+a_5+a_6+a_7$	the sum of a sub i for i equals 2 to i equals 7	$\sum_{i=2}^{7} a_i$

Table 1.2: How to read the sigma notation.

x	$f(x)$	$g(x)$	$h(x)$
1	2	4	3
2	3	1	3
3	1	−2	3
4	0	3	3
5	3	5	3

Table 1.3: Domain values and their images under the functions f, g, and h.

Practice 1.3.1. *Write the summation denoted by each of the following:*

(a) $\sum_{k=1}^{5} k^3$, *(b)* $\sum_{j=2}^{7} (-1)^j \frac{1}{j}$, *(c)* $\sum_{m=0}^{4} (2m+1)$.

In practice, the sigma notation is frequently used with the standard function notation:

$$\sum_{k=1}^{3} f(k+2) = f(1+2) + f(2+2) + f(3+2) = f(3) + f(4) + f(5)$$

and

$$\sum_{k=1}^{4} f(x_i) = f(x_1) + f(x_2) + f(x_3) + f(x_4).$$

Example 1.3.1. *Use the values in Table 1.3 to evaluate (i)* $\sum_{k=2}^{5} 2f(k)$*; and (ii)* $\sum_{j=3}^{5} \big(5 + f(j-2)\big)$.

Solution. (i) Using the values of $f(k)$ for $k = 2, 3, 4, 5$, we have

$$\begin{aligned} \sum_{k=2}^{5} 2f(k) &= 2f(2) + 2f(3) + 2f(4) + 2f(5) \\ &= 2(3) + 2(1) + 2(0) + 2(3) \\ &= 14 \end{aligned} \tag{1.3}$$

On the other hand, note that in expression (1.3) we can factor out the value 2 to get an equivalent expression

$$\sum_{k=2}^{5} 2f(k) = 2\sum_{k=2}^{5} f(k)$$

This expression evaluates to the same value as the original expression.

(ii) For $j = 3, 4, 5$ we have

$$f(3-2) = f(1), \qquad f(4-2) = f(2), \qquad f(5-2) = f(3)$$

Then we get

$$\begin{aligned} & \sum_{j=3}^{5} \big(5 + f(j-2)\big) \\ = \; & \big(5 + f(3-2)\big) + \big(5 + f(4-2)\big) + \big(5 + f(5-2)\big) \\ = \; & \big(5 + f(1)\big) + \big(5 + f(2)\big) + \big(5 + f(3)\big) \\ = \; & (5+2) + (5+3) + (5+1) \\ = \; & 21 \end{aligned} \tag{1.4}$$

Now carefully look at expression (1.4) again. When the index j takes on each of the three values 3, 4, and 5, the term 5 is added three times in evaluating the whole sum. Using this observation, we can write (1.4) in the equivalent form

$$\sum_{j=3}^{5} \big(5 + f(j-2)\big) = 3 \times 5 + \sum_{j=3}^{5} f(j-2)$$

This observation will be formalized later on. □

Practice 1.3.2. *Use the values of f, g and h in Table 1.3 to evaluate the following:*

$$(a) \quad \sum_{k=2}^{5} g(k), \qquad (b) \quad \sum_{j=1}^{4} h(j), \qquad (c) \quad \sum_{i=3}^{5} \big[f(i-1) + g(i)\big].$$

Since the sigma notation is simply a notation for addition, it has all of the familiar properties of addition. Theorem 1.3.1 states a number of these familiar addition properties for the Σ notation.

Theorem 1.3.1. *(Summation Properties) Let C be a fixed constant and let $m, n \in \mathbb{Z}$ be positive. Then we have the following summation properties for the Σ notation.*

1. *Sum of constants:* $\sum_{k=1}^{n} C = \underbrace{C + C + C + \cdots + C}_{n \text{ terms}} = nC$.

2. *Addition:* $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$.
3. *Subtraction:* $\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$.
4. *Constant multiple:* $\sum_{k=1}^{n} Ca_k = C\sum_{k=1}^{n} a_k$.
5. *Preserves positivity: If* $b_k \geq a_k$ *for all* k*, then* $\sum_{k=1}^{n} b_k \geq \sum_{k=1}^{n} a_k$*. In particular, if* $a_k \geq 0$ *for all* k *then* $\sum_{k=1}^{n} a_k \geq 0$.
6. *Additivity of ranges: If* $1 \leq m \leq n$ *then* $\sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k = \sum_{k=1}^{n} a_k$.

1.3.1 Sums of areas of rectangles

Later, we will approximate the areas under curves by building rectangles as high as the curve, calculating the area of each rectangle, and then adding the rectangular areas together.

Example 1.3.2. *Evaluate the sum of the rectangular areas in Figure 1.19, and write the sum using the sigma notation.*
Solution: We have

$$\begin{aligned}\text{sum of the rectangular areas} &= \text{sum of (base)} \times \text{(height) for each rectangle}\\ &= (1)(1/3) + (1)(1/4) + (1)(1/5) = 47/60.\end{aligned}$$

Using the sigma notation,

$$(1)(1/3) + (1)(1/4) + (1)(1/5) = \sum_{k=1}^{3} \frac{1}{k}.$$

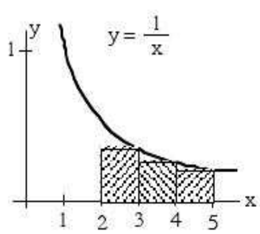

Figure 1.19: Area and summation notation.

Practice 1.3.3. *Evaluate the sum of the rectangular areas in Figure 1.20, and write the sum using the sigma notation.*

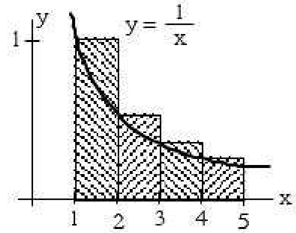

Figure 1.20: Area and summation notation.

Example 1.3.3. *Write the sum of the areas of the rectangles in Figure 1.21 using the sigma notation.*
Solution: The area of each rectangle is (base)×(height).

rectangle	*base*	*height*	*area*
1	$x_1 - x_0$	$f(x_1)$	$(x_1 - x_0)f(x_1)$
2	$x_2 - x_1$	$f(x_2)$	$(x_2 - x_1)f(x_2)$
3	$x_3 - x_2$	$f(x_3)$	$(x_3 - x_2)f(x_3)$

The area of the k-th rectangle is $(x_k - x_{k-1})f(x_k)$*, and the total area of the rectangles is the sum* $\sum_{k=1}^{3}(x_k - x_{k-1})f(x_k)$.

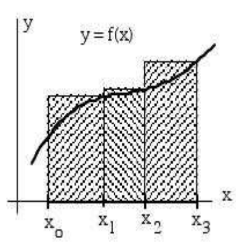

Figure 1.21: Area and summation notation.

1.3.2 Area under a curve: Riemann sums

Suppose we want to calculate the area between the graph of a positive function f and the interval $[a, b]$ on the x–axis (Fig. 7). The Riemann Sum method is to build several rectangles with $y = f(x)$ bases on the interval $[a, b]$ and sides that reach up to the graph of f (Fig. 8). Then the areas of the rectangles can be calculated and added together to get a number called a **Riemann sum of** f **on** $[a, b]$. The area of the region formed by the rectangles is an *approximation* of the area we want.

Example 1.3.4. *Approximate the area in Figure 1.22(a) between the graph of f and the interval $[2,5]$ on the x–axis by summing the areas of the rectangles in Figure 1.22(b).*

Solution: The total area of rectangles is $(2)(3)+(1)(5)=11$ square units.

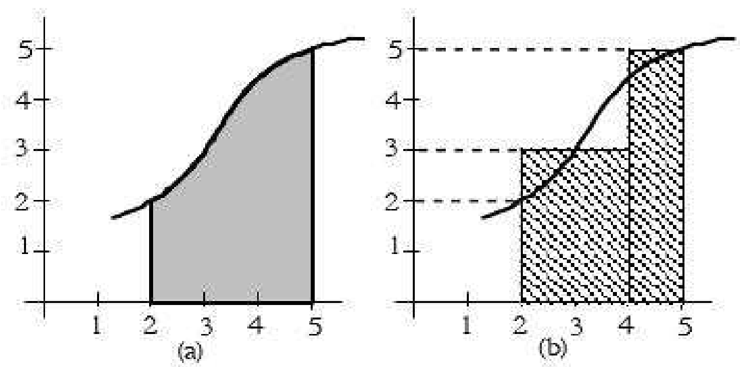

Figure 1.22: Illustration for Example 1.3.4.

In order to effectively describe this process, some new vocabulary is helpful: a "partition" of an interval and the mesh of the partition. A **partition** P of a closed interval $[a,b]$ into n subintervals is a set of $n+1$ points $\{x_0 = a, x_1, x_2, x_3, ..., x_{n-1}, x_n = b\}$ in increasing order, $a = x_0 < x_1 < x_2 < x_3 < ... < x_n = b$. (A partition is a collection of points on the axis and it does not depend on the function in any way.)

The points of the partition P divide the interval into n subintervals (Figure 1.23): $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$, $\ldots$, and $[x_{n-1},x_n]$ with lengths $\Delta x_1 = x_1 - x_0$, $\Delta x_2 = x_2 - x_1$, $\ldots$, $\Delta x_n = x_n - x_{n-1}$. The points x_k of the partition P are the locations of the vertical lines for the sides of the rectangles, and the bases of the rectangles have lengths Δx_k for $k = 1,2,3,...,n$.

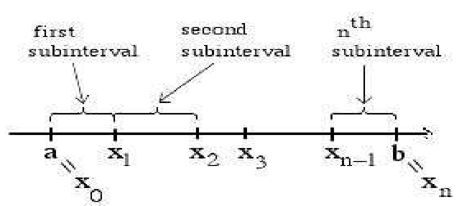

Figure 1.23: Partition of the interval $[a,b]$.

The **mesh** or **norm** of the partition is the length of the longest of the subintervals $[x_{k-1},x_k]$, or, equivalently, the maximum of the Δx_k for $k=1,2,3,...,n$. For example, the set $P = \{2,3,4.6,5.1,6\}$ is a partition of the interval $[2,6]$ and divides it into 4 subintervals with lengths $\Delta x_1 = 1$, $\Delta x_2 = 1.6$, $\Delta x_3 = 0.5$ and

$\Delta x_4 = 0.9$. The mesh of this partition is 1.6, the maximum of the lengths of the subintervals. (If the mesh of a partition is "small," then the length of each one of the subintervals is the same or smaller.)

A function, a partition, and a point in each subinterval determine a Riemann sum. Suppose f is a positive function on the interval $[a, b]$, $P = \{x_0 = a, x_1, x_2, x_3, ..., x_{n-1}, x_n = b\}$ is a partition of $[a, b]$, and c_k is an $x$value in the k-th subinterval $[x_{k-1}, x_k] : x_{k-1} \leq c_k \leq x_k$. Then the area of the k-th rectangle is $f(c_k) \cdot (x_k - x_{k-1}) = f(c_k)\Delta x_k$. (Figure 1.24)

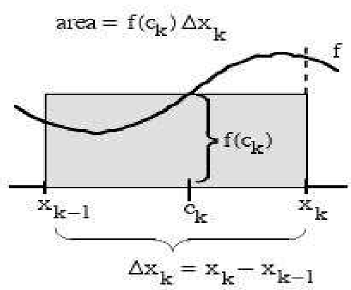

Figure 1.24: Part of a Riemann sum.

Definition 1.3.1. *A summation of the form* $\sum_{k=1}^{n} f(c_k)\Delta x_k$ *is called a* **Riemann sum** *of* f *for the partition* P.

This Riemann sum is the total of the areas of the rectangular regions and is an approximation of the area between the graph of f and the x–axis.

Example 1.3.5. *Find the Riemann sum for* $f(x) = 1/x$ *and the partition* $\{1, 4, 5\}$ *using the values* $c_1 = 2$ *and* $c_2 = 5$.

Solution: The two subintervals are $[1, 4]$ *and* $[4, 5]$ *so* $\Delta x_1 = 3$ *and* $\Delta x_2 = 1$. *Then the Riemann sum for this partition is*

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 = f(2)(3) + f(5)(1) = \frac{1}{2}(3) + \frac{1}{5}(1) = 1.7.$$

Practice 1.3.4. *Calculate the Riemann sum for* $f(x) = 1/x$ *on the partition* $\{1, 4, 5\}$ *using the values* $c_1 = 3$, $c_2 = 4$.

Practice 1.3.5. *What is the smallest value a Riemann sum for* $f(x) = 1/x$ *and the partition* $\{1, 4, 5\}$ *can have? (You will need to select values for* c_1 *and* c_2*.) What is the largest value a Riemann sum can have for this function and partition?*

Here is a Sage example.

Example 1.3.6. *Using* Sage*, we construct the Riemann sum of the function* $y = x^2$ *using a partition of* 6 *equally spaced points, where the* c_k*'s are taken to be the midpoints.*

```
sage: f1(x) = x^2
sage: f = Piecewise([[(-1,1),f1]])
sage: g = f.riemann_sum(6,mode="midpoint")
sage: P = f.plot(rgbcolor=(0.7,0.1,0.5), plot_points=40)
sage: Q = g.plot(rgbcolor=(0.7,0.6,0.6), plot_points=40)
sage: L = add([line([[pf[0][0],0],[pf[0][0],pf[1](x=pf[0][0])]],\
      rgbcolor=(0.7,0.6,0.6)) for pf in g.list()])
sage: show(P+Q+L)
```

Here is the plot:

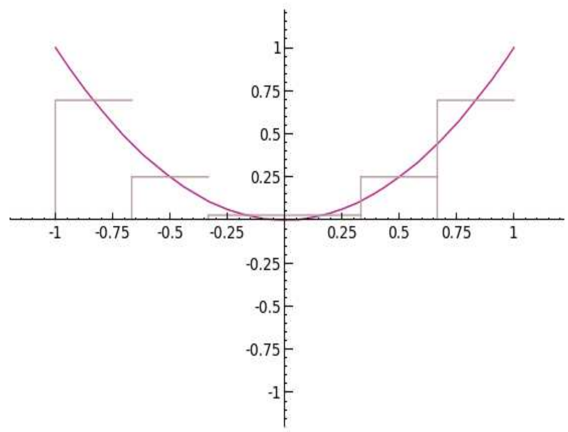

Figure 1.25: Plot using Sage of a Riemann sum for $y = x^2$.

At the end of this section is a Python[2] program listing for calculating Riemann sums of $f(x) = 1/x$ on the interval $[1, 5]$ using 100 subintervals. It can be modified easily to work for different functions, different endpoints, and different numbers of subintervals. Table 1.26 shows the results of running the program with different numbers of subintervals and different ways of selecting the points c_i in each subinterval. When the mesh of the partition is small (and the number of subintervals large), all of the ways of selecting the c_i lead to approximately the same number for the Riemann sums.

[2]Python is a general purpose cross-platform, free and open source computer programming language. It is widely used in industry and academia, and is available for download at `http://www.python.org`. Alternatively, you can download the mathematical software system Sage from `http://www.sagemath.org`. It comes with Python pre-installed.

Here is a Python program to calculate Riemann sums of $f(x) = 1/x$ on $[1, 5]$ using 100 equal length subintervals, based on the "left-hand" endpoints.

```
f = lambda x: 1/x                             # define the function
a = 1.0                                       # left endpoint of integral
b = 5.0                                       # right endpoint of integral
n = 100                                       # number of subintervals
Dx = (b-a)/n                                  # width of each subinterval
rsum = sum([f(a+i*Dx)*Dx for i in range(n)])  # compute the Riemann sum
print rsum                                    # print the Riemann sum
```

Other Riemann sums can be calculated by replacing the "rsum" line with one of:

`rsum = sum([f(a+(i+0.5)*Dx)*Dx for i in range(n)])` "midpoint"

`rsum = sum([f(a+(i+1)*Dx)*Dx for i in range(n)])` "right-hand"

Written as Python "functions", these three are written as below[3]

```
def rsum_lh(n):
    f = lambda x: 1/x
    a = 1.0
    b = 5.0
    Dx = (b-a)/n
    return sum([f(a+i*Dx)*Dx for i in range(n)])

def rsum_mid(n):
    f = lambda x: 1/x
    a = 1.0
    b = 5.0
    Dx = (b-a)/n
    return sum([f(a+(i+0.5)*Dx)*Dx for i in range(n)])

def rsum_rh(n):
    f = lambda x: 1/x
    a = 1.0
    b = 5.0
    Dx = (b-a)/n
    return sum([f(a+(i+1)*Dx)*Dx for i in range(n)])
```

[3]If you have an electronic copy of this file, and "copy-and-paste" these into Python, keep in mind indenting is very important in Python syntax.http://www.python.org/doc/essays/styleguide.html

n	Δx_i	left–hand Riemann sum	midpoint Riemann sum	right–hand Riemann sum
5	0.8	1.9779070602600015	1.5861709609993364	1.3379070602600014
10	0.4	1.7820390106296689	1.6032106782106783	1.462039010629669
20	0.2	1.6926248444201737	1.6078493243021688	1.5326248444201738
100	0.04	1.6255658911511259	1.6093739310551827	1.5935658911511259
1000	0.004	1.6110391924319691	1.6094372724359669	1.607839192431969

Figure 1.26: Table for Python example.

The command

```
sizes = [5, 10, 20, 100, 1000]
table = [[n, (b-a)/n, rsum_lh(n),rsum_mid(n), rsum_rh(n)] for n in sizes]
```

yields the following data:

In fact, the exact value is $\log(5) = 1.609437...$, so these last few lines yielded pretty good approximations.

Practice 1.3.6. *Replace $1/x$ by x^2 and $[a,b] = [1,5]$ by $[a,b] = [-1,1]$ in the Python code above and find the Riemann sum for the new function and $n = 100$. Use the midpoint approximation. (You may use* Sage *or Python, whichever you prefer.)*

Example 1.3.7. *Find the Riemann sum for the function $f(x) = \sin(x)$ on the interval $[0,\pi]$ using the partition $\{0,\pi/4,\pi/2,\pi\}$ with $c_1 = \pi/4$, $c_2 = \pi/2$, $c_3 = 3\pi/4$.*

Solution: The 3 subintervals are $[0,\pi/4]$, $[\pi/4,\pi/2]$, and $[\pi/2,\pi]$ so $\Delta x_1 = \pi/4$, $\Delta x_2 = \pi/4$ and $\Delta x_3 = \pi/2$. The Riemann sum for this partition is

$$\begin{aligned}\sum_{k=1}^{3} f(c_k)\Delta x_k &= \sin(\pi/4)(\pi/4) + \sin(\pi/2)(\pi/4) + \sin(3\pi/4)(\pi/2)\\ &= \frac{1}{\sqrt{2}}\frac{\pi}{4} + 1\cdot\frac{\pi}{4} + \frac{1}{\sqrt{2}}\frac{\pi}{2}\\ &= 2.45148...\ .\end{aligned}$$

Practice 1.3.7. *Find the Riemann sum for the function and partition in the previous example, but use $c_1 = 0$, $c_2 = \pi/2$, $c_3 = \pi/2$.*

1.3.3 Two special Riemann sums: lower and upper sums

Two particular Riemann sums are of special interest because they represent the extreme possibilities for Riemann sums for a given partition.

Definition 1.3.2. *Suppose f is a positive function on $[a,b]$, and P is a partition of $[a,b]$. Let m_k be the xvalue in the k-th subinterval so that $f(m_k)$ is the* *minimum value of f in that interval, and let M_k be the xvalue in the k-th subinterval so that $f(M_k)$ is the maximum value of f in that interval.*

lower sum: $LS = \sum_{k=1}^{n} f(m_k)\Delta x_k$.

upper sum: $US = \sum_{k=1}^{n} f(M_k)\Delta x_k$.

Geometrically, the lower sum comes from building rectangles under the graph of f (Figure 1.27(a)), and the lower sum (every lower sum) is less than or equal to the exact area A: $LS \leq A$ for every partition P. The upper sum comes from building rectangles over the graph of f (Figure 1.27(b)), and the upper sum (every upper sum) is greater than or equal to the exact area A: $US \geq A$ for every partition P. The lower and upper sums provide bounds on the size of the exact area: $LS \leq A \leq US$.

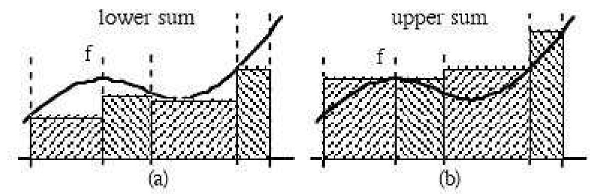

Figure 1.27: Lower and upper Riemann sums.

Unfortunately, finding minimums and maximums can be a timeconsuming business, and it is usually not practical to determine lower and upper sums for "arbitrary" functions. If f is monotonic, however, then it is easy to find the values for m_k and M_k , and sometimes we can explicitly calculate the limits of the lower and upper sums.

For a monotonic bounded function we can guarantee that a Riemann sum is within a certain distance of the exact value of the area it is approximating.

Theorem 1.3.2. *If f is a positive, monotonically increasing, bounded function on $[a, b]$, then for any partition P and any Riemann sum for P,*

distance between the Riemann sum and the exact area $\leq$ distance between the upper sum (US) and the lower sum (LS) $\leq (f(b) - f(a)) \cdot (\text{mesh of } P)$.

Proof: The Riemann sum and the exact area are both between the upper and lower sums so the distance between the Riemann sum and the exact area is less than or equal to the distance between the upper and lower sums. Since f is monotonically increasing, the areas representing the difference of the upper and lower sums can be slid into a rectangle whose height equals $f(b)-f(a)$ and whose base equals the mesh of P. Then the total difference of the upper and lower sums is less than or equal to the area of the rectangle, $(f(b)-f(a))\cdot(\text{mesh of } P)$. □

1.3.4 Problems

For problems the next four problems, sketch the function and find the smallest possible value and the largest possible value for a Riemann sum of the given function and partition.

1. $f(x) = 1 + x^2$
   (a) $P = \{1, 2, 4, 5\}$
   (b) $P = \{1, 2, 3, 4, 5\}$
   (c) $P = \{1, 1.5, 2, 3, 4, 5\}$

2. $f(x) = 7 - 2x$
   (a) $P = \{0, 2, 3\}$
   (b) $P = \{0, 1, 2, 3\}$
   (c) $P = \{0, .5, 1, 1.5, 2, 3\}$

3. $f(x) = \sin(x)$
   (a) $P = \{0, \pi/2, \pi\}$
   (b) $P = \{0, \pi/4, \pi/2, \pi\}$
   (c) $P = \{0, \pi/4, 3\pi/4, \pi\}$.

4. $f(x) = x^2 - 2x + 3$
   (a) $P = \{0, 2, 3\}$
   (b) $P = \{0, 1, 2, 3\}$
   (c) $P = \{0, .5, 1, 2, 2.5, 3\}$.

5. Suppose we divide the interval $[1, 4]$ into 100 equally wide subintervals and calculate a Riemann sum for $f(x) = 1 + x^2$ by randomly selecting a point c_i in each subinterval.

   (a) We can be certain that the value of the Riemann sum is within what distance of the exact value of the area between the graph of f and the interval $[1, 4]$?

   (b) What if we take 200 equally long subintervals?

6. If f is monotonic decreasing on $[a, b]$ and we divide the interval $[a, b]$ into n equally wide subintervals, then we can be certain that the Riemann sum is within what distance of the exact value of the area between f and the interval $[a, b]$?

7. Suppose $LS = 7.362$ and $US = 7.402$ for a positive function f and a partition P of the interval $[1, 5]$.

   (a) We can be certain that every Riemann sum for the partition P is within what distance of the exact value of the area under the graph of f over the interval $[1, 5]$?

   (b) What if $LS = 7.372$ and $US = 7.390$?

1.3.5 The trapezoidal rule

This section includes several techniques for getting approximate numerical values for definite integrals without using antiderivatives. Mathematically, exact answers are preferable and satisfying, but for most applications, a numerical answer with several digits of accuracy is just as useful. For instance, suppose you are a automotive or aircraft designer. You may wish to know how much metal is required to build your design, created using a computer-aided design graphics program. Due to the fluctuations in the price for metal, you only need the approximate cost based on a piecewise-linear approximation to your design. Numerical techniques such as those discussed in this section can be used for that.

The methods in this section approximate the definite integral of a function f by building "easy" functions close to f and then exactly evaluating the definite integrals of the "easy" functions. If the "easy" functions are close enough to f, then the sum of the definite integrals of the "easy" functions will be close to the definite integral of f. The Left, Right and Midpoint approximations fit horizontal lines to f , the "easy" functions are piecewise constant functions, and the approximating regions are rectangles. The Trapezoidal Rule fits slanted lines to f , the "easy" functions are piecewise linear, and the approximating regions are trapezoids. Finally, Simpson's Rule fits parabolas to f, and the "easy" functions are piecewise quadratic polynomials.

All of the methods divide the interval $[a, b]$ into n equally long subintervals. Each subinterval has length $h = \Delta x_i = \frac{b-a}{n}$, and the points of the partition are $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, $\ldots$, $x_i = a + ih$, $\ldots$, $x_n = a + nh = a + n(\frac{b-a}{n}) = b$.

If the graph of f is curved, then slanted lines typically come closer to the graph of f than horizontal ones do, and the slanted lines lead to trapezoidal regions.

The area of a trapezoid is (base)$\times$(average height), so the area of the trapezoid with coordinates $(x_0, 0)$, (x_0, y_0), $(x_1, 0)$, (x_1, y_1), is (see Figure 1.28),

$$(x_1 - x_0)\frac{y_0 + y_1}{2}.$$

Example 1.3.8. *Here is how to create and plot a piecewise linear function describing the trapezoidal approximation to the area under $y = x^3 - 3x^2 + 2x$.*

```
sage: x = var("x")
sage: f1 = lambda x: x^3-3*x^2+2*x
sage: f = Piecewise([[(0,2),f1]])
sage: tf = f.trapezoid(4)
sage: P3 = list_plot([(1/2,0),(0.5,f(0.5))],plotjoined=True,linestyle=":")
sage: P4 = list_plot([(3/2,0),(1.5,f(1.5))],plotjoined=True,linestyle=":")
sage: show(P1+P2+P3+P4)
sage: f.trapezoid_integral_approximation(4)
0
```

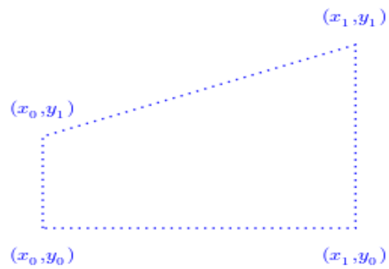

Figure 1.28: Trapezoid.

```
sage: integrate(x^3-3*x^2+2*x, x, 0, 2)
0
```

Here is the plot:

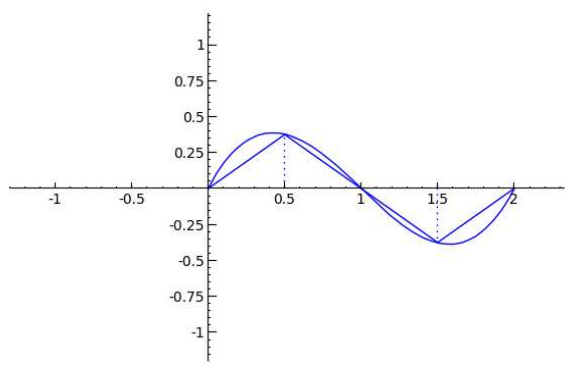

Figure 1.29: Plot using Sage of a trapezoidal approximation to the integral $\int_0^2 x^3 - 3x^2 + 2x\,dx$.

We got lucky here since both the integral and its trapezoidal approximation actually have the same value.

Theorem 1.3.3. *("Trapezoidal Approximation Rule") If f is integrable on $[a, b]$, and $[a, b]$ is partitioned into n subintervals of length $h = \frac{b-a}{n}$, then the Trapezoidal approximation of $\int_a^b f(x)\,dx$ is*

$$T_n = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)].$$

x	$f(x)$
1.0	4.2
1.5	3.4
2.0	2.8
2.5	3.6
3.0	3.2

Figure 1.30: Table for 1.3.9

Proof: The area of the trapezoidal with coordinates $(x_i, 0)$, (x_i, y_i), $(x_{i+1}, 0)$, (x_{i+1}, y_{i+1}), where $y_i = f(x_i)$, is $(x_{i+1} - x_i)\frac{y_i+y_{i+1}}{2} = \frac{h}{2} \cdot (y_i + y_{i+1})$. Therefore, the sum of the trapezoidal areas approximating $\int_a^b f(x)\,dx$ is

$$\sum_{i=}^{n}(x_{i+1} - x_i)\frac{y_i + y_{i+1}}{2} = \frac{h}{2} \cdot \sum_{i=}^{n}(y_i + y_{i+1}) = \frac{h}{2} \cdot (y_0 + 2y_1 + ... + 2y_{n-1} + y_n),$$

as desired. □

Example 1.3.9. *Calculate T_4, the Trapezoidal approximation of $\int_1^3 f(x)\,dx$, for the function values tabulated in Figure 1.30.*

Solution: The step size is $h = (b - a)/n = (3 - 1)/4 = 1/2$. Then

$$\begin{aligned} T_4 &= \tfrac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)] \\ &= \tfrac{1}{4}[4.2 + 2(3.4) + 2(2.8) + 2(3.6) + (3.2)] \\ &= (0.25)(27) = 6.75. \end{aligned}$$

Example 1.3.10. *Let's see how well the trapezoidal rule approximates an integral whose exact value we know, $\int_1^3 x^2\,dx = \frac{26}{3}$. Calculate T_4, the Trapezoidal approximation of $\int_1^3 x^2\,dx$.*

Solution: As in the example above, $h = 0.5$ and $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, and $x_4 = 3$. Then

$$\begin{aligned} T_4 &= \tfrac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)] \\ &= 0.5[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)] \\ &= (0.25)[1 + 2(2.25) + 2(4) + 2(6.25) + 9] = 8.75. \end{aligned}$$

Using Sage*, one can show that $T_{10} = 217/25 = 8.68$, $T_{100} = 21667/2500 = 8.6668$, and $T_{1000} = 2166667/250000 = 8.666668$. These trapezoidal approximations are indeed approaching the value 8.666... of the integral.*

Example 1.3.11. *Here is a Python program illustrating the Trapezoidal rule.*

```
f = lambda x: sin(x)
def trapezoidal_rule(fcn,a,b,n):
```

```
"""
Does computation of the Trapezoidal rule approx of int_a^b fcn(x) dx
using n steps.

"""
Deltax = (b-a)*1.0/n
coeffs = [2]*(n-1)
coeffs = [1]+coeffs+[1]
valsf = [f(a+Deltax*i) for i in range(n+1)]
return (Deltax/2)*sum([coeffs[i]*valsf[i] for i in range(n+1)])
```

Now we paste this into Sage *(you may instead paste into Python if you wish) and see how it works.*

```
sage: f = lambda x: sin(x)
sage: def trapezoidal_rule(fcn,a,b,n):
....:       """
....:    Does computation of the Trapezoidal rule approx of int_a^b fcn(x) dx
....:    using n steps.
....:
....:    """
....:    Deltax = (b-a)*1.0/n
....:    coeffs = [2]*(n-1)
....:    coeffs = [1]+coeffs+[1]
....:    valsf = [f(a+Deltax*i) for i in range(n+1)]
....:    return (Deltax/2)*sum([coeffs[i]*valsf[i] for i in range(n+1)])
....:

sage: integral(f(x),x,0,1)
1 - cos(1)
sage: RR(integral(f(x),x,0,1))
0.459697694131860
sage: trapezoidal_rule(f(x),0,1,4)
0.457300937571502
```

You see $\int_0^1 \sin(x)\,dx = 1 - \cos(1) = 0.459...$, *whereas the trapezoidal rule gives the approximation* $T_4 = 0.457....$

1.3.6 Simpson's rule and Sage

If the graph of f is curved, even the slanted lines may not fit the graph of f as closely as we would like, and a large number of subintervals may still be needed with the Trapezoidal rule to get a good approximation of the definite integral. Curves typically fit the graph of f better than straight lines, and the easiest

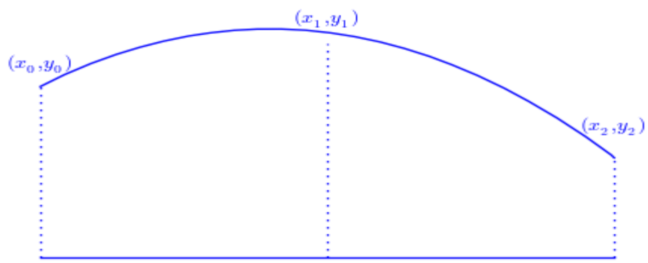

Figure 1.31: Piece of a parabola.

Theorem 1.3.4. *Three points (x_0, y_0), (x_1, y_1), (x_2, y_2) are needed to determine the equation of a parabola, and the area under a parabolic region with evenly spaced x_i values (Figure 1.31) is*

$$\frac{\Delta x}{3}(y_0 + 4y_1 + y_2).$$

Theorem 1.3.5. *("Simpson's Rule") If f is integrable on $[a, b]$, and $[a, b]$ is partitioned into an even number n of subintervals of length $h = \Delta x = \frac{b-a}{n}$, then the Parabolic approximation of $\int_a^b f(x)\,dx$ is*

$$S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + ... + 4f(x_{n-1}) + f(x_n)].$$

Example 1.3.12. *Calculate S_4, the Simpson's rule approximation of $\int_1^3 f(x)\,dx$, for the function values tabulated in Figure 1.30.*

Solution: The step size is $h = (b - a)/n = (3 - 1)/4 = 1/2$. Then

$$\begin{aligned} S_4 &= \tfrac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \\ &= \tfrac{1}{6}[4.2 + 4(3.4) + 2(2.8) + 4(3.6) + (3.2)] \\ &= \tfrac{41}{6} = 6.83\ldots. \end{aligned}$$

Example 1.3.13. *Here is a Python program illustrating Simpson's rule.*

```
f = lambda x: sin(x)
def simpsons_rule(fcn,a,b,n):
    """
```

```
    Does computation of the Simpson's rule approx of int_a^b fcn(x) dx
    using n steps. Here n must be an even integer.

    """
    Deltax = (b-a)*1.0/n
    n2 = int(n/2)
    coeffs = [4,2]*n2
    coeffs = [1]+coeffs[:n-1]+[1]
    valsf = [f(a+Deltax*i) for i in range(n+1)]
    return (Deltax/3)*sum([coeffs[i]*valsf[i] for i in range(n+1)])
```

Now we paste this into Sage *and see how it works:*

```
sage: f = lambda x: sin(x)
sage: def simpsons_rule(fcn,a,b,n):
....:        """
....:    Does computation of the Simpson's rule approx of int_a^b fcn(x) dx
....:    using n steps. Here n must be an even integer.
....:
....:    """
....:    Deltax = (b-a)*1.0/n
....:    n2 = int(n/2)
....:    coeffs = [4,2]*n2
....:    coeffs = [1]+coeffs[:n-1]+[1]
....:    valsf = [f(a+Deltax*i) for i in range(n+1)]
....:    return (Deltax/3)*sum([coeffs[i]*valsf[i] for i in range(n+1)])
....:
sage: integral(f(x),x,0,1)
1 - cos(1)
sage: RR(integral(f(x),x,0,1))
0.459697694131860
sage: simpsons_rule(f(x),0,1,4)
(sin(1) + 4*sin(3/4) + 2*sin(1/2) + 4*sin(1/4))/12
sage: RR(simpsons_rule(f(x),0,1,4))
0.459707744927311
sage: RR(simpsons_rule(f(x),0,1,10))
0.459697949823821
```

To paste this into Python, you mus first import the sin function.

```
>>> from math import sin
>>> f = lambda x: sin(x)
```

```
>>> def simpsons_rule(fcn,a,b,n):
...     """
...     Does computation of the Simpson's rule approx of int_a^b fcn(x) dx
...     using n steps. Here n must be an even integer.
...
...     """
...     Deltax = (b-a)*1.0/n
...     n2 = int(n/2)
...     coeffs = [4,2]*n2
...     coeffs = [1]+coeffs[:n-1]+[1]
...     valsf = [f(a+Deltax*i) for i in range(n+1)]
...     return (Deltax/3)*sum([coeffs[i]*valsf[i] for i in range(n+1)])
...
>>> simpsons_rule(f,0,1,4)
0.45970774492731092
```

Using this and the trapezoidal approximation function, we can compare which is best in this example.

```
sage: simpsons_rule(f(x),0,1,4)
0.459707744927311
sage: trapezoidal_rule(f(x),0,1,4)
0.457300937571502
sage: simpsons_rule(f(x),0,1,10)
0.459697949823821
sage: trapezoidal_rule(f(x),0,1,10)
0.459314548857976
sage: integral(f(x),x,0,1)
1 - cos(1)
sage: integral(f(x),x,0,1)*1.0
1.00000000000000*(1 - cos(1))
sage: RR(integral(f(x),x,0,1))
0.459697694131860
```

We see Simpson's rule wins every time.

1.3.7 Trapezoidal vs. Simpson: Which Method Is Best?

The hardest and slowest part of these approximations, whether by hand or by computer, is the evaluation of the function at the x_i values. For n subintervals, all of the methods require about the same number of function evaluations. The rest of this section discusses "error bounds" of the approximations so we can know how close our approximation is to the exact value of the integral even if we don't know the exact value.

Theorem 1.3.6. *(Error Bound for Trapezoidal Approximation) If the second derivative of f is continuous on $[a, b]$ and $M_2 \geq \max_{x\in[a,b]} |f''(x)|$, then the "error of the T_n approximation" is*

$$|\int_a^b f(x)\,dx - T_n| \leq \frac{(ba)^3}{12n^2} M_2.$$

The "error bound" formula $\frac{(ba)^3}{12n^2} M_2$ for the Trapezoidal approximation is a "guarantee": the actual error is guaranteed to be no larger than the error bound. In fact, the actual error is usually much smaller than the error bound. The word "error" does not indicate a mistake, it means the deviation or distance from the exact answer.

Example 1.3.14. *How large must n be to be certain that T_n is within 0.001 of $\int_0^1 \sin(x)\,dx$?*

Solution: We want to pick n so that $\frac{(ba)^3}{12n^2} M_2 \leq 1/1000$. We may take $M_2 = 1$, so $\frac{(ba)^3}{12n^2} M_2 = \frac{1}{12n^2} \leq 1/1000$, or $n^2 \geq 1000/12$. Taking $n = 10$ will work.

Theorem 1.3.7. *(Error Bound for Simpson's Rule Approximation) If the fourth derivative of f is continuous on $[a, b]$ and $M_4 \geq \max_{x\in[a,b]} |f^{(4)}(x)|$, then the "error of the S_n approximation" is*

$$|\int_a^b f(x)\,dx - S_n| \leq \frac{(ba)^5}{180n^4} M_4.$$

Example 1.3.15. *How large must n be to be certain that S_n is within 0.001 of $\int_0^1 \sin(x)\,dx$?*

Solution: We want to pick n so that $\frac{(ba)^5}{180n^4} M_2 \leq 1/1000$. We may take $M_2 = 1$, so $\frac{(ba)^5}{180n^4} M_4 = \frac{1}{180n^4} \leq 1/1000$, or $n^4 \geq 1000/180$. Taking $n = 2$ will work.

1.4 The definite integral

Each particular Riemann sum depends on several things: the function f, the interval $[a, b]$, the partition P of the interval, and the values chosen for c_k in each subinterval. Fortunately, for most of the functions needed for applications, as the approximating rectangles get thinner (as the mesh of P approaches 0 and the number of subintervals gets bigger) the values of the Riemann sums approach the same value independently of the particular partition P and the points c_k. For these functions, the *limit* (as the mesh approaches 0) of the Riemann sums is the same number no matter how the c_k's are chosen. This limit of the Riemann sums is the next big topic in calculus, the definite integral. Integrals arise throughout the rest of this book and in applications in almost every field that uses mathematics.

Definition 1.4.1. *If* $\lim_{\mathrm{mesh}(P)\to 0}\sum_{k=1}^{n} f(c_k)\Delta x_k$ *equals a finite number* I *then* f *is said to be* **(Riemann) integrable** *on the interval* $[a,b]$.

The number I *is called the* **definite integral** *of* f *over* $[a,b]$ *and is written* $\int_a^b f(x)\,dx$.

The symbol $\int_a^b f(x)\,dx$ is read "the integral from a to b of f of x dee x"or "the integral from a to b of $f(x)$ with respect to x." The **lower limit** is a, **upper limit** is b, the **integrand** is $f(x)$, and x is sometimes called the **dummy variable**. Note that $\int_a^b f(u)\,du$ numerically means exactly the same thing, but with a different dummy variable. The value of a definite integral $\int_a^b f(x)\,dx$ depends only on the function f being integrated and on the endpoints a and b. The following integrals each represent the integral of the function f on the interval $[a,b]$, and they are all equal:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du = \int_a^b f(z)\,dz.$$

Also, note that when the upper limit and the lower limit are the same then the integral is always 0:

$$\int_a^a f(x)dx = 0.$$

There are many other properties, as we will see later.

Example 1.4.1. *(Relation between velocity and area)*

Suppose you're reading a car magazine and there is an article about a new sports car that has this table in it:

Time (seconds)	*0*	*1*	*2*	*3*	*4*	*5*	*6*
Speed (mph)	*0*	*5*	*15*	*25*	*40*	*50*	*60*

They claim the car drove $1/8$*th of a mile after* 6 *seconds, but this just "feels" wrong... Hmmm... Let's estimate the distance driven using the formula*

$$\textit{distance} \;=\; \textit{rate} \;\times\; \textit{time}.$$

We overestimate by assuming the velocity is a constant equal to the max on each interval:

$$\textit{estimate} \;= 5\cdot 1 + 15\cdot 1 + 25\cdot 1 + 40\cdot 1 + 50\cdot 1 + 60\cdot 1 = \frac{195}{3600}\ \textit{miles}\ = 0.054...$$

(Note: there are 3600 *seconds in an hour.) But* $1/8 \sim 0.125$*, so the article is inconsistent. (Doesn't this sort of thing just bug you? By learning calculus you'll be able to double-check things like this much more easily.)*

Insight! The formula for the estimate of distance traveled above looks exactly like an approximation for the area under the graph of the speed of the car! *In* *fact, if an object has velocity $v(t)$ at time t, then the net change in position from time a to b is*

$$\int_a^b v(t)dt.$$

If f is a velocity, then the integrals on the intervals where f is positive measure the distances moved forward; the integrals on the intervals where f is negative measure the distances moved backward; and the integral over the whole time interval is the total (net) change in position, the distance moved forward minus the distance moved backward.

Practice 1.4.1. *A bug starts at the location $x = 12$ on the x–axis at 1 pm and walks along the axis in the positive direction with the velocity shown in Figure 1.32. How far does the bug travel between 1 pm and 3 pm, and where is the bug at 3 pm?*

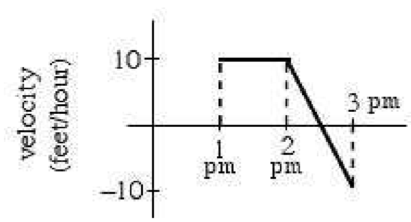

Figure 1.32: Velocity of a bug on the x–axis.

Practice 1.4.2. *A car is driven with the velocity west shown in Figure 1.33.*
(a) Between noon and 6 pm how far does the car travel?
(b) At 6 pm, where is the car relative to its starting point (its position at noon)?

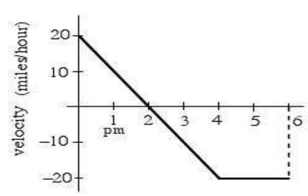

Figure 1.33: Velocity of a car on the x–axis.

Units For the Definite Integral We have already seen that the "area" under a graph can represent quantities whose units are not the usual geometric units of square meters or square feet. In general, the units for the definite integral $\int_a^b f(x)dx$ are (units for $f(x)$)×(units for x). A quick check of the units can help avoid errors in setting up an applied problem.

For example, if x is a measure of time in seconds and $f(x)$ is a velocity with units feet/second, then Δx has the units seconds and $f(x)\Delta x$ has the units (feet/second)(seconds) = feet, a measure of distance. Since each Riemann sum $\sum f(x_k)\Delta x_k$ is a sum of feet and the definite integral is the limit of the Riemann sums, the definite integral $\int_a^b f(x)dx$, has the same units, feet.

If $f(x)$ is a force in grams, and x is a distance in centimeters, then $\int_a^b f(x)dx$ is a number with the units "gram·centimeters," a measure of work.

1.4.1 The Fundamental Theorem of Calculus

Example 1.4.2. *For the function $f(t) = 2$, define $A(x)$ to be the area of the region bounded by the graph of f, the t–axis, and vertical lines at $t = 1$ and $t = x$.*

(a) Evaluate $A(1)$, $A(2)$, $A(3)$, $A(4)$.

(b) Find an algebraic formula for $A(x)$, for $x \geq 1$.

(c) Calculate $\frac{d}{dx}A(x)$.

(d) Describe $A(x)$ as a definite integral.

Solution : (a) $A(1) = 0$, $A(2) = 2$, $A(3) = 4$, $A(4) = 6$. *(b)* $A(x) =$ *area of a rectangle* $=$ *(base)×(height)* $= (x-1)\cdot(2) = 2x-2$. *(c)* $\frac{d}{dx}A(x) = \frac{d}{dx}(2x-2) = 2$. *(d)* $A(x) = \int_1^x 2\,dt$.

Practice 1.4.3. *Answer the questions in the previous Example for $f(t) = 3$.*

A curious "coincidence" appeared in this Example and Practice problem: the derivative of the function defined by the integral was the same as the integrand, the function "inside" the integral. Stated another way, the function defined by the integral was an "antiderivative" of the function "inside" the integral. We will see that this is no coincidence: it is an important property called The Fundamental Theorem of Calculus.

Let f be a continuous function on the interval $[a, b]$.

Theorem 1.4.1. *("Fundamental Theorem of Calculus") If $F(x)$ is any differentiable function on $[a, b]$ such that $F'(x) = f(x)$, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

The above theorem is *incredibly* useful in mathematics, physics, biology, etc. One reason this is amazing, is because it says that the area under the entire curve is completely determined by the values of a ("magic") auxiliary function *at only* 2 *points*. It's hard to believe. It reduces computing (1.4.1) to finding a single function F, which one can often do algebraically, in practice. Whether or not one should use this theorem to evaluate an integral depends a lot on the application at hand, of course. One can also use a partial limit via a computer for certain applications (numerical integration).

Example 1.4.3. *I've always wondered exactly what the area is under a "hump" of the graph of* $\sin$*. Let's figure it out, using* $F(x) = -\cos(x)$*.*

$$\int_0^\pi \sin(x)dx = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 2.$$

In Sage*, you can do this both "algebraically" and "numerically" as follows.*

```
sage: f = lambda x: sin(x)
sage: integral(f(x),x,0,pi)
2
sage: numerical_integral(f(x),0,pi)
(1.9999999999999998, 2.2204460492503128e-14)
```

For the "algebraic" computation, Sage *knows how to integrate* $\sin(x)$ *exactly, so can compute* $\int_0^\pi \sin(x)dx = 2$ *using its* `integral` *command*[4]*. On the last line of output, the first entry is the approximation, and the second is the error bound. For the "numerical" computation,* Sage *obtains*[5] *the approximation* $\int_0^\pi \sin(x)dx \approx 1.99999...$ *by taking enough terms in a Riemann sum to achieve a very small error. (A lot of theory of numerical integration goes into why* `numerical_integral` *works correctly, but that would take us too far afield to explain here.)*

Example 1.4.4. *Let* $[...]$ *denote the "greatest integer" (or "floor") function, so* $[1/2] = [0.5] = 0$ *and* $[3/2] = [1.5] = 1$*. Evaluate* $\int_{1/2}^{3/2} [x]\, dx$*. (The function of* $y = [x]$ *is sometimes called the "staircase function" because of the look of its discontinuous graph, Figure 1.34.)*

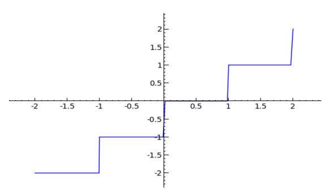

Figure 1.34: Plot of the "greatest integer" function.

Solution: $f(x) = [x]$ *is not continuous at* $x = 1$ *in the interval* $[1/2.3/2]$ *so the Fundamental Theorem of Calculus can not be used. We can, however, use our understanding of the meaning of an integral as an area to get* $\int_{1/2}^{3/2} [x]\, dx$

[4]In fact, Sage includes Maxima (`http://maxima.sf.net`) and calls Maxima to compute this integral.

[5]In fact, Sage includes the GNU Scientific Library (`http://www.gnu.org/software/gsl/`) and calls it to approximate this integral.

= (area under $y = 0$ between 0.5 and 1) + (area under $y = 1$ between 1 and 1.5)$=0 + 1/2 = 1/2$.

Now, let's try something illegal - using the Fundamental Theorem of Calculus to evaluate this. Pretend for the moment that the Fundamental Theorem of Calculus is valid for discontinuous functions too. Let

$$F(x) = \begin{cases} 1, & 1/2 \le x \le 1, \\ x, & 1 < x \le 3/2. \end{cases}$$

This function F is continuous and satisfies $F'(x) = [x]$ for all x in $[1/2, 3/2]$ except $x = 1$ (where $f(x) = [x]$ is discontinuous), so this F could be called an "antiderivative" of f. If we use it to evaluate the integral we get $\int_{1/2}^{3/2} [x]\, dx = F(x)|_{1/2}^{3/2} = 3/2 - 1 = 1/2$. This is correct. (Surprised?) Let's try another antiderivative. Let

$$F(x) = \begin{cases} 2, & 1/2 \le x \le 1, \\ x, & 1 < x \le 3/2. \end{cases}$$

This function F also satisfies $F'(x) = [x]$ for all x in $[1/2, 3/2]$ except $x = 1$. If we use it to evaluate the integral we get $\int_{1/2}^{3/2} [x]\, dx = F(x)|_{1/2}^{3/2} = 3/2-2 = -1/2$. This doesn't even have the right sign (the integral of a non-negative function must be non-negative!), so it must be wrong. Moral of the story: In general, the Fundamental Theorem of Calculus is false for discontinuous functions.

But does such an F as in the fundamental theorem of calculus (Theorem 1.4.1) always exist? The surprising answer is "yes".

Theorem 1.4.2. *Let $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$ for all $x \in [a, b]$.*

Note that a "nice formula" for F can be hard to find or even provably non-existent.

The proof of Theorem 1.4.2 is somewhat complicated but is given in complete detail in many calculus books, and you should definitely (no pun intended) read and understand it.

Proof: [Sketch of Proof] We use the definition of derivative.

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \to 0} \left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right) /h \\ &= \lim_{h \to 0} \left(\int_x^{x+h} f(t)dt \right) /h \end{aligned}$$

Intuitively, for h sufficiently small f is essentially constant, so $\int_x^{x+h} f(t)dt \sim hf(x)$ (this can be made precise using the extreme value theorem). Thus

$$\lim_{h\to 0}\left(\int_x^{x+h} f(t)dt\right)/h = f(x),$$

which proves the theorem. □

1.4.2 Problems

In problems 1 – 4 , rewrite the limit of each Riemann sum as a definite integral.

1. $\lim_{\text{mesh}(P)\to 0}\sum_{k=1}^{n}(2+3c_k)\Delta x_k$ on the interval $[0,4]$.
2. $\lim_{\text{mesh}(P)\to 0}\sum_{k=1}^{n}\cos(5c_k)\Delta x_k$ on the interval $[0,11]$.
3. $\lim_{\text{mesh}(P)\to 0}\sum_{k=1}^{n}c_k^3\Delta x_k$ on the interval $[2,5]$.
4. $\lim_{\text{mesh}(P)\to 0}\sum_{k=1}^{n}c_k\Delta x_k$ on the interval $[2,5]$.
5. Write as a definite integral (don't evaluate it though): The region bounded by $y = x^3$, the x–axis, the line $x = 1$, and $x = 5$.
6. Write as a definite integral (don't evaluate it though): The region bounded by $y = \sqrt{x}$, the x–axis, and the line $x = 9$.
7. Write as a definite integral (*do* evaluate it, using geometry formulas): The region bounded by $y = 2x$, the x–axis, the line $x = 1$, and $x = 3$.
8. Write as a definite integral (*do* evaluate it, using geometry formulas): The region bounded by $y = |x|$, the x–axis, and the line $x = -1$.
9. For $f(x) = 3+x$, partition the interval $[0,2]$ into n equally wide subintervals of length $\Delta x = 2/n$.

   (a) Write the lower sum for this function and partition, and calculate the limit of the lower sum as $n \to \infty$. (b) Write the upper sum for this function and partition and find the limit of the upper sum as $n \to \infty$.

10. For $f(x) = x^3$, partition the interval $[0,2]$ into n equally wide subintervals of length $\Delta x = 2/n$.

    (a) Write the lower sum for this function and partition, and calculate the limit of the lower sum as $n \to \infty$.

    (b) Write the upper sum for this function and partition and find the limit of the upper sum as $n \to \infty$.

1.4.3 Properties of the definite integral

Definite integrals are defined as limits of Riemann sums, and they can be interpreted as "areas" of geometric regions. This section continues to emphasize this geometric view of definite integrals and presents several properties of definite integrals. These properties are justified using the properties of summations and the definition of a definite integral as a Riemann sum, but they also have natural interpretations as properties of areas of regions. These properties are used in this section to help understand functions that are defined by integrals. They will be used in future sections to help calculate the values of definite integrals.

Since integrals are a lot like sums (they are, after all, limits of them), their properties are similar too. Here is the integral analog of Theorem 1.3.1.

Theorem 1.4.3. *(Integral Properties)*

- *Integral of a constant function:* $\int_a^b c\,dx = c \cdot (b-a)$.
- *Addition:* $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
- *Subtraction:* $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.
- *Constant Multiple:* $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$.
- *Preserves positivity: If* $f(x) \geq g(x)$ *on for all* $x \in [a,b]$, *then*

  $$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

  In particular, if $f(x) \geq 0$ *on for all* $x \in [a,b]$, *then*

  $$\int_a^b f(x)\,dx \geq 0.$$

- *Additivity of ranges:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.

Here are some other properties.

Theorem 1.4.4.

$$(b-a)\cdot(\min_{x\in[a,b]} f(x)) \leq \int_a^b f(x)\,dx \leq (b-a)\cdot(\min_{x\in[a,b]} f(x)).$$

Which Functions Are Integrable? This important question was finally answered in the 1850s by Georg Riemann, a name that should be familiar by now. Riemann proved that a function must be badly discontinuous to not be integrable.

Theorem 1.4.5. *Every continuous function is integrable. If f is continuous on the interval $[a,b]$, then $\lim_{\mathrm{mesh}(P)\to 0}(\sum_{k=1}^n f(c_k)\Delta x_k)$ is always the same finite number, namely, $\int_a^b f(x)\,dx$, so f is integrable on $[a,b]$.*

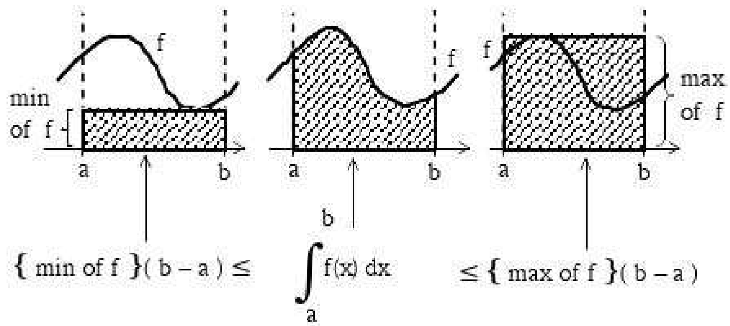

Figure 1.35: Plot illustrating Theorem 1.4.4.

In fact, a function can even have any finite number of breaks and still be integrable.

Theorem 1.4.6. *Every bounded, piecewise continuous function is integrable. If f is defined and bounded (for all x in $[a, b]$, $M \leq f(x) \leq M$ for some $M > 0$), and continuous except at a finite number of points in $[a, b]$, then $\lim_{\text{mesh}(P)\to 0}(\sum_{k=1}^{n} f(c_k)\Delta x_k)$ is always the same finite number, namely, $\int_a^b f(x)\,dx$, so f is integrable on $[a, b]$.*

Example 1.4.5. *(A Nonintegrable Function)*
Though rarely encountered in "everyday practice", there are functions for which the limit of the Riemann sums does not exist, and those functions are not integrable.
A nonintegrable function: The function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is a rational number,} \\ 0, & \text{if } x \text{ is an irrational number} \end{cases}$$

is not integrable on $[0, 1]$.
Proof: *For any partition P, suppose that you, a very rational (pun intended) person, always select values of c_k which are rational numbers. (Every subinterval contains rational numbers and irrational numbers, so you can always pick c_k to be a rational number.) Then $f(c_k) = 1$, and your Riemann sum is always*

$$Y_P = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} \Delta x_k = x_n - x_0 = 1.$$

Suppose your friend, however, always selects values of c_k which are irrational numbers. Then $f(c_k) = 0$, and your friend's Riemann sum is always

$$F_P = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} 0 \cdot \Delta x_k = 0.$$

Now, take finer and finer partitions P so that $\text{mesh}(P) \to 0$*. Keep in mind that, no matter how you refine P, you can always make "rational choices" for c_k and your friend can always make "irrational choices". We have* $\lim_{\text{mesh}(P)\to 0} Y_P = 1$ *and* $\lim_{\text{mesh}(P)\to 0} F_P = 0$*, so the limit of the Riemann sums doesn't have a unique value. Therefore the limit*

$$\lim_{\text{mesh}(P)\to 0}\left(\sum_{k=1}^{n} f(c_k)\Delta x_k\right)$$

does not exist, so f is not integrable.

1.4.4 Problems

Problems 1 – 20 refer to the graph of f in Figure 1.36. Use the graph to determine the values of the definite integrals. (The bold numbers represent the area of each region.)

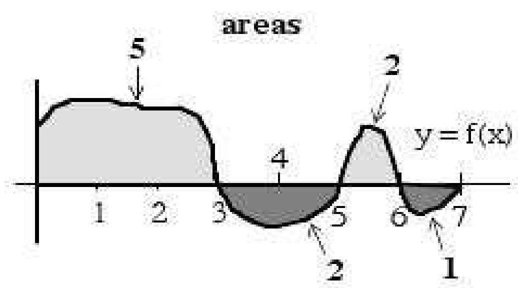

Figure 1.36: Plot for problems.

1. $\int_0^3 f(x)\,dx$
2. $\int_3^5 f(x)\,dx$
3. $\int_2^2 f(x)\,dx$
4. $\int_6^7 f(x)\,dx$
5. $\int_0^5 f(x)\,dx$
6. $\int_0^7 f(x)\,dx$
7. $\int_3^6 f(x)\,dx$
8. $\int_5^7 f(x)\,dx$
9. $\int_3^0 f(x)\,dx$

10. $\int_5^3 f(x)\,dx$

11. $\int_6^0 f(x)\,dx$

12. $\int_0^3 2f(x)\,dx$

13. $\int_4^4 f(x)^2\,dx$

14. $\int_0^3 1+f(t)\,dt$

15. $\int_0^3 x+f(x)\,dx$

16. $\int_3^5 3+f(x)\,dx$

17. $\int_0^5 2+f(x)\,dx$

18. $\int_3^5 |f(x)|\,dx$

19. $\int_7^3 1+|f(x)|\,dx$

For problems 21–28, sketch the graph of the integrand function and use it to help evaluate the integral. (|...| denotes the absolute value and [...] denotes the integer part.)

21. $\int_0^4 |x|\,dx$,

22. $\int_0^4 1+|x|\,dx$,

23. $\int_{-1}^2 |x|\,dx$,

24. $\int_1^2 |x|-1\,dx$,

25. $\int_1^3 [x]\,dx$,

26. $\int_1^{3.5} [x]\,dx$,

27. $\int_1^3 2+[x]\,dx$,

28. $\int_3^1 [x]\,dx$.

1.5 Areas, integrals, and antiderivatives

This section explores properties of functions defined as areas and examines some of the connections among areas, integrals and antiderivatives. In order to focus on the geometric meaning and connections, all of the functions in this section are nonnegative, but the results are generalized in the next section and proved true for all continuous functions. This section also introduces examples to illustrate how areas, integrals and antiderivatives can be used. When f is a continuous, nonnegative function, then the "area function" $A(x) = \int_a^x f(t)\,dt$ represents the area between the graph of f, the t–axis, and between the vertical lines at $t = a$ and $t = x$ (Figure 1.37), and the derivative of $A(x)$ represents the rate of change (growth) of $A(x)$.

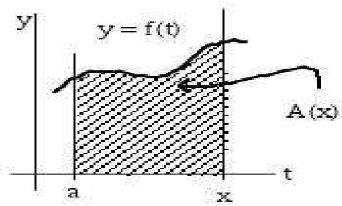

Figure 1.37: Plot of an "area function".

Let $F(x)$ be a differentiable function. Call $F(x)$ an **antiderivative** of $f(x)$ if $\frac{d}{dx}F(x) = f(x)$. We have seen examples which showed that, at least for *some* functions f, the derivative of $A(x)$ was equal to f so $A(x)$ was an antiderivative of f. The next theorem says the result is true for every continuous, nonnegative function f.

Theorem 1.5.1. *("The Area Function is an Antiderivative") If f is a continuous nonnegative function, $x \geq a$, and $A(x) = \int_a^x f(t)\,dt$ then $\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}A(x) = f(x)$, so $A(x)$ is an antiderivative of $f(x)$.*

This result relating integrals and antiderivatives is a special case (for nonnegative functions f) of the Fundamental Theorem of Calculus. This result is important for two reasons:

- it says that a large collection of functions have antiderivatives, and
- it leads to an easy way of exactly evaluating definite integrals.

x

Example 1.5.1. *Let $G(x) = \frac{d}{dx}\int_0^x \cos(t)dt$. Evaluate $G(x)$ for $x = \pi/4$, $\pi/2$, and $3\pi/4$.*

Solution: It is not hard to plot the graph of $A(x) = \int_0^x \cos(t)dt = \sin(x)$ (Figure 1.38). By the theorem, $A'(x) = G(x) = \cos(x)$ so $A'(\pi/4) = \cos(\pi/4) = .707...$, $A'(\pi/2) = \cos(\pi/2) = 0$, and $A'(3\pi/4) = \cos(3/4) = -0.707...$.

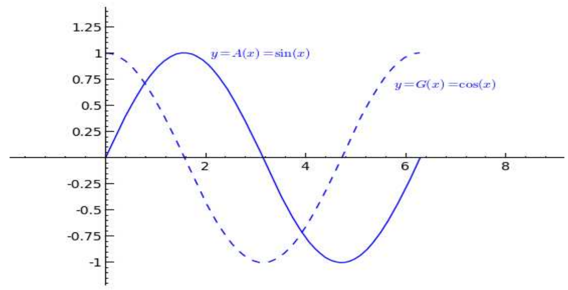

Figure 1.38: Plot of $y = \int_0^x G(t)\,dt$ and $y = G(x)$.

Here is the plot of $y = A(x)$ and $y = G(x)$:

Incidentally, this was created using the following Sage *commands.*

```
sage: P = plot(cos(x),x,0,2*pi,linestyle="--")
sage: Q = plot(sin(x),x,0,2*pi)
sage: R = text("$y=A(x) = \sin(x)$",(3.1,1))
sage: S = text("$y=G(x) = \cos(x)$",(6.8,0.7))
sage: show(P+Q+R+S)
```

Theorem 1.5.2. *("Antiderivatives and Definite Integrals") If f is a continuous, nonnegative function and F is any antiderivative of f ($F'(x) = f(x)$) on the interval $[a, b]$, then*

$$\boxed{\begin{array}{c}\textit{area bounded between the graph}\\ \textit{of } f \textit{ and the } x\textit{–axis and}\\ \textit{vertical lines at } x = a \textit{ and } x = b\end{array}} = \int_a^b f(x)\,dx = F(b) - F(a).$$

The problem of finding the exact value of a definite integral reduces to finding some (any) antiderivative F of the integrand and then evaluating $F(b)F(a)$. Even finding one antiderivative can be difficult, and, for now, we will stick to functions which have easy antiderivatives. Later we will explore some methods for finding antiderivatives of more difficult functions.

The evaluation $F(b) - F(a)$ is represented by the symbol $F(x)|_a^b$.

Example 1.5.2. *Evaluate $\int_1^3 x\,dx$ in two ways:*

(a) By sketching the graph of $y = x$ and geometrically finding the area.

(b) By finding an antiderivative of $F(x)$ of f and evaluating $F(3) - F(1)$.

Solution: (a) The graph of $y = x$ is a straight line, so the area is a triangle which geometrical formulas (area=$\frac{1}{2}bh$) tell us has area 4.

(b) One antiderivative of x is $F(x) = \frac{1}{2}x^2$ (check that $\frac{d}{dx}(\frac{1}{2}x^2) = x$), and

$$F(x)|_1^3 = F(3) - F(1) = \frac{1}{2}3^2 - \frac{1}{2}1^2 = 4,$$

which agrees with (a). Suppose someone chose another antiderivative of x, say $F(x) = \frac{1}{2}x^2 + 7$ (check that $\frac{d}{dx}(\frac{1}{2}x^2 + 7) = x$), then

$$F(x)|_1^3 = F(3) - F(1) = (\frac{1}{2}3^2 + 7) - (\frac{1}{2}1^2 + 7) = 4.$$

No matter which antiderivative F is chosen, $F(3) - F(1)$ equals 4.

Practice 1.5.1. *Evaluate $\int_1^3 (x-1)\,dx$ in the two ways of the previous example.*

Practice 1.5.2. *Find the area between the graph of $y = 3x^2$ and the horizontal axis for x between 1 and 2.*

1.5.1 Integrals, Antiderivatives, and Applications

The antiderivative method of evaluating definite integrals can also be used when we need to find an "area," and it is useful for solving applied problems.

Example 1.5.3. *Suppose that t minutes after putting 1000 bacteria on a petri plate the rate of growth of the population is $6t$ bacteria per minute.*

(a) How many new bacteria are added to the population during the first 7 minutes?

(b) What is the total population after 7 minutes?

(c) When will the total population be 2200 bacteria?

Solution: (a) The number of new bacteria is the area under the rate of growth graph, and one antiderivative of $6t$ is $3t^2$ (check that $\frac{d}{dx}(3t^2) = 6t$) so new bacteria = $\int_0^7 6t\,dt = 3t^2|_0^7 = 147$.

(b) The new population = (old population) + (new bacteria) = $1000 + 147 = 1147$ bacteria.

(c) If the total population is 2200 bacteria, then there are $2200 - 1000 = 1200$ new bacteria, and we need to find the time T needed for that many new bacteria to occur. 1200 new bacteria = $\int_0^T 6t\,dt = 3t^2|_0^T = 3(T)^2 - 3(0)^2 = 3T^2$ so $T^2 = 400$ and $T = 20$ minutes. After 20 minutes, the total bacteria population will be $1000 + 1200 = 2200$.

Practice 1.5.3. *A robot has been programmed so that when it starts to move, its velocity after t seconds will be $3t^2$ feet/second.*

(a) How far will the robot travel during its first 4 seconds of movement?

(b) *How far will the robot travel during its next 4 seconds of movement?*

(c) *How many seconds before the robot is 729 feet from its starting place?*

(Hint: an antiderivative of $3t^2$ *is* t^3*.)*

Practice 1.5.4. *The velocity of a car after* t *seconds is* $2t$ *feet per second.*

(a) *How far does the car travel during its first* 10 *seconds?*

(b) *How many seconds does it take the car to travel half the distance in part (a)?*

1.5.2 Indefinite Integrals and net change

We've seen how integrals can be interpreted using area. In this section, we will see how integrals can be interpreted physically as the "net change" of a quantity.

The notation $\int f(x)dx = F(x)$ means that $F'(x) = f(x)$ on some (usually specified) domain of definition of $f(x)$. Recall, we call such an $F(x)$ an antiderivative of $f(x)$.

Proposition 1.5.1. *Suppose* f *is a continuous function on an interval* (a, b)*. Then any two antiderivatives differ by a constant.*

Proof: If $F_1(x)$ and $F_2(x)$ are both antiderivatives of a function $f(x)$, then

$$(F_1(x) - F_2(x))' = F_1'(x) - F_2'(x) = f(x) - f(x) = 0.$$

Thus $F_1(x) - F_2(x) = c$ from some constant c (since only constant functions have slope 0 everywhere). Thus $F_1(x) = F_2(x) + c$ as claimed. □

We thus often write

$$\int f(x)dx = F(x) + C,$$

where C is an unspecified constant.

Note that the proposition need not be true if f is not defined on a whole interval. For example, $f(x) = 1/x$ is not defined at 0. For any pair of constants c_1, c_2, the function

$$F(x) = \begin{cases} \ln(|x|) + c_1 & x < 0, \\ \ln(x) + c_2 & x > 0, \end{cases}$$

satisfies $F'(x) = f(x)$ for all $x \neq 0$. We often still just write $\int 1/x = \ln(|x|) + c$ anyways, meaning that this formula is supposed to hold only on one of the intervals on which $1/x$ is defined (e.g., on $(-\infty, 0)$ or $(0, \infty)$).

We pause to emphasize the notation difference between definite and indefinite integration.

$$\begin{aligned} \int_a^b f(x)dx &= \text{a specific number} \\ \int f(x)dx &= \text{a (family of) functions} \end{aligned}$$

There are no small families in the world of antiderivatives: if f has one antiderivative F (as it always does, unless f is a really unusual function), then f has an infinite number of antiderivatives and every one of them has the form $F(x) + C$.

Example 1.5.4. *There are many ways to write a particular indefinite integral and some of them may look very different. You can check that* $F(x) = \sin(x)^2$*,* $G(x) = -\cos(x)^2$*, and* $H(x) = 2\sin(x)^2 + \cos(x)^2$ *all have the same derivative* $f(x) = 2\sin(x)\cos(x)$*, so the indefinite integral of* $2\sin(x)\cos(x)$*,* $\int 2\sin(x)\cos(x)\,dx$*, can be written in several ways:* $\sin(x)^2+C$*, or* $-\cos(x)^2+C$*, or* $2\sin(x)^2 + \cos(x)^2 + C$*.*

One of the *main goals* of this course is to help you to get really good at computing $\int f(x)dx$ for various functions $f(x)$. It is useful to memorize a table of examples, such as the one in the appendix below, since often the trick to integration is to relate a given integral to one you know. Integration is like solving a puzzle or playing a game, and often you win by moving into a position where you know how to defeat your opponent, e.g., relating your integral to integrals that you already know how to do. If you know how to do a basic collection of integrals, it will be easier for you to see how to get to a known integral from an unknown one.

Whenever you successfully compute $F(x) = \int f(x)dx$, then you've constructed a *mathematical gadget* that allows you to very quickly compute $\int_a^b f(x)dx$ for any a, b (in the interval of definition of $f(x)$). The gadget is $F(b) - F(a)$. This is really powerful.

Example 1.5.5.

$$\begin{aligned}\int x^2 + 1 + \frac{1}{x^2+1}dx &= \int x^2 dx + \int 1dx + \int \frac{1}{x^2+1}dx \\ &= \frac{1}{3}x^2 + x + \tan^{-1}(x) + c.\end{aligned}$$

Example 1.5.6.

$$\int \sqrt{\frac{5}{x}}dx = \int \sqrt{5}x^{-1/2}dx = 2\sqrt{5}x^{1/2} + c.$$

Example 1.5.7.

$$\int \frac{\sin(2x)}{\sin(x)}dx = \int \frac{2\sin(x)\cos(x)}{\sin(x)} = \int 2\cos(x) = 2\sin(x) + c$$

Particular Antiderivatives: You can verify the following yourself.

- Constant Function: $\int k\,dx = kx + C$
- Powers of x: $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$,
  $n \neq -1$.
  $\int x^{-1}\,dx = \ln(x) + C$.
  Common special cases:

  - $\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$.
  - $\int \frac{1}{\sqrt{x}}\,dx = 2x^{1/2} + C$.

- Trigonometric Functions:
  $\int \cos(ax)\,dx = \frac{1}{a}\sin(x) + C$.
  $\int \sin(ax)\,dx = -\frac{1}{a}\cos(x) + C$.
  $\int \sec(ax)^2\,dx = \frac{1}{a}\tan(x) + C$.
  $\int \csc(ax)^2\,dx = -\frac{1}{a}\cot(x) + C$.
  $\int \sec(ax)\tan(x)\,dx = \frac{1}{a}\sec(x) + C$.
  $\int \csc(ax)\cot(x)\,dx = -\frac{1}{a}\csc(x) + C$.
  Common special cases:

  - $\int \cos(x)\,dx = \sin(x) + C$.
  - $\int \sin(x)\,dx = -\cos(x) + C$.

1.5.3 Physical Intuition

In the previous lecture we mentioned a relation between velocity, distance, and the meaning of integration, which gave you a physical way of thinking about integration. In this section we generalize our previous observation.

The following is a restatement of the fundamental theorem of calculus.

Theorem 1.5.3. *(Net Change Theorem) The definite integral of the rate of change $f'(x)$ of some quantity $f(x)$ is the net change in that quantity:*

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

For example, if $p(t)$ is the population of your hometown at time t, then $p'(t)$ is the rate of change. If $p'(t)$ is positive then your hometown is growing. The net change interpretation of integration is that

$$\int_{t_1}^{t_2} p'(t)\,dt = p(t_2) - p(t_1) = \text{ change in number of residents from time } t_1 \text{ to } t_2.$$

Another very common example you'll seen in problems involves water flow into or out of something. If the volume of water in your bathtub is $V(t)$ gallons at time t (in seconds), then the rate at which your tub is draining is $V'(t)$ gallons per second. If you have the geekiest drain imaginable, it prints out the drainage rate $V'(t)$. You can use that printout to determine how much water drained out from time t_1 to t_2:

$$\int_{t_1}^{t_2} V'(t)\,dt \;=\; \text{water that drained out from time } t_1 \text{ to } t_2$$

Some problems will try to confuse you with different notions of change. A standard example is that if a car has **velocity** $v(t)$, and you drive forward, then slam it in reverse and drive backward to where you start (say 10 seconds total elapse), then $v(t)$ is positive some of the time and negative some of the time. The integral $\int_0^{10} v(t)dt$ is not the total distance registered on your odometer, since $v(t)$ is partly positive and partly negative. If you want to express how far you actually drove going back and forth, compute $\int_0^{10} |v(t)|dt$. The following example emphasizes this distinction:

Example 1.5.8. A bug is pacing back and forth, and has velocity $v(t) = t^2 - 2t - 8$. Find

(1) the **displacement** of the bug from time $t = 1$ until time $t = 6$ (i.e., how far the bug is at time 6 from where it was at time 1), and

(2) the *total distance* the bug paced from time $t = 1$ to $t = 6$.

For (1), we compute

$$\int_1^6 (t^2 - 2t - 8)\, dt = \left[\frac{1}{3}t^3 - t^2 - 8t\right]_1^6 = -\frac{10}{3}.$$

For (2), we compute the integral of $|v(t)|$*:*

$$\int_1^6 |t^2 - 2t - 8|\, dt = \left[-\left(\frac{1}{3}t^3 - t^2 - 8t\right)\right]_1^4 + \left[\frac{1}{3}t^3 - t^2 - 8t\right]_4^6 = 18 + \frac{44}{3} = \frac{98}{3}.$$

1.5.4 Problems

1. Two objects start from the same location and travel along the same path with velocities $v_A(t) = t + 3$ and $v_B(t) = t^2 4t + 3$ meters per second. How far ahead is A after 3 seconds? After 5 seconds?

2. Sketch the graph of each function and find the area between the graphs of $f(x) = x^2 + 3$, $g(x) = 1$ and $-1 \le x \le 2$.

3. Sketch the graph of each function and find the area between the graphs of $f(x) = x^2 + 3$, $g(x) = 1 + x$ and $0 \le x \le 3$.

4. Sketch the graph of each function and find the area between the graphs of $f(x) = x^2$, $g(x) = x$ and $0 \le x \le 2$.

5. Sketch the graph of each function and find the area between the graphs of $f(x) = x + 1$, $g(x) = \cos(x)$ and $0 \le x \le \pi/4$.

6. If $f(t)$ denoted the velocity of a bug traveling along a line at time t, find the distance traveled in the first 4 seconds.

7. If $f(t)$ denoted the velocity of a bug traveling along a line at time t, find the distance traveled in the first 4 seconds.

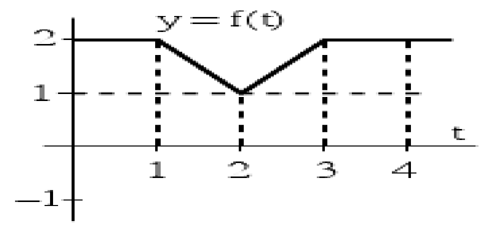

Figure 1.39: Velocity of bug at time t.

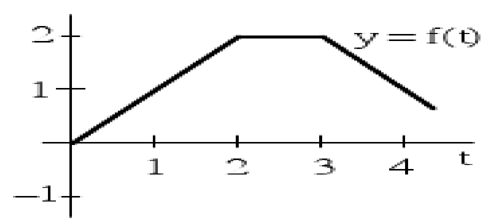

Figure 1.40: Velocity of bug at time t.

1.6 Substitution and Symmetry

Remarks:

1. The **total distance traveled** is $\int_{t_1}^{t_2} |v(t)|dt$ since $|v(t)|$ is the rate of change of $F(t) =$ distance traveled (your speedometer displays the rate of change of your odometer).

2. How to compute $\int_a^b |f(x)|dx$.

   (a) Find the zeros of $f(x)$ on $[a, b]$, and use these to break the interval up into subintervals on which $f(x)$ is always ≥ 0 or always ≤ 0.

   (b) On the intervals where $f(x) \geq 0$, compute the integral of f, and on the intervals where $f(x) \leq 0$, compute the integral of $-f$.

   (c) The sum of the above integrals on intervals is $\int |f(x)|dx$.

This section is primarily about a powerful technique for computing definite and indefinite integrals.

1.6.1 The Substitution Rule

In first quarter calculus you learned numerous methods for computing derivatives of functions. For example, the **power rule** asserts that

$$(x^a)' = a \cdot x^{a-1}.$$

We can turn this into a way to compute certain integrals:

$$\int x^a dx = \frac{1}{a+1} x^{a+1} \qquad \text{if } a \neq -1.$$

Just as with the power rule, many other rules and results that you already know yield *techniques* for integration. In general integration is potentially much trickier than differentiation, because it is often not obvious which technique to use, or even how to use it. *Integration is a more exciting than differentiation!*

Recall the **chain rule**, which asserts that

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

We turn this into a technique for integration as follows:

Proposition 1.6.1. *(Substitution Rule) Let $u = g(x)$, we have*

$$\int f(g(x))g'(x)dx = \int f(u)du,$$

assuming that $g(x)$ is a function that is differentiable and whose range is an interval on which f is continuous.

Proof: Since f is continuous on the range of g, Theorem 1.4.2 (the fundamental theorem of Calculus) implies that there is a function F such that $F' = f$. Then

$$\begin{aligned}
\int f(g(x))g'(x)dx &= \int F'(g(x))g'(x)dx \\
&= \int \left(\frac{d}{dx} F(g(x))\right) dx \\
&= F(g(x)) + C \\
&= F(u) + C = \int F'(u)du = \int f(u)du.
\end{aligned}$$

□

If $u = g(x)$ then $du = g'(x)dx$, and the substitution rule simply says if you let $u = g(x)$ formally in the integral everywhere, what you naturally would hope to be true based on the notation actually is true. The substitution rule illustrates how the notation Leibniz invented for Calculus is *incredibly brilliant*. It is said that Leibniz would often spend days just trying to find the right notation for a concept. He succeeded.

As with all of Calculus, the best way to start to get your head around a new concept is to see severally clearly worked out examples. (And the best way to actually be able to use the new idea is to *do* lots of problems yourself!) In this section we present examples that illustrate how to apply the substitution rule to compute indefinite integrals.

Example 1.6.1.

$$\int x^2(x^3+5)^9 dx$$

Let $u = x^3+5$. Then $du = 3x^2 dx$, hence $dx = du/(3x^2)$. Now substitute it all in:

$$\int x^2(x^3+5)^9 dx = \int \frac{1}{3}u^9 = \frac{1}{30}u^{10} = \frac{1}{30}(x^3+5)^{10}.$$

There's no point in expanding this out: "only simplify for a purpose*!"*

Example 1.6.2.

$$\int \frac{e^x}{1+e^x} dx$$

Substitute $u = 1+e^x$. Then $du = e^x dx$, and the integral above becomes

$$\int \frac{du}{u} = \ln|u| = \ln|1+e^x| = \ln(1+e^x).$$

Note that the absolute values are not needed, since $1+e^x > 0$ for all x.

Example 1.6.3.

$$\int \frac{x^2}{\sqrt{1-x}} dx$$

Keeping in mind the power rule, we make the substitution $u = 1-x$. Then $du = -dx$. Noting that $x = 1-u$ by solving for x in $u = 1-x$, we see that the above integral becomes

$$\begin{aligned}
\int -\frac{(1-u)^2}{\sqrt{u}} du &= -\int \frac{1-2u+u^2}{u^{1/2}} du \\
&= -\int u^{-1/2} - 2u^{1/2} + u^{3/2} du \\
&= -\left(2u^{1/2} - \frac{4}{3}u^{3/2} + \frac{2}{5}u^{5/2}\right) \\
&= -2(1-x)^{1/2} + \frac{4}{3}(1-x)^{3/2} - \frac{2}{5}(1-x)^{5/2}.
\end{aligned}$$

The steps of the "change of variable" method can be summarized as

1. set a new variable, say u , equal to some function of the original variable x (usually u is set equal to some part of the original integrand function),

2. calculate the differential du as a function of dx,
3. rewrite the original integral in terms of u and du,
4. integrate the new integral to get an answer in terms of u,
5. replace the u in the answer to get an answer in terms of the original variable.

A "Rule of thumb" for changing the variable: If part of the integrand is a composition of functions, $f(g(x))$, then try setting $u = g(x)$, the "inner" function.

Example 1.6.4. *elect a function for u for each integral and rewrite the integral in terms of u and du.*
(a) $\int \cos(3x)\sqrt{2+\sin(3x)}\, dx$, *(b)* $\int \frac{5e^x}{2+e^x}\, dx$, *(c)* $\int e^x \sin(e^x)\, dx$.
Solution: (a) Put $u = 2+\sin(3x)$. *Then* $du = 3\cos(3x)\, dx$, *and the integral becomes* $\int \frac{1}{3}\sqrt{u}\, du$.
(b) Put $u = 2+e^x$. *Then* $du = e^x\, dx$, *and the integral becomes* $\int \frac{5}{u}\, du$.
(c) Put $u = e^x$. *Then* $du = e^x\, dx$, *and the integral becomes* $\int \sin(u)\, du$.

1.6.2 Substitution and definite integrals

Once an antiderivative in terms of u is found, we have a choice of methods. We can

(a) rewrite our antiderivative in terms of the original variable x, and then evaluate the antiderivative at the integration endpoints and subtract, or

(b) change the integration endpoints to values of u, and evaluate the antiderivative in terms of u before subtracting.

If the original integral had endpoints $x = a$ and $x = b$, and we make the substitution $u = g(x)$ and $du = g'(x)dx$, then the new integral will have endpoints $u = g(a)$ and $u = g(b)$:

$$\int_{x=a}^{x=b} (\text{original integrand})\, dx \text{ becomes } \int_{u=g(a)}^{u=g(b)} (\text{new integrand})\ du.$$

Example 1.6.5. *To evaluate*

$$\int_0^1 (3x-1)^4\, dx,$$

we can, in line with the "Rule of thumb" above, use the substitution $u = 3x-1$. *Then* $du = \frac{d}{dx}(3x-1)dx = 3dx$, *so the indefinite integral* $\int (3x-1)^4\, dx$ *becomes* $\int \frac{1}{3}u^4\, du = \frac{1}{15}u^5 + C$.
(a) Converting our antiderivative back to the variable x and evaluating with the original endpoints:

$$\int_0^1 (3x-1)^4\,dx = (\frac{1}{15}(3x-1)^5 + C)|_0^1 = \frac{32}{15} - \frac{-1}{15} = \frac{11}{5} = 2.2.$$

(b) Converting the integration endpoints to values of u : when $x = 0$, then $u = 3x - 1 = 3 \cdot 0 - 1 = -1$, and when $x = 1$, then $u = 3x - 1 = 3 \cdot 1 - 1 = 2$ so

$$\int_0^1 (3x-1)^4\,dx = \int_{-1}^2 \frac{1}{3}u^4\,du = (\frac{1}{15}u^5 + C)|_{-1}^2 = \frac{11}{5} = 2.2.$$

Both approaches typically involve about the same amount of work and calculation. Of course, these approaches lead to the same numerical answer, by the "substitution rule" (Proposition 1.6.1).

Here's how to do this using Sage. *Note that the area under the two curves plotted below, $y = (3x-1)^4$, $0 < x < 1$, and $y = x^4/3$, $-1 < x < 2$, are the same.*

```
sage: x,u = var("x,u")
sage: integral((3*x-1)^4,x,0,1)
11/5
sage: integral(u^4/3,u,-1,2)
11/5
sage: P = plot((3*x-1)^4,x,0,1,rgbcolor=(0.7,0.1,0.5), plot_points=40)
sage: Q = plot(u^4/3,u,-1,2,linestyle=":")
sage: R = text("$y=(3x-1)^4$",(1.4,12))
sage: S = text("$y=x^4/3$",(2,2.5))
sage: plot(P+Q+R+S)
```

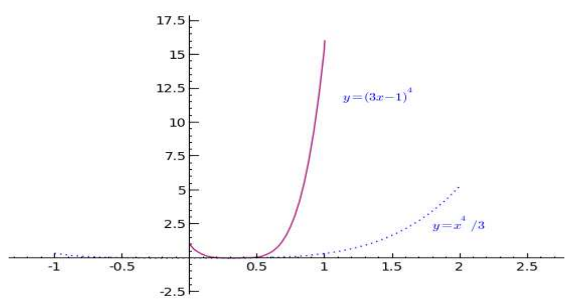

Figure 1.41: Plots of $y = (3x-1)^4$ and $y = x^4/3$.

1.6.3 Symmetry

An **odd function** is a function $f(x)$ (defined for all reals) such that $f(-x) = -f(x)$, and an **even function** one for which $f(-x) = f(x)$. If f is an odd function, then for any a,

$$\int_{-a}^{a} f(x)dx = 0.$$

If f is an even function, then for any a,

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

Both statements are clear if we view integrals as computing the signed area between the graph of $f(x)$ and the x-axis.

Example 1.6.6. *An even example,*

$$\int_{-1}^{1} x^2 dx = 2\int_{0}^{1} x^2 dx = 2\left[\frac{1}{3}x^3\right]_0^1 = \frac{2}{3},$$

and an odd example,

$$\int_{-1}^{1} x^3 dx = \left[\frac{1}{4}x^4\right]_{-1}^{1} = 0.$$

These computations are consistent with the symmetry (or "anti-symmetry") of the graphs and what you know about the relationship between the integral and area.

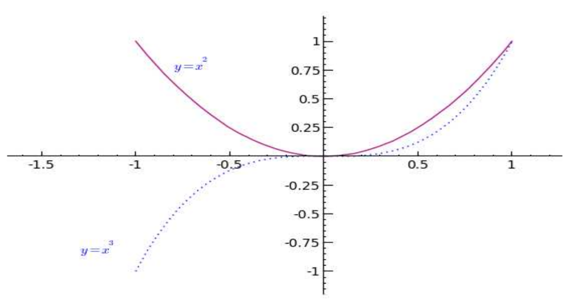

Figure 1.42: Plots of $y = x^2$ and $y = x^3$.

1.6.4 Problems

For the problems below, let $f(x) = x^2$ and $g(x) = x$ and verify that

1. $\int f(x) \cdot g(x)\, dx \neq \int f(x)\, dx \cdot \int g(x)\, dx$.
2. $\int f(x)/g(x)\, dx \neq \int f(x)\, dx / \int g(x)\, dx$.
3. $\sqrt{\int f(x)\, dx} \neq \int \sqrt{f(x)}\, dx$.

4. $\frac{1}{\int f(x)\,dx} \neq \int \frac{1}{f(x)}\,dx$.
5. $\int \cos(3x)\,dx, \quad u = 3x$.
6. $\int \sin(7x)\,dx, \quad u = 7x$.
7. $\int e^{5x}\,dx, \quad u = 5x$.
8. $\int e^{3x} + \cos(2x)\,dx, \quad u = 3x$ and $u = 2x$.

Chapter 2

Applications

2.1 Applications of the integral to area

The development of calculus by Newton and Leibniz was a vital step in the advancement of pure mathematics, but Newton also advanced the applied sciences and mathematics. Not only did he discover theoretical results, but he immediately used those results to answer important applied questions about gravity and motion. The success of these applications of mathematics to the physical sciences helped establish what we now take for granted: mathematics can and should be used to answer questions about the world. Newton applied mathematics to the outstanding problems of his day, problems primarily in the field of physics. In the intervening 300 years, thousands of people have continued these theoretical and applied traditions and have used mathematics to help develop our understanding of all of the physical and biological sciences as well as the behavioral sciences and business. Mathematics is still used to answer new questions in physics and engineering, but it is also important for modeling ecological processes, for understanding the behavior of DNA, for determining how the brain works, and even for devising strategies for voting effectively. The mathematics you are learning now can help you become part of this tradition, and you might even use it to add to our understanding of different areas of life. It is important to understand the successful applications of integration in case you need to use those particular applications. It is also important that you understand the process of building models with integrals so you can apply it to new problems. Conceptually, converting an applied problem to a Riemann sum is the most valuable and the most difficult step.

2.1.1 Using integration to determine areas

This section is about how to compute the area of fairly general regions in the plane. Regions are often described as the area enclosed by the graphs of several curves. ("My land is the plot enclosed by that river, that fence, and the highway.")

Recall that the integral $\int_a^b f(x)dx$ has a geometric interpretation as the signed area between the graph of $f(x)$ and the x-axis. We defined area by subdividing, adding up approximate areas (use points in the intervals) as *Riemann sums*, and taking the limit. Thus we defined area as a limit of Riemann sums. The fundamental theorem of calculus asserts that we can compute areas exactly when we can finding antiderivatives.

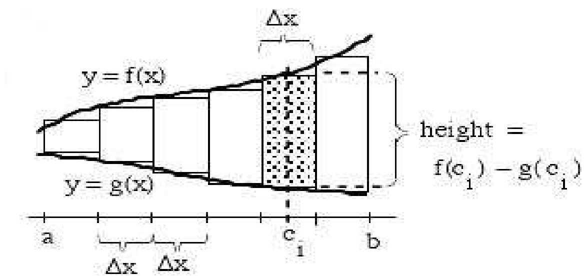

Figure 2.1: Area between $y = f(x)$ and $y = g(x)$.

Instead of considering the area between the graph of $f(x)$ and the x-axis, we consider more generally two graphs, $y = f(x)$, $y = g(x)$, and assume for simplicity that $f(x) \geq g(x)$ on an interval $[a, b]$. Again, we approximate the area *between* these two curves as before using Riemann sums. Each approximating rectangle has width $(b - a)/n$ and height $f(x) - g(x)$, so

$$\text{Area bounded by graphs} \sim \sum [f(c_i) - g(c_i)]\Delta x.$$

Note that $f(x) - g(x) \geq 0$, so the area is nonnegative. From the definition of integral we see that the exact area is

$$\text{Area bounded by graphs } = \int_a^b (f(x) - g(x))dx. \tag{2.1}$$

Why did we make a big deal about approximations instead of just writing down (2.1)? Because having a sense of how this area comes directly from a Riemann sum is very important. But, what is the point of the Riemann sum if all we're going to do is write down the integral? The sum embodies the geometric manifestation of the integral. If you have this picture in your mind, then the Riemann sum has *done its job*. If you understand this, you're more likely to know what integral to write down; if you don't, then you might not.

Remark 2.1.1. *By the linearity property of integration, our sought for area is the integral of the "top" function minus the integral of the "bottom" function,*

$$\int_a^b f(x)dx - \int_a^b g(x)dx,$$

of two signed areas.

Example 2.1.1. *Find the area enclosed by* $y = x + 1$, $y = 9 - x^2$, $x = -1$, $x = 2$ *(see Figure 2.2).*

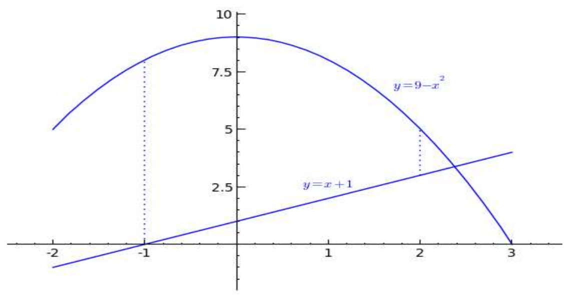

Figure 2.2: Plots of $y = x + 1$ and $y = 9 - x^2$.

$$Area = \int_{-1}^{2} \Big[(9 - x^2) - (x + 1)\Big] dx\,.$$

We have reduced the problem to a computation:

$$\int_{-1}^{2} [(9 - x^2) - (x + 1)]dx = \int_{-1}^{2} (8 - x - x^2)dx = \left[8x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2} = \frac{39}{2}.$$

Here is this plot and computation in Sage*:*

```
sage: x = var("x")
sage: P1 = plot(x+1, x, -2, 3)
sage: P2 = plot(9 - x^2, x, -2, 3)
sage: T1 = text("$y = x+1$", (1,2.6))
sage: T2 = text("$y = 9-x^2$", (2,7))
sage: show(P1+P2+T1+T2)
sage: integrate((9-x^2) - (x+1),x, -1, 2)
39/2
```

The above example illustrates the simplest case. In practice more interesting situations often arise. The next example illustrates finding the boundary points a, b when they are not explicitly given.

Example 2.1.2. *Find area enclosed by the two parabolas* $y = 12 - x^2$ *and* $y = x^2 - 6$.

Problem: We didn't tell you what the boundary points a, b *are. We have to figure that out. How? We must find* exactly *where the two curves intersect, by setting the two curves equal and finding the solution. We have*

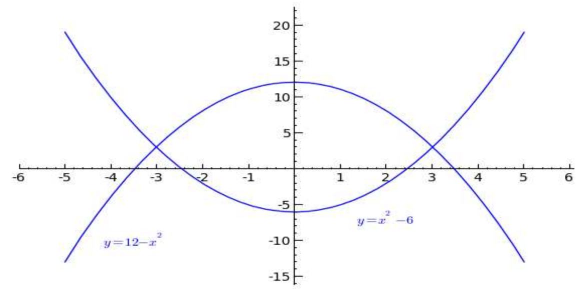

Figure 2.3: Plots of $y = 12 - x^2$ and $y = x^2 - 6$.

$$x^2 - 6 = 12 - x^2,$$

so $0 = 2x^2 - 18 = 2(x^2 - 9) = 2(x-3)(x+3)$, *hence the intersect points are at* $a = -3$ *and* $b = 3$. *We thus find the area by computing*

$$\int_{-3}^{3} \left[12 - x^2 - (x^2 - 6)\right] dx = \int_{-3}^{3} (18 - 2x^2) dx = 4 \int_{0}^{3} (9 - x^2) dx = 4 \cdot 18 = 72.$$

Here is this plot and computation in Sage*:*

```
sage: P1 = plot(12-x^2, x, -5, 5)
sage: P2 = plot(x^2-6, x, -5, 5)
sage: T1 = text("$y = 12-x^2$", (-3.5,-10))
sage: T2 = text("$y = x^2-6$", (2,-7))
sage: show(P1+P2+T1+T2)
sage: integrate((12-x^2) - (x^2-6),x, -3, 3)
72
```

Of course, if you had mistakenly computed $\int_{-3}^{3} \left[(x^2 - 6) - (12 - x^2)\right] dx$, *then don't worry. You would've gotten* -72 *as your answer. However, always remember* **areas are non-negative**, *so the correct answer is* 72.

Example 2.1.3. *A common way in which you might be tested to see if you* really *understand what is going on, is to be asked to find the area between two graphs* $x = f(y)$ *and* $x = g(y)$. *If the two graphs are vertical, subtract off the right-most curve. Or, just "switch* x *and* y*" everywhere (i.e., reflect about* $y = x$*). The area is unchanged.*

For instance, consider the area between the two parabolas $x = 12 - y^2$ *and* $x = y^2 - 6$.

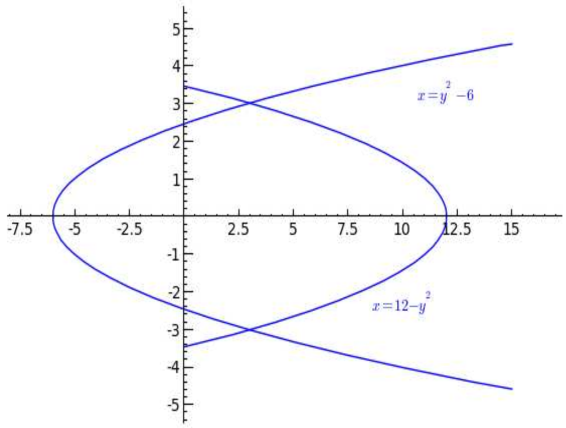

Figure 2.4: Plots of $x = 12 - y^2$ and $x = y^2 - 6$.

Swapping x *and* y *amounts to reflecting the plot in Figure 2.4 above about the* 45^o *line* $y = x$. *The reflected graph coincides with that in Figure 2.3 above. Therefore, by Example 2.1.2, the answer is* 72.

Example 2.1.4. *Find the area (*not signed area!*) enclosed by* $y = \sin(\pi x)$, $y = x^2 - x$, *and* $x = 2$.

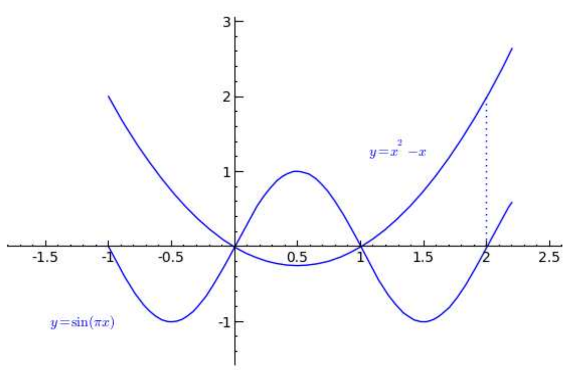

Figure 2.5: Plots of $y = \sin(\pi x)$ and $y = x^2 - x$.

Write $x^2 - x = (x - 1/2)^2 - 1/4$, *so that we can obtain the graph of the parabola by shifting the standard graph. The area comes in two pieces, and the upper and lower curve switch in the middle. Technically, what we're doing is integrating* *the* absolute value *of the difference. The area is*

$$\int_0^1 \sin(\pi x) - (x^2 - x)dx - \int_1^2 (x^2 - x) - \sin(\pi x)dx = \frac{4}{\pi} + 1$$

Here is this plot and computation in Sage*:*

```
sage: P1 = plot(sin(pi*x), x, -1, 2.2)
sage: P2 = plot(x^2-x, x, -1, 2.2)
sage: P3 = list_plot([(2,0),(2,2)],plotjoined=True,linestyle=":")
sage: T1 = text("$y = \sin(\pi x)$", (-1.2,-1))
sage: T2 = text("$y = x^2-x$", (1.3,1.3))
sage: show(P1+P2+P3+T1+T2)
```

Something to take away from this is that in order to solve this sort of problem, you need some facility with graphing functions. If you aren't comfortable with this, review.

2.2 Computing Volumes of Surfaces of Revolution

The last section emphasized a geometric interpretation of definite integrals as "areas" in two dimensions. This section emphasizes another geometrical use of integration, calculating volumes of solid threedimensional objects, such as a volume of revolution. Our basic approach is to cut the whole solid into thin "slices" whose volumes can be approximated, add the volumes of these "slices" together (a Riemann sum), and finally obtain an exact answer by taking a limit of the sums to get a definite integral.

Practice 2.2.1. *Most people have a body density between* 0.95 *and* 1.05 *times the density of water which is* 62.5 *pounds per cubic foot. Use your weight (in lbs) to estimate the volume of your body (in cubic ft). (If you float in fresh water, your body density is less than* 1.*)*

First, we introduce the building blocks of this section, right solids. A *right solid* is a three–dimensional shape swept out by moving a planar region A some distance h along a line perpendicular to the plane of A. For instance, if A is a rectangle, then the "right solid" formed by moving A along the line is a 3dimensional solid box B and, of course, the volume of B is

$$\text{(area of A)} \times \text{(distance along the line)} = \text{(base)} \times \text{(height)} \times \text{(width)}.$$

The region A is called a *face* of the solid. The word "right" is simply used to indicate that the movement is along a line perpendicular (at a right angle) to the plane of A. Two parallel cuts though the shape produce a slice with two faces.

Example 2.2.1. *Suppose there is a fine, uniform mist in the air, and every cubic foot of mist contains* 0.02 *ounces of water droplets. If you run* 50 *feet in a straight line through this mist, how wet do you get? Assume that the front (or a cross section) of your body has an area of* 8 *square feet.*

Solution: As you run, the front of your body sweeps out a "tunnel" through the mist. The volume of the tunnel is the (cross sectional) area of the front of your body multiplied by the length of the tunnel: volume $=$ *(*8 *ft)(*50 *ft)* $= 400$ *ft. Since each cubic foot of mist held* 0.02 *ounces of water which is now on you, you swept out a total of (*400 *ft)(*0.02 *oz/ft)* $= 8$ *ounces of water.*

A general solid can be cut into slices which are almost right solids. An individual slice may not be exactly a right solid since its cross sections may have different areas. However, if the cuts are close together, then the cross sectional areas will not change much within a single slice. Each slice will be almost a right solid, and its volume will be almost the volume of a right solid.

Suppose an $x$axis is positioned below the solid shape, and let $A(x)$ be the area of the face formed when the solid is cut at x perpendicular to the $x$axis. If $P = \{x_0 = a, x_1, x_2, ..., x_n = b\}$ is a partition of $[a, b]$, and the solid is cut at each x_i, then each slice of the solid is almost a right solid, and the volume of each slice is approximately

$$\text{(area of a face of the slice)} \times \text{(thickness of the slice)} \cong A(x_i)\Delta x_i.$$

The total volume V of the solid is approximately the sum of the volumes of the slices:

$$V = \sum \text{volume of each slice} \cong \sum_i A(x_i)\Delta x_i,$$

which is a Riemann sum. The limit, as the mesh of the partition approaches 0 (taking thinner and thinner slices), of the Riemann sum is the definite integral of $A(x)$:

$$V \cong \sum_i A(x_i)\Delta x_i \to \int_a^b A(x)\, dx.$$

Theorem 2.2.1. *(Volume By Slices Formula) If S is a solid and $A(x)$ is the area of the face formed by a cut at x and perpendicular to the x–axis, then the volume V of the part of S above the interval $[a, b]$ is $V = \int_a^b A(x)\, dx$.*

Everybody knows that the volume of a solid box is

$$\text{volume} = \text{length} \times \text{width} \times \text{height}.$$

More generally, the volume of cylinder is $V = \pi r^2 h$ (cross sectional area times height). Even more generally, if the base of a prism has area A, the volume of the prism is $V = Ah$.

But what if our solid object looks like a complicated blob? How would we compute the volume? We'll do something that by now should seem familiar, which is to chop the object into small pieces and take the limit of approximations. If these small pieces are cross sections then the corresponding method of computing the volume of revolution is called the "disc method". If these small pieces are cylindrical shells then the corresponding method of computing the volume of revolution is called the "shell method". We look in detail into the disc method first, followed by the shell method.

2.2.1 Disc method

Assume that we have a function

$$A(x) = \text{cross sectional area at } x.$$

The volume of our potentially complicated blob is approximately $\sum A(x_i)\Delta x$. Thus

$$\begin{aligned}\text{volume of blob} &= \lim_{n\to\infty}\sum_{i=1}^{n} A(x_i)\Delta x \\ &= \int_a^b A(x)dx\end{aligned}$$

Here is the plot a picture of solid sliced vertically into a bunch of vertical thin solid discs:

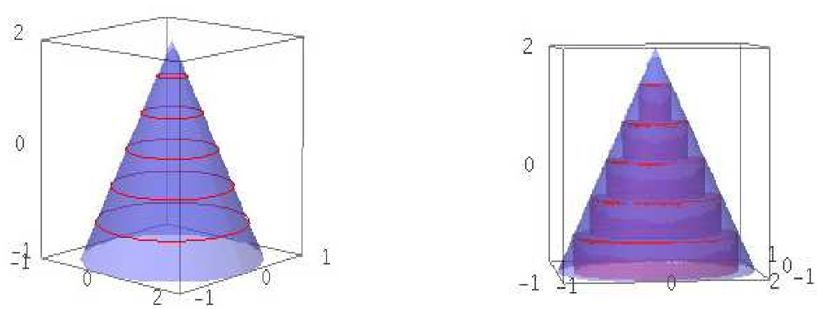

Figure 2.6: Plot of the cone $z = \frac{3}{2}(1-\sqrt{x^2+y^2})$ sliced into thin "shells", which are approximated by thin discs.

Example 2.2.2. *Find the volume of the pyramid with height H and square base with sides of length L.*

For convenience look at pyramid on its side, with the tip of the pyramid at the origin. We need to figure out the cross sectional area as a function of x, for

$0 \leq x \leq H$. The function that gives the distance $s(x)$ from the x–axis to the edge is a line, with $s(0) = 0$ and $s(H) = L/2$. The equation of this line is thus $s(x) = \frac{L}{2H}x$. Thus the cross sectional area is

$$A(x) = (2s(x))^2 = \frac{x^2L^2}{H^2}.$$

The volume is then

$$\int_0^H A(x)dx = \int_0^H \frac{x^2L^2}{H^2}dx = \left[\frac{x^3L^2}{3H^2}\right]_0^H = \frac{H^3L^2}{3H^2} = \frac{1}{3}HL^2.$$

Figure 2.7: How big is Pharaoh's place?. (Photo found on `http://en.wikipedia.org/wiki/Egyptian_pyramids`, taken by Ricardo Liberato.)

When a region is revolved around a line (Figure 2.8) a right solid is formed. When the face of each slice of the revolved region is a circle then the formula for the area of the face is easy: $A(x) =$ area of a circle $= \pi(\text{radius})$, where the radius is often a function of the location x. Finding a formula for the changing radius requires care.

Theorem 2.2.2. *(Volumes of Revolved Regions by Discs) If the region formed between f, the horizontal line $y = L$, and the interval $[a, b]$ is revolved about the horizontal line $y = L$ (see Figure 2.8) then the volume is $V = \int_a^b A(x)\,dx = \int_a^b \pi\,(\text{radius})\,dx = \int_a^b \pi(f(x) - L)\,dx$.*

Example 2.2.3. *Find the volume of the solid obtained by rotating the following "flower pot" region about the x axis: the region enclosed by $y = x^2$ and $y = x^3$ between $x = 0$ and $x = 1$.*

The cross section is a "washer", and the area as a function of x is

$$A(x) = \pi(r_{out}(x)^2 - r_{in}(x)^2) = \pi(x^4 - x^6).$$

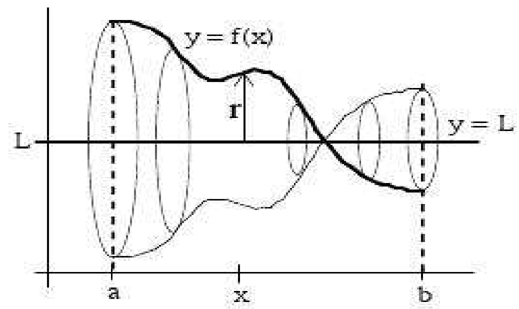

Figure 2.8: Disc method for computing a volume of revolution.

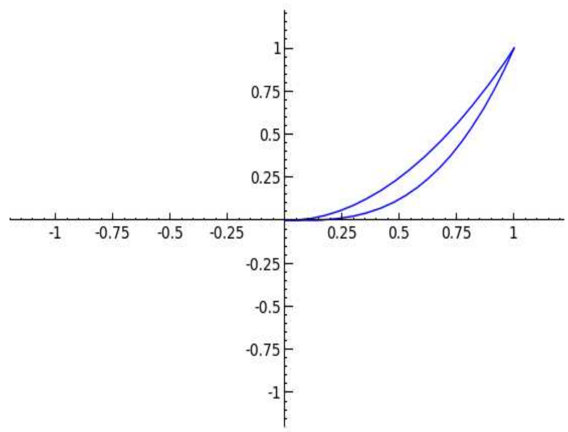

Figure 2.9: Plots of $y = x^2$ and $y = x^3$.

The volume is thus

$$\int_0^1 A(x)dx = \int_0^1 \left(\frac{1}{5}x^5 - \frac{1}{7}x^7\right) dx = \pi \left[\frac{1}{5}x^5 - \frac{1}{7}x^7\right]_0^1 = \frac{2}{35}\pi.$$

Practice 2.2.2. *Find the volumes swept out when*

(a) the region between $f(x) = x$ and the x–axis, for $0 \le x \le 2$, is revolved about the x–axis, and

(b) the region between $f(x) = x$ and the line $y = 2^x$, for $0 \le x \le 2$, is revolved about the line $y = 2$.

Example 2.2.4. *One of the most important examples of a volume is the volume V of a sphere of radius r. Let's find it! We'll just compute the volume of a half* *and multiply by* 2*. The cross sectional area is*

$$A(x) = \pi r(x)^2 = \pi(\sqrt{r^2 - x^2})^2 = \pi(r^2 - x^2).$$

Then

$$\frac{1}{2}V = \int_0^r \pi(r^2 - x^2)dx = \pi \left[r^2 x - \frac{1}{3}x^3\right]_0^r = \pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3.$$

Thus $V = (4/3)\pi r^3$.

Example 2.2.5. *Find volume of intersection of two spheres of radius* r*, where the center of each sphere lies on the edge of the other sphere.*

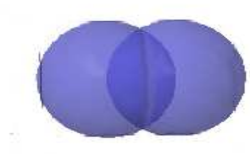

Figure 2.10: Plot of two spheres.

From the picture (Figure 2.10) we see that the answer is

$$2\int_{r/2}^{r} A(x),$$

where $A(x)$ *is* exactly *as in Example 2.2.4. We have*

$$2\int_{r/2}^{r} \pi(r^2 - x^2)dx = \frac{5}{12}\pi r^3.$$

The previous ideas and techniques can also be used to find the volumes of solids with holes in them. If $A(x)$ is the area of the face formed by a cut at x, then it is still true that the volume is $V = \int_a^b A(x)\,dx$. However, if the solid has holes, then some of the faces will also have holes and a formula for $A(x)$ may be more complicated. Sometimes it is easier to work with two integrals and then subtract: (i) calculate the volume S of the solid *without* the hole, (ii) calculate the volume H of the hole, and (iii) subtract H from S. This is what was done in Example 2.2.3.

2.2.2 Shell method

The disk method can be cumbersome if we want the volume when the region in the figure is revolved about the y–axis or some other vertical line. To revolve the region about the y–axis, the disk method requires that we represent the original equation $y = f(x)$ as a function of y: $x = g(y)$. Sometimes that is easy: if $y = 3x$ then $x = y/3$. But sometimes it is not easy at all: if $y = x + e^x$, then we can not solve for y as an "elementary" function of x. The "shell" method lets us use the original equation $y = f(x)$ to find the volume when the region is revolved about a vertical line. We partition the x–axis to cut the region into thin, almost rectangular "slices." When the thin "slice" at x_i is revolved about the y–axis (Figure 2.11(a)), the volume of the resulting "tube" (or cylindrical "shell") can be approximated by cutting the wall of the tube and laying it out flat (Figure 2.11(b)) to get a thin, solid rectangular box.

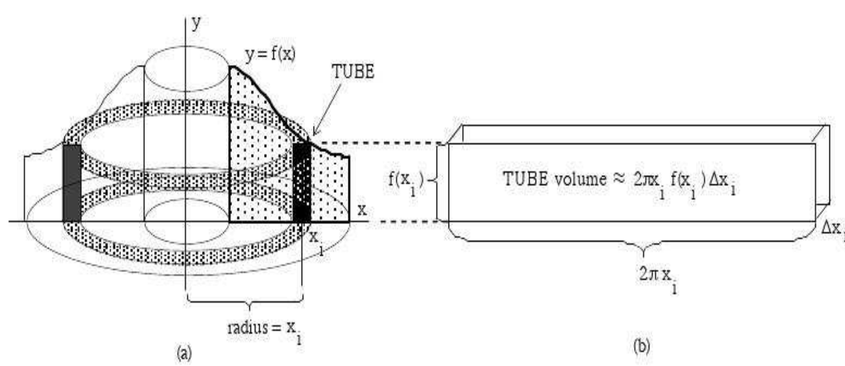

Figure 2.11: Shell method for computing a volume of revolution.

The volume of the tube is approximately the same as the volume of the solid box:

$$\begin{aligned}\text{Vol. tube} \cong \text{Vol. box} \quad &= (\text{length}) \times (\text{height}) \times (\text{thickness}) \\ &= (2\pi\text{radius}) \times (\ \text{height}\ ) \times (\Delta x_i) \\ &= 2\pi x_i f(x_i)\Delta x_i.\end{aligned}$$

The volume swept out when the whole region is revolved is the sum of the volumes of these "tubes", a Riemann sum. The limit of the Riemann sum is

$$\text{volume of rotation about the } y\text{–axis } = \int_a^b 2\pi x f(x)\,dx.$$

Theorem 2.2.3. *(Volume of Revolution Using Shells) If region R is bounded between the functions $f(x) \geq g(x)$ for $0 \leq a \leq b$ (see Figure 2.12), then*

$$\textit{volume obtained when } R \textit{ is revolved about the } y\textit{–axis} = \int_a^b 2\pi x(f(x) - g(x))\,dx.$$

Example 2.2.6. *Find the volume when the region R in between $x = 2$, $x = 4$, $y = x$ and $y = x^2$ is revolved about the y–axis.*

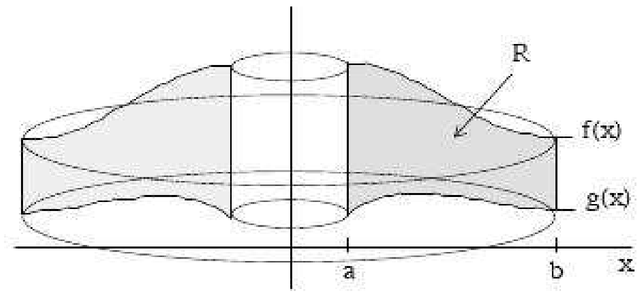

Figure 2.12: Shell method for computing a volume of revolution.

Solution: We can partition the interval $[2,4]$ *on the* x*–axis to get thin slices of* R*. When the slice at* x_i *is revolved around the* y*–axis, a tube is swept out, and the volume* V_i *of this* i*-th tube is*

$$\begin{aligned} V_i &\cong (2\pi \cdot \textit{radius}) \times \textit{(height)} \times \textit{(thickness)} \\ &\cong 2\pi x_i(x_i^2 - x_i)\Delta x_i \\ &\cong 2\pi(x_i^3 - x_i^2)\Delta x_i. \end{aligned}$$

The total approximate volume is the sum of the volumes of the tubes. As the partition gets finer and finer, we get

$$V = \sum_i V_i \to \int_2^4 2\pi(x^3 - x^2)\,dx = 2\pi(\frac{x^4}{4} - \frac{x^3}{3})|_2^4 = 2\pi\frac{124}{3} = 259.7\ldots\ .$$

2.2.3 Problems

1. For the solid in Figure 2.13, the face formed by a cut at x is a triangle with a base of 4 inches and a height of x^2 inches. Write and evaluate an integral for the volume of the solid for x between 1 and 2.

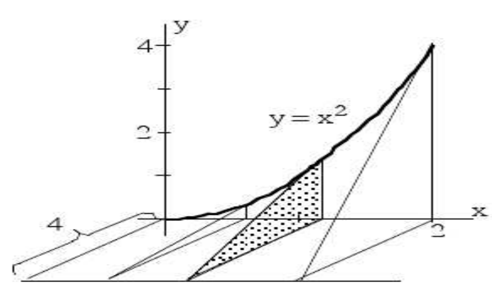

Figure 2.13: Volume of a solid.

2. Find the volume of the squarebased pyramid in Figure 2.14.

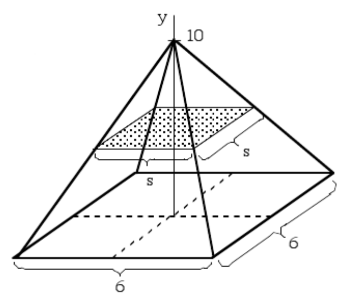

Figure 2.14: Volume of a pyramid.

3. Find the volume generated when the region between one arch of the sine curve ($0 \le x \le \pi$) and the x-axis is revolved about (a) the x-axis and (b) the line $y = 1/2$.

4. Given that $\int_1^5 f(x)\,dx = 4$ and $\int_1^5 f(x)^2\,dx = 7$. Represent the volumes of the solids (a), (b), (c) and (d) in Figure 2.15 as definite integrals and evaluate the integrals.

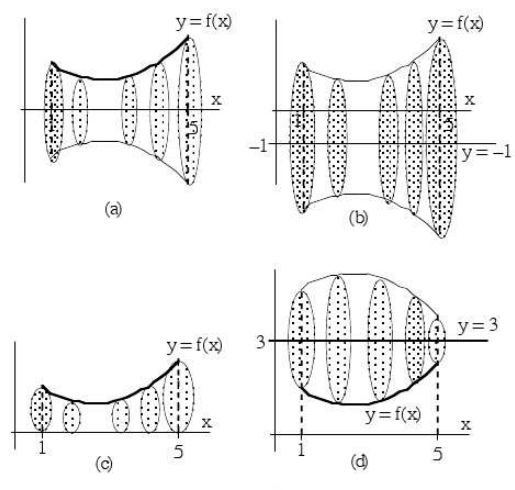

Figure 2.15: Four volumes.

5. Figure 2.16. For $0 \leq x \leq 3$, each face is a circle with height (diameter) $4 - x$ meters.

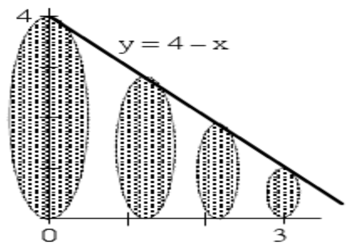

Figure 2.16: Volume with circular cross-sections.

6. Suppose A and B are solids so that every horizontal cut produces faces of A and B that have equal areas. What (if anything) can we conclude about the volumes of A and B? Justify your answer.

7. Calculate the volume of a sphere of radius 2.

8. Let $0 < r < R$ be fixed. Revolve the circle $x^2 + (y - R)^2 = r^2$ about the x-axis. Compute the volume of this "donut" solid.

9. (a) Find the area between $f(x) = 1/x$ and the x-axis for $1 \leq x \leq 10$, $1 \leq x \leq 100$, and $1 \leq x \leq A$. What is the limit of the area for $1 \leq x \leq A$ as $A \to \infty$? If $A = 1000000$ and your think of this area as a long, flat wall, estimate the amount of paint (in square feet) you need to paint this surface.
   (b) Find the volume swept out when the region in part (a) is revolved about the x-axis for $1 \leq x \leq 10$, $1 \leq x \leq 100$, and $1 \leq x \leq A$. What is the limit of the volumes for $1 \leq x \leq A$ as $A \to \infty$? If $A = 1000000$ and your think of this volume as a room constructed by revolving the wall in (a) about an axis, estimate the amount of paint (in cubic feet) you need to completely *fill* the room.
   (c) Which is larger: the paint needed to paint the wall of the paint needed to completely fill the room?

10. The region between $y = 2x - x^2$ and the $x$axis for $0 \leq x \leq 2$. Sketch the region and calculate the volume swept out when the region is revolved about the $y$axis.

11. The region between $y = \sqrt{1 - x^2}$ and the $x$axis for $0 \leq x \leq 1$. Sketch the region and calculate the volume swept out when the region is revolved about the $y$axis.

12. The region between $y = \frac{1}{1+x^2}$ and the $x$axis for $0 \leq x \leq 1$. Sketch the region and calculate the volume swept out when the region is revolved about the $y$axis.

2.3 Average Values

In this section we use Riemann sums to extend the familiar notion of an average, which provides yet another physical interpretation of integration.

Recall: Suppose $y_1, \ldots, y_n$ are the amount of rain each day in you hometown so far this year. The average rainfall per day is

$$y_{\text{avg}} = \frac{y_1 + \cdots + y_n}{n} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Definition 2.3.1 (Average Value of Function)**.** *Suppose f is a continuous function on an interval $[a, b]$. The* **average value** *of f on $[a, b]$ is*

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Motivation: If we sample f at n points x_i, then

$$f_{\text{avg}} \sim \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \frac{(b-a)}{n(b-a)}\sum_{i=1}^{n} f(x_i) = \frac{1}{(b-a)}\sum_{i=1}^{n} f(x_i)\Delta x,$$

since $\Delta x = \dfrac{b-a}{n}$. This is a Riemann sum!

$$\frac{1}{(b-a)}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{(b-a)}\int_a^b f(x)dx.$$

This explains why we defined f_{avg} as above.

Example 2.3.1. *What is the average value of* $\sin(x)$ *on the interval* $[0, \pi]$*?*

$$\begin{aligned}\frac{1}{\pi-0}\int_0^\pi \sin(x)dx &= \frac{1}{\pi-0}\Big[-\cos(x)\Big]_0^\pi \\ &= \frac{1}{\pi}\Big[-(-1)-(-1)\Big]_0^\pi = \frac{2}{\pi}\end{aligned}$$

Observation: If you multiply both sides by $(b-a)$ in Definition 2.3.1, you see that the average value times the length of the interval is the area, i.e., the average value gives you a rectangle with the same area as the area under your function. In particular, in Figure 2.17 the area between the x-axis and $\sin(x)$ is exactly the same as the area between the horizontal line of height $2/\pi$ and the x-axis.

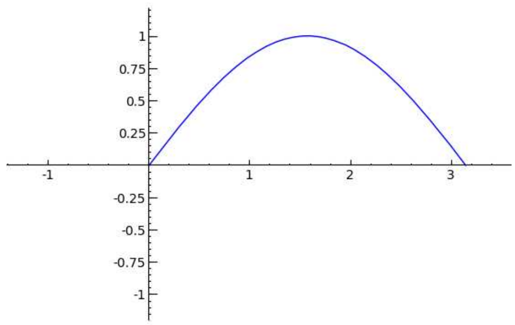

Figure 2.17: What is the average value of $\sin(x)$?

Theorem 2.3.1 (Mean Value Theorem)**.** *Suppose f is a continuous function on $[a, b]$. Then there is a number c in $[a, b]$ such that $f(c) = f_{\text{avg}}$.*

This says that f assumes its average value. It is a used very often in understanding why certain statements are true. Notice that in Example 2.3.1 it is just the assertion that the graphs of the function and the horizontal line $y = f_{\text{avg}}$ intersect.

Proof: Let $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$. By the mean value theorem for derivatives, there is $c \in [a, b]$ such that $f(c) = F'(c) = (F(b) - F(a))/(b - a)$. But by the fundamental theorem of calculus,

$$f(c) = \frac{F(b) - F(a)}{b - a} = \frac{1}{b - a}\int_a^b f(x)dx = f_{\text{avg}}.$$

□

2.3.1 Problems

In problems 1-4, use the values in Figure 2.18 to estimate the average values.

1. Estimate the average value of f on the interval [0.5, 4.5].

2. Estimate the average value of f on the interval [0.5, 6.5].

3. Estimate the average value of f on the interval [1.5, 3.5].

4. Estimate the average value of f on the interval [3.5, 6.5].

5. Find the average value of $\sin(x)$, $0 \le x \le \pi$.

6. Find the average value of x^2, $-1 \le x \le 1$.

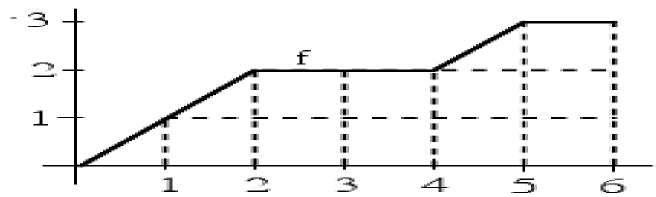

Figure 2.18: Table of values of the function $f(x)$.

2.4 Moments and centers of mass

This section develops a method for finding the center of mass of a thin, flat shape the point at which the shape will balance without tilting. Centers of mass are important because in many applied situations an object behaves as though its entire mass is located at its center of mass. For example, if you are riding in a car with a high center of mass (such as an SUV) and you make a sudden sharp turn, you are more likely to tip over than if you are riding in a car with a high center of mass (such as a sports car). As another example, the work done to pump the water in a tank to a higher point is the same as the work to move the center of mass of the water to the higher point (Figure 2.19), a much easier problem, if we know the mass and the center of mass of the water. Also, volumes and surface areas of solids of revolution can be easy to calculate, if we know the center of mass of the region being revolved.

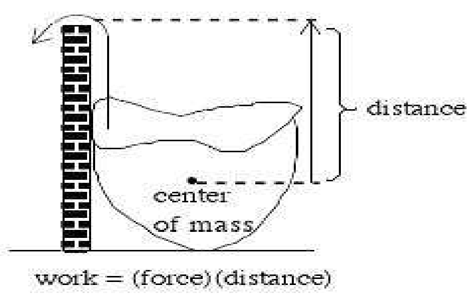

Figure 2.19: Work depends on the center of mass.

Before looking for the centers of mass of complicated regions, we consider point masses and systems of point masses, first in one dimension and then in two dimensions.

2.4.1 Point Masses

First we discuss point masses along a line.

Two people with different masses can position themselves on a seesaw so that the seesaw balances (Figure 2.20). The person on the right causes the seesaw to "want to turn" clockwise about the fulcrum, and the person on the left causes it to "want to turn" counterclockwise. If these two "tendencies" are equal, the seesaw will balance. A measure of this tendency to turn about the fulcrum is called the **moment** about the fulcrum of the system, and its magnitude is the mass multiplied by the distance from fulcrum.

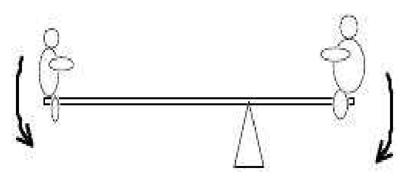

Figure 2.20: Balance on a see-saw depends on the center of mass.

In general, the moment about the origin, M_0, produced by a mass m at a location x is mx, the product of the mass and the "signed distance" of the mass from the origin. For a system of masses m_1 , m_2, $\ldots$, m_n at locations x_1, x_2, $\ldots$, x_n, respectively,

$$M = \text{total mass of the system} = \sum_{i=1}^{n} m_i,$$

and

$$M_0 = \text{moment about the origin} = x_1m_1 + x_2m_2 + \cdots + x_nm_n = \sum_{i=1}^{n} x_im_i.$$

If the moment about the origin is positive then the system tends to rotate clockwise about the origin. If the moment about the origin is negative then the system tends to rotate counterclockwise about the origin. If the moment about the origin is zero, then the system does not tend to rotate in either direction about the origin; it balances on a fulcrum at the origin. The moment about the point p, M_p, produced by a mass m at the location x is the signed distance of x from p times the mass m: $(x - p) \cdot m$. The moment about the point p produced by masses m_1, m_2, $\ldots$, m_n at locations x_1, x_2, $\ldots$, x_n, respectively, is

$$M_p = \text{moment about } p = (x_1-p)m_1+(x_2-p)m_2+\cdots+(x_n-p)m_n = \sum_{i=1}^{n}(x_i-p)m_i.$$

The point at which the system balances is called the **center of mass** of the system and is written $\overline{x}$ (pronounced "x bar"). Since the system balances at $\overline{x}$, the moment about $p = \overline{x}$ must be zero. Using this fact and properties of summation, we can find a formula for $\overline{x}$:

$$0 = M_{\overline{x}} = \sum_{i=1}^{n}(x_i - \overline{x})m_i = \sum_{i=1}^{n} x_i m_i - \overline{x})\sum_{i=1}^{n} m_i,$$

so

$$\overline{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i}.$$

This is summarized as follows.

Theorem 2.4.1. *The center of mass of a system of masses m_1, m_2, $\ldots$, m_n at locations x_1, x_2, $\ldots$, x_n, is given by*

$$\overline{x} = \frac{\textit{moment about the origin}}{\textit{total mass}} = M_0/M.$$

Now we discuss point masses in the plane.

The ideas of moments and centers of mass extend nicely from one dimension to a system of masses located at points in the plane.

For a system of masses m_i located at the points (x_i, y_i),

$$\begin{aligned} M \quad &= \text{total mass of particles} = \textstyle\sum_{i=1}^{n} m_i, \\ M_y &= \text{moment about the } y\text{–axis} = \textstyle\sum_{i=1}^{n} m_i x_i, \\ M_x &= \text{moment about the } x\text{–axis} = \textstyle\sum_{i=1}^{n} m_i y_i. \end{aligned}$$

Theorem 2.4.2. *The center of mass of a system of masses m_1, m_2, $\ldots$, m_n at locations (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), is given by $(\overline{x}, \overline{y})$, where*

$$\begin{aligned} \overline{x} &= \frac{\textit{moment about the } y\textit{–axis}}{\textit{total mass}} = M_y/M, \\ \overline{y} &= \frac{\textit{moment about the } x\textit{–axis}}{\textit{total mass}} = M_x/M. \end{aligned}$$

Example 2.4.1. *Consider a regular hexagon in the plane centered at the origin*[1] *Suppose that all the vertices of the hexagon have equal mass 1. The center of mass of this hexagon is the same as the average value of the vertices. Therefore, by construction, $(\overline{x}, \overline{y}) = (0, 0)$.*

[1] To draw this, simply draw a circle and slice it up fairly into 6 equal "pie pieces". The points on the "crust" where your slices start are the vertices of the hexagon.

2.4.2 Center of mass of a region in the plane

When we move from discrete point masses to whole, continuous regions in the plane, we move from finite sums and arithmetic to limits of Riemann sums, definite integrals, and calculus. The following material extends the ideas and calculations from point masses to uniformly thin, flat plates that have a constant density given as mass per area. The center of mass of one of these plates is the point $(\overline{x}, \overline{y})$ at which the plate balances without tilting. It turns out that the center of mass $(\overline{x}, \overline{y})$ of such a plate depends only on the region of the plane covered by the plate and not on its density. In this situation, the point $(\overline{x}, \overline{y})$ is also called the **centroid** of the region. In the following discussion, you should notice that each finite sum that appeared in the discussion of point masses has a counterpart for these thin plates in terms of integrals.

The rectangle is the basic shape used to extend the point mass ideas to regions. The total mass of a rectangular plate is the area of the plate multiplied by the density constant: mass M = area $\times$ density. We assume that the center of mass of a thin, rectangular plate is located half way up and half way across the rectangle, at the point where the diagonals of the rectangle cross. Then the moments of the rectangle can be found by treating the rectangle as a point with mass M located at the center of mass of the rectangle.

To find the moments and center of mass of a plate made up of several rectangular regions, just treat each of the rectangular pieces as a point mass concentrated at its center of mass. Then the plate is treated as a system of discrete point masses.

Example 2.4.2. *The plate in Figure 2.21 can be divided into two rectangular plates, one with mass* 24 *g and center of mass* $(1, 4)$*, and one with mass* 12 *g and center of mass* $(3, 3)$.

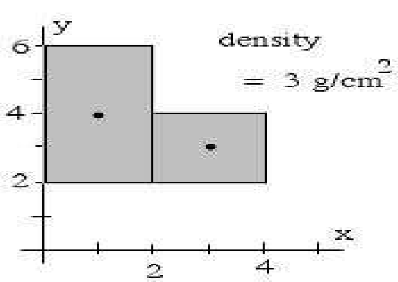

Figure 2.21: Centroid of two rectangles.

The total mass of the pair is $M = 36$ *g, and the moments about the axes are* $M_x = (24\ g)(4\ cm) + (12\ g)(3\ cm) = 132\ gcm$, *and* $M_y = (24\ g)(1\ cm) + (12\ g)(3\ cm) = 60\ gcm$.

Then $\overline{x} = M_y/M = (60\ gcm)/(36\ g) = 5/3\ cm$ *and* $\overline{y} = M_x/M = (132\ gcm)/(36\ g) = 11/3\ cm$ *so the center of mass of the plate is at* $(\overline{x}, \overline{y}) = (5/3, 11/3)$.

To find the center of mass of a thin plate, we will "slice" the plate into narrow rectangular plates and treat the collection of rectangular plates as a system of point masses located at the centers of mass of the rectangles. The total mass and moments about the axes for the system of point masses will be Riemann sums. Then, by taking limits as the widths of the rectangles approach 0, we will obtain exact values for the mass and moments as definite integrals

2.4.3 $\overline{x}$ For A Region

Suppose $f(x) \geq g(x)$ on $[a, b]$ and R is a plate on the region between the graphs of f and g for $a \leq x \leq b$ (Figure 2.22).

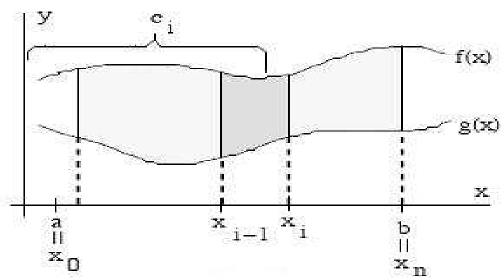

Figure 2.22: Centroid of a region.

If the interval $[a, b]$ is partitioned into subintervals $[x_{i-1}, x_i]$ and the point c_i is the midpoint of each subinterval, then the slice between vertical cuts at x_{i-1} and x_i is approximately rectangular and has mass approximately equal to

$$\begin{aligned}
\text{(area)} \times \text{(density)} &= \text{(height)} \times \text{(width)} \times \text{(density)} \\
&\cong (f(c_i) - g(c_i))(x_{i-1} x_i)k \\
&= (f(c_i) - g(c_i))\Delta x_i k,
\end{aligned}$$

where k denotes the density. The mass of the whole plate is approximately

$$\begin{aligned}
M &= \sum_i (f(c_i) - g(c_i))\Delta x_i k \to k \int_a^b (f(x)g(x))\, dx \\
&= k \cdot \text{(area of the region between } f \text{ and } g\text{)}.
\end{aligned}$$

The moment about the y–axis of each rectangular piece is

$$\begin{aligned}
M_y &= \text{(distance from } y\text{–axis to center of mass of piece)} \times \text{(mass)} \\
&= c_i(f(c_i) - g(c_i))(\Delta x_i)k,
\end{aligned}$$

so

$$M_y = \sum_i c_i(f(c_i) - g(c_i))(\Delta x_i)k \to k \int_a^b x(f(x) - g(x))\,dx.$$

The x–coordinate of the center of mass of the plate is

$$\overline{x} = \frac{M_y}{M} = \frac{\int_a^b x(f(x) - g(x))\,dx}{\int_a^b f(x) - g(x)\,dx},$$

since the common factor of k on top and bottom cancel.

Practice 2.4.1. *Find the x–coordinate of the center of mass of the region between $f(x) = x^2$ and the x–axis for $0 \leq x \leq 2$. (In this case, $g(x) = 0$.)*

2.4.4 $\overline{y}$ For a Region

Again, suppose $f(x) \geq g(x)$ on $[a, b]$ and R is a plate on the region between the graphs of f and g for $a \leq x \leq b$ (Figure 2.22).

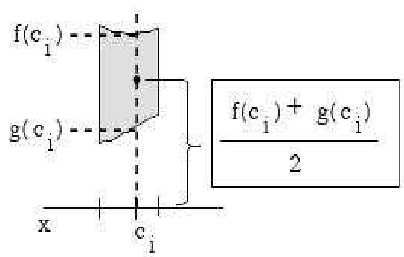

Figure 2.23: Finding the y–coordinate of the centroid of a region.

To find $\overline{y}$, the y–coordinate of the center of mass of the plate R, we need to find M_x , the moment of the plate about the x–axis. When R is partitioned vertically (Figure 2.23), the moment of each (very narrow) strip about the x–axis, M_x, is

(signed distance from x–axis to the center of mass of strip) $\times$ (mass of strip).

Since each thin strip is approximately rectangular, the y–coordinate of the center of mass of each strip is approximately half way up the strip: $\overline{y}_i \cong (f(c_i) + g(c_i))/2$. Then

$$\begin{aligned}
M_x \text{ for the strip} &= \text{(signed distance from the } x\text{–axis} \\
&\quad \text{to the center of mass of the strip)} \times \text{(mass of strip)} \\
&= \text{(signed distance from xaxis)} \times \text{(height of strip)} \\
&\quad \times \text{(width of strip)} \times \text{(density constant)} \\
&= \tfrac{f(c_i)+g(c_i)}{2}(f(c_i) - g(c_i))(\Delta x_i)k.
\end{aligned}$$

The moment about the x–axis of each rectangular piece is

$$\begin{aligned}
&\text{(distance from the } x\text{–axis to the center of mass of the piece)} \times \text{(mass)} \\
&= \tfrac{f(c_i)+g(c_i)}{2}(f(c_i) - g(c_i))(\Delta x_i)k,
\end{aligned}$$

so

$$\begin{aligned}
M_x = &\textstyle\sum_i \frac{f(c_i)+g(c_i)}{2}(f(c_i) - g(c_i))(\Delta x_i)k \\
&\to k \textstyle\int_a^b \frac{f(x)+g(x)}{2}(f(x) - g(x))\, dx = \frac{k}{2}\int_a^b f(x)^2 - g(x)^2\, dx.
\end{aligned}$$

Practice 2.4.2. *Show that the centroid of a triangular region with vertices $(0,0)$, $(0,h)$ and $(b,0)$ is $(\overline{x}, \overline{y}) = (b/3, h/3)$.*

Example 2.4.3. *Find the centroid of the region bounded between the graphs of $y = x$ and $y = x^2$, for $0 \le x \le 1$.*

Solution: $M = k\int_0^1 (x - x^2)\, dx = k/6$, $M_y = k\int_0^1 x(x - x^2)\, dx = k/12$ and $M_x = \frac{k}{2}\int_0^1 (x^2 - x^4)\, dx = k/15$. Then $\overline{x} = M_y/M = 1/2$ and $\overline{y} = M_x/M = 2/5$.

2.4.5 Theorems of Pappus

When location of the center of mass of an object is known, the theorems of Pappus make some volume and surface area calculations very easy.

Volume of Revolution: If a plane region with area A and centroid $(\overline{x}, \overline{y})$ is revolved around a line in the plane which does not go through the region (touching the boundary is alright), then the volume swept out by one revolution is the area of the region times the distance traveled by the centroid (Figure 2.24):

Theorem 2.4.3. *(Pappas' theorem for volume)*

$$\textit{Volume about line } L = A \cdot 2\pi \textit{ distance of } (\overline{x}, \overline{y}) \textit{ from the line } L\ ,$$

so in particular,

$$\textit{Volume about } x\textit{–axis} = A \cdot 2\pi\overline{y},$$

and

$$\textit{Volume about } y\textit{–axis} = A \cdot 2\pi\overline{x}.$$

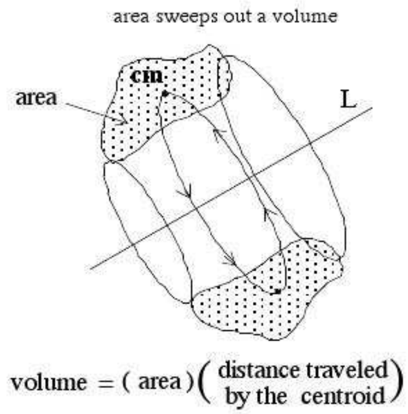

Figure 2.24: Pappas' theorem for a volume of revolution.

Surface Area of Revolution If a plane region with perimeter P and centroid of the edge $(\overline{x}, \overline{y})$ is revolved around a line in the plane which does not go through the region (touching the boundary is alright), then the surface area swept out by one revolution is the perimeter of the region times the distance traveled by the centroid (Figure 2.25):

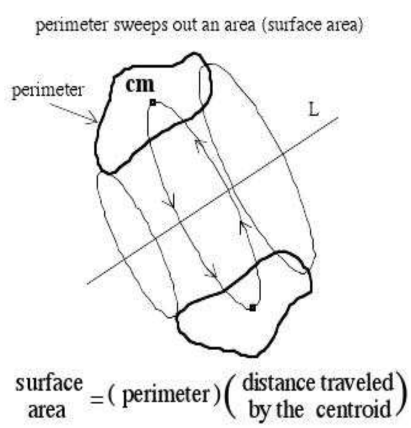

Figure 2.25: Pappas' theorem for a surface area of revolution.

Theorem 2.4.4. *(Pappas' theorem for surface area)*

$$\textit{Surface area about line } L = P \cdot 2\pi \textit{ distance of } (\overline{x}, \overline{y}) \textit{ from the line } L\ ,$$

so in particular,

$$\textit{Surface area about } x\textit{–axis} = P \cdot 2\pi \overline{y},$$

and

$$\textit{Surface area about } y\text{–}axis = P \cdot 2\pi\overline{x}.$$

Example 2.4.4. *The center of a square region with* 2 *foot sides is at the point* $(3, 4)$*. Use the Theorems of Pappus to find the volume and surface area swept out when the square is rotated (a) about the* x*–axis, (b) about the* y*–axis, and (c) about the horizontal line* $y = 6$*.*

Solution: (a) Volume about x*–axis* $= A \cdot 2\pi\overline{y} = 32\pi$*, Surface area about* x*–axis* $= P \cdot 2\pi\overline{y} = 64\pi$*.*

(b) Volume about y*–axis* $= A \cdot 2\pi\overline{x} = 24\pi$*, Surface area about* y*–axis* $= P \cdot 2\pi\overline{x} = 48\pi$*.*

(c) Volume about line $y = 6$ $= A \cdot 2\pi$*(distance of* $(3, 4)$ *to the line* $y = 6$*)* $=16\pi$*, Surface area about line* $y = 6$ $= P \cdot 2\pi$*(distance of* $(3, 4)$ *to the line* $y = 6$*)* $=32\pi$*.*

2.5 Arc lengths

This section introduces another geometric applications of integration: finding the length of a curve, i.e., the total distance you travel if you are moving along a curve. The general strategy is the same as before: partition the problem into small pieces, approximate the solution on each small piece, add the small solutions together in the form of a Riemann sum, and finally, take the limit of the Riemann sum to get a definite integral.

2.5.1 2-D Arc length

Suppose C is a curve, and we pick some points (x_i, y_i) along C and connect the points with straight line segments. Then the sum of the lengths of the line segments will approximate the length of C. We can think of this as pinning a string to the curve at the selected points, and then measuring the length of the string as an approximation of the length of the curve. Of course, if we only pick a few points, then the total length approximation will probably be rather poor, so eventually we want lots of points (x_i, y_i), close together all along C. Suppose the points are labeled so (x_0, y_0) is one endpoint of C and (x_n, y_n) is the other endpoint and that the subscripts increase as we move along C. Then the distance between the successive points (x_{i-1}, y_{i-1}) and (x_i, y_i) is $\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$, and the total length of the line segments is simply the sum of the successive lengths. This is an important approximation of the length of C, and all of the integral representations for the length of C come from it. The length of the curve C is approximately

$$\sum_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_i \sqrt{1 + (\Delta y_i/\Delta x_i)^2}\Delta x_i.$$

This is a Riemann sum. We could have factored out a Δy_i instead: the arc length of C is approximately

$$\sum_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_i \sqrt{1 + (\Delta x_i/\Delta y_i)^2}\Delta y_i.$$

Theorem 2.5.1.
- *The length of the curve C described by the graph of the function $y = f(x)$, $a \le x \le b$, is given by*

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

- *The length of the curve C described by the graph of the function $x = g(y)$, $a \le y \le b$, is given by*

$$\int_a^b \sqrt{1 + g'(y)^2}\, dy.$$

Example 2.5.1. *Use the points $(0, 0)$, $(1, 1)$, and $(3, 9)$ to approximate the length of $y = x^2$, for $0 \le x \le 3$.*

Solution: The lengths of the pieces are $\sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$ and $\sqrt{(3-1)^2 + (9-1)^2} = \sqrt{68}$, so the total length is approximately $\sqrt{2} + \sqrt{68} =$ 9.66... .

We can use Sage *to compute some more Riemann sum approximations and also this arc length exactly.*

```
sage: f1 = lambda x: sqrt(1+4*x^2)
sage: f = Piecewise([[(0,3),f1]])
sage: n = 10; RR(f.riemann_sum_integral_approximation(n))
8.99946939777166
sage: n = 50; RR(f.riemann_sum_integral_approximation(n))
9.59519771936512
sage: n = 100; RR(f.riemann_sum_integral_approximation(n))
9.67099527976211
sage: n = 200; RR(f.riemann_sum_integral_approximation(n))
9.70900502940468
sage: integral(sqrt(1+(2*x)^2),x,0,3)
(arcsinh(6) + 6*sqrt(37))/4
sage: RR(integral(sqrt(1+(2*x)^2),x,0,3))
9.74708875860856
```

In other words,

$$\int_0^3 \sqrt{1 + (2x)^2}\, dx = \frac{\sinh^{-1} 6 + 6\sqrt{37}}{4} = 9.74... \, .$$

Note that if we reflect this curve about the $y = x$ line then the resulting part of the curve must have the same arc length. Agreed? Do you see that the reflected *curve is $y = \sqrt{x}$, $0 \le x \le 9$? In this case the reflected points are $(0,0)$, $(1,1)$, and $(9,3)$. The lengths of the reflected pieces are $\sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$ and $\sqrt{(9-1)^2 + (3-1)^2} = \sqrt{68}$, so the total length is (still) approximately $\sqrt{2} + \sqrt{68} = 9.66...$. The integral describing the arc length is*

$$\int_0^9 \sqrt{1 + (\frac{1}{2}x^{-1/2})^2}\, dx = \int_0^9 \sqrt{1 + \frac{1}{4}x^{-1}}\, dx.$$

This is a harder integral to compute[2] *but we can use* Sage *to compute this reflected arc length fairly accurately.*

```
sage: f1 = lambda x: sqrt(1+1/(4*x))
sage: numerical_integral(f1, 0, 9, max_points=100)
(9.7470886680795221, 7.9546440984616276e-06)
```

The output is a pair, the first coordinate is the approximate numerical value of the integral and the second is an upper bound for the error term. This is in agreement with the above answer.

Parametric equations: When the curve C is described by pairs (x, y), where x and y are functions of t, $x = x(t)$ and $y = y(t)$, for $\alpha \le t \le \beta$, we can factor (Δt_i) from inside the radical and simplify:

$$\sum_i \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_i \sqrt{(\Delta x_i/\Delta t_i)^2 + (\Delta y_i/\Delta t_i)^2}\Delta t_i.$$

This is a Riemann sum. Taking limits, we get the following formula.

Theorem 2.5.2. *The length of the curve C described by the graph of the parametric equations $x = x(t)$ and $y = y(t)$, for $\alpha \le t \le \beta$, is given by*

$$\int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

Example 2.5.2. *Represent the length of each curve as a definite integral.*
(a) The length of $y = e^x$ between $(0,1)$ and $(1,e)$.
(b) The length of the parametric curve $x(t) = \cos(t)$ and $y(t) = \sin(t)$ for $0 \le t \le 2\pi$.
Solution: (a) $\int_0^1 \sqrt{1 + e^{2x}}\, dx$. This looks complicated (and it is) but amazingly enough Sage *has no problem with it:*

```
sage: f1 = lambda x: sqrt(1+exp(2*x))
sage: integral(f1(x), x, 0, 1)
-arcsinh(e^-1) + arcsinh(1) + sqrt(e^2 + 1) - sqrt(2)
```

[2]Of course, there is no need to, since we already know its value!

```
sage: RR(integral(f1(x), x, 0, 1))
2.00349711162735
```

(b) $\int_0^{2\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt = \int_0^{2\pi} \sqrt{1}\, dt = 2\pi$.

2.5.2 3-D Arc length

The parametric equation form of arc length extends very nicely to 3 dimensions. If a curve C in 3-dimensions (Figure 2.26) is given parametrically by $x = x(t)$, $y = y(t)$, and $z = z(t)$ for $a \le t \le b$, then the distance between the successive points $(x_{i-1}, y_{i-1}, z_{i-1})$ and (x_i, y_i, z_i) is $\sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2}$.

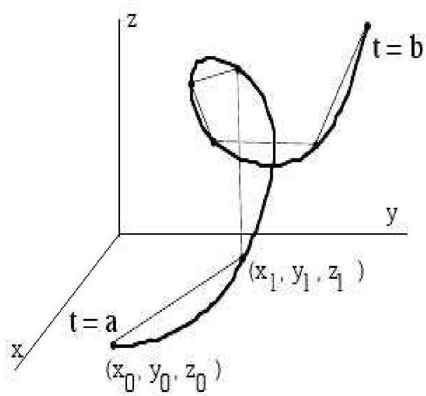

Figure 2.26: The arc length or a space curve.

We can, as before, factor $(\Delta t_i)^2$ from each term under the radical, sum the pieces to get a Riemann sum, and take a limit of the Riemann sum to get a definite integral representing the length of the curve C.

$$\sum_i \sqrt{\Delta x_i^2 + \Delta y_i^2 + \Delta z_i^2} = \sqrt{(\Delta x_i/\Delta t_i)^2 + (\Delta y_i/\Delta t_i)^2 + (\Delta z_i/\Delta t_i)^2}\Delta t_i$$

$$\to \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Theorem 2.5.3. *If a curve C in 3-dimensions is given parametrically by $x = x(t)$, $y = y(t)$, and $z = z(t)$ for $a \le t \le b$, then the arc length is*

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt.$$

Example 2.5.3. *Find the arc length of the helix $x = \cos(t)$, $y = \sin(t)$, $z = t$ for $0 \leq t \leq 4\pi$.*

Solution: We have

$$\begin{aligned}
\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt &= \int_0^{4\pi} \sqrt{\sin(t)^2 + \cos(t)^2 + 1}\, dt \\
&= \int_0^{4\pi} \sqrt{2}\, dt = 4\sqrt{2}\pi.
\end{aligned}$$

Chapter 3

Polar coordinates and trigonometric integrals

The rectangular coordinate system is immensely useful, but it is not the only way to assign an address to a point in the plane and sometimes it is not the most useful. In applications to physical problems where there is some "cylindrical symmetry", such as a vibrating drum or water moving along a pipe, the most natural coordinates are often polar coordinates rather than rectangular coordinates.

In many experimental situations, our location is fixed and we, using sonar or radar, take readings in different directions (Figure 3.2); this information can be graphed using rectangular coordinates (e.g., with the angle on the horizontal axis and the measurement on the vertical axis).

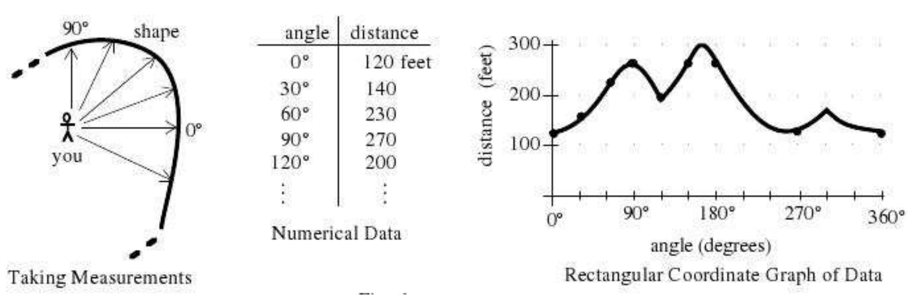

Figure 3.1: Sonar and radar use polar coordinates.

Sometimes, however, it is more useful to plot the information in a way similar to the way in which it was collected, as magnitudes along radial lines (Figure 3.2). This system is called the Polar Coordinate System.

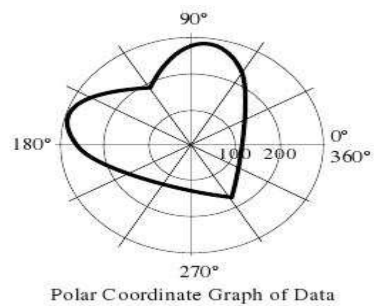

Figure 3.2: Polar coordinates.

Example 3.0.4. *SOS! You've just received a distress signal from a ship located at A on your radar screen (Figure 3.3). Describe its location to your captain so your vessel can speed to the rescue.*

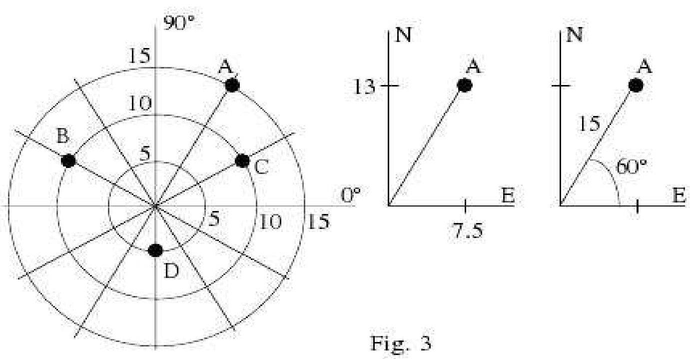

Figure 3.3: Polar coordinate figure for Example 3.0.4.

Solution: You could convert the relative location of the other ship to rectangular coordinates and then tell your captain to go due east for 7.5 miles and north for 13 miles, but that certainly is not the quickest way to reach the other ship. It is better to tell the captain to sail for 15 miles in the direction of 60^o*. If the distressed ship was at B on the radar screen, your vessel should sail for 10 miles in the direction* 150^o*. (Real radar screens have* 0^o *at the top of the screen, but the convention in mathematics is to put* 0^o *in the direction of the positive x–axis and to measure positive angles counterclockwise from there. And of course a real sailor speaks of "bearing" and "range" instead of direction and magnitude.)*

Practice 3.0.1. *Describe the locations of the ships at C and D in Figure 3.3 by giving a distance and a direction to those ships from your current position at the center of the radar screen.*

Points in Polar Coordinates: To construct a polar coordinate system we need a starting point (called the origin or pole) for the magnitude measurements and a starting direction (called the polar axis) for the angle measurements. A **polar coordinate pair** for a point P in the plane is an ordered pair (r, θ), where r is the directed distance along a radial line from O to P, and θ is the angle formed by the polar axis and the segment OP. The angle θ is positive when the angle of the radial line OP is measured counterclockwise from the polar axis, and θ is negative when measured clockwise.

Degree or Radian Measure for θ?: Either degree or radian measure can be used for the angle in the polar coordinate system, but when we differentiate and integrate trigonometric functions of θ we will *always* want all of the angles to be given in *radian measure*. From now on, we will primarily use radian measure. *You should assume that all angles are given in radian measure unless the units "^{o}" ("degrees") are shown.*

In the rectangular coordinate system, the derivative dy/dx measured both the rate of change of y with respect to x and the slope of the tangent line. In the polar coordinate system $dr/d\theta$ measures the rate of change of r with respect to θ. The sign of $dr/d\theta$ tells us whether r is increasing or decreasing as θ increases.

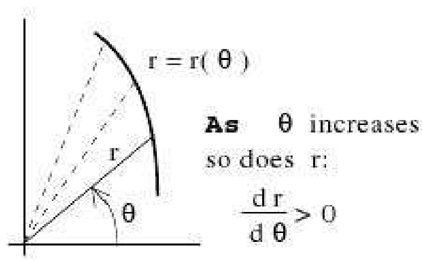

Figure 3.4: r changing as a function of θ.

3.1 Polar Coordinates

Rectangular coordinates allow us to describe a point (x, y) in the plane in a different way, namely

$$(x, y) \leftrightarrow (r, \theta),$$

where r is any real number and θ is an angle.

Polar coordinates are extremely useful, especially when thinking about complex numbers. Note, however, that the (r, θ) representation of a point is very non-unique.

First, θ is not determined by the point. You could add 2π to it and get the same point:

$$\left(2, \frac{\pi}{4}\right) = \left(2, \frac{9\pi}{4}\right) = \left(2, \frac{\pi}{4} + 389 \cdot 2\pi\right). = \left(2, \frac{-7\pi}{4}\right)$$

Also that r can be negative introduces further non-uniqueness:

$$\left(1, \frac{\pi}{2}\right) = \left(-1, \frac{3\pi}{2}\right).$$

Think about this as follows: facing in the direction $3\pi/2$ and backing up 1 meter gets you to the same point as looking in the direction $\pi/2$ and walking forward 1 meter.

We can convert back and forth between Cartesian and polar coordinates using that

$$x = r\cos(\theta) \tag{3.1}$$

$$y = r\sin(\theta), \tag{3.2}$$

and in the other direction

$$r^2 = x^2 + y^2 \tag{3.3}$$

$$\tan(\theta) = \frac{y}{x} \tag{3.4}$$

(Thus $r = \pm\sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.)

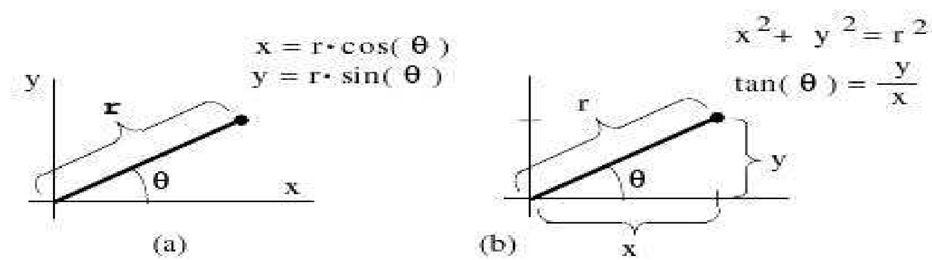

Figure 3.5: Rectangular to polar coordinate conversion.

Example 3.1.1. *Sketch $r = \sin(\theta)$, which is a circle sitting on top the x axis.*

We plug in points for one period of the function we are graphing—in this case $[0, 2\pi]$*:*

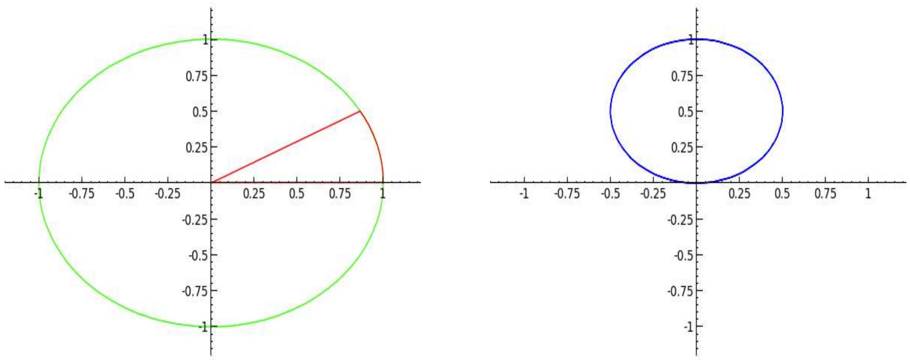

Figure 3.6: Plot of (a) $r = 1$, $0 < \theta < \pi/6$, and (b) $r = \sin(\theta)$, $0 < \theta < 2\pi$.

0	$\sin(0) = 0$
$\pi/6$	$\sin(\pi/6) = 1/2$
$\pi/4$	$\sin(\pi/4) = \frac{\sqrt{2}}{2}$
$\pi/2$	$\sin(\pi/2) = 1$
$3\pi/4$	$\sin(3\pi/4) = \frac{\sqrt{2}}{2}$
π	$\sin(\pi) = 0$
$\pi + \pi/6$	$\sin(\pi + \pi/6) = -1/2$

Notice it is nice to allow r to be negative, so we don't have to restrict the input. BUT it is really painful to draw this graph by hand.

To more accurately draw the graph, let's try converting the equation to one involving polar coordinates. This is easier if we multiply both sides by r:

$$r^2 = r\sin(\theta).$$

Note that the new equation has the extra solution ($r = 0, \theta =$ anything), so we have to be careful not to include this point. Now convert to Cartesian coordinates using (3.1) to obtain (3.3):

$$x^2 + y^2 = y. \tag{3.5}$$

The graph of (3.5) is the same as that of $r = \sin(\theta)$. To confirm this we complete the square:

$$\begin{aligned} x^2 + y^2 &= y \\ x^2 + y^2 - y &= 0 \\ x^2 + (y - 1/2)^2 &= 1/4 \end{aligned}$$

Thus the graph of (3.5) is a circle of radius $1/2$ centered at $(0, 1/2)$.

Actually *any* polar graph of the form $r = a\sin(\theta) + b\cos(\theta)$ is a circle (exercise for the interested reader).

3.2 Areas in Polar Coordinates

The previous section introduced the polar coordinate system and discussed how to plot points, how to create graphs of functions (from data, a rectangular graph, or a formula), and how to convert back and forth between the polar and rectangular coordinate systems. This section examines calculus in polar coordinates: rates of changes, slopes of tangent lines, areas, and lengths of curves. The results we obtain may look different, but they all follow from the approaches used in the rectangular coordinate system.

We know how to compute the area of a sector, i.e., piece of a circle with angle θ. [[draw picture]]. This is the basic polar region. The area is

$$A = (\text{fraction of the circle}) \cdot (\text{area of circle}) = \left(\frac{\theta}{2\pi}\right) \cdot \pi r^2 = \frac{1}{2} r^2 \theta.$$

We now imitate what we did before with Riemann sums. We chop up, approximate, and take a limit. Break the interval of angles from a to b into n subintervals. Choose θ_i^* in each interval. The area of each slice is approximately $(1/2) f(\theta_i^*)^2 \theta_i^2$. Thus

$$A = \text{Area of the shaded region} \sim \sum_{i=1}^{n} \frac{1}{2} f(\theta_i^*)^2 \Delta(\theta).$$

Taking the limit, we see that

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2} f(\theta_i^*)^2 \Delta(\theta) = \frac{1}{2} \cdot \int_a^b f(\theta)^2 d\theta. \tag{3.6}$$

Amazing! By understanding the definition of Riemann sum, we've derived a formula for areas swept out by a polar graph. But does it work in practice?

Example 3.2.1. *Find the area enclosed by one leaf of the four-leaved rose $r = \cos(2\theta)$.*

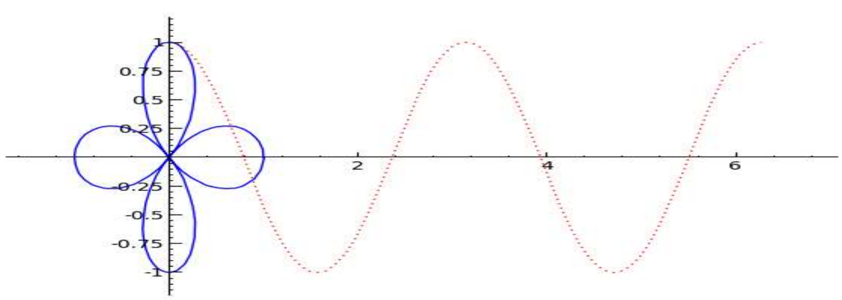

Figure 3.7: Graph of $y = \cos(2x)$ and $r = \cos(2\theta)$.

This was plotted in Sage *using these commands:*

```
sage: P1 = polar_plot(lambda x:cos(2*x), 0, 2*pi, rgbcolor=(0,0,1))
sage: P2 = plot(lambda x:cos(2*x), 0, 2*pi, rgbcolor=(1,0,0),linestyle=":")
sage: show(P1+P2)
```

To find the area using the methods we know so far, we would need to find a function $y = f(x)$ that gives the "height" of the leaf.

Multiplying both sides of the equation $r = \cos(2\theta)$ by r yields

$$r^2 = r\cos(2\theta) = r(\cos^2\theta - \sin^2\theta) = \frac{1}{r}((r\cos\theta)^2 - (r\sin\theta)^2).$$

Because $r^2 = x^2 + y^2$ and $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we have

$$x^2 + y^2 = \frac{1}{\sqrt{x^2+y^2}}(x^2 - y^2).$$

Solving for y is a crazy mess, and then integrating? It seems impossible!

But it isn't... if we remember the basic idea of integral calculus: integral equals area.

We need the boundaries of integration to determine the area. Start at $\theta = -\pi/4$ and go to $\theta = \pi/4$. As a check, note that $\cos((-\pi/4)\cdot 2) = 0 = \cos((\pi/4)\cdot 2)$. We evaluate

$$\begin{aligned}
\frac{1}{2}\cdot\int_{-\pi/4}^{\pi/4}\cos(2\theta)^2 d\theta &= \int_0^{\pi/4}\cos(2\theta)^2 d\theta \qquad \textit{(even function)}\\
&= \frac{1}{2}\int_0^{\pi/4}(1+\cos(4\theta))d\theta\\
&= \frac{1}{2}\left[\theta + \frac{1}{4}\cdot\sin(4\theta)\right]_0^{\pi/4}\\
&= \frac{\pi}{8}.
\end{aligned}$$

We used that

$$\cos^2(x) = (1+\cos(2x))/2 \qquad \textit{and} \qquad \sin^2(x) = (1-\sin(2x))/2, \tag{3.7}$$

which follow from

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x).$$

Therefore, by (3.6), the area is $A = \frac{\pi}{8}$.

Example 3.2.2. *Find area of region inside the curve $r = 3\cos(\theta)$ and outside the cardiod curve $r = 1 + \cos(\theta)$.*

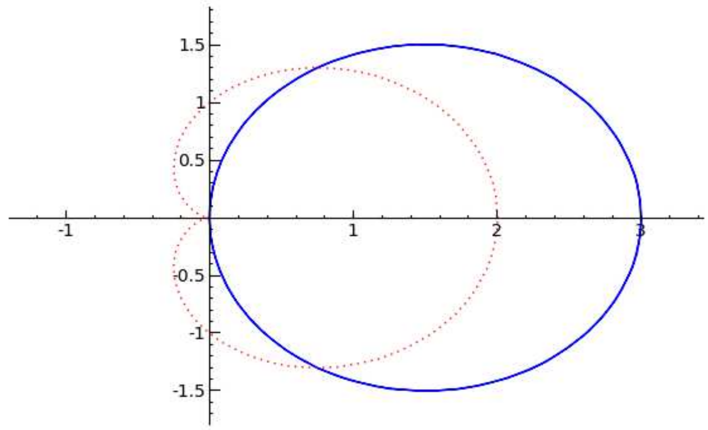

Figure 3.8: Graph of $r = 3\cos(x)$ and $r = 1 + \cos(\theta)$.

Figure 3.8 was plotted in Sage *using these commands:*

```
sage: P1 = polar_plot(lambda x:3*cos(x), 0, 2*pi, rgbcolor=(0,0,1))
sage: P2 = polar_plot(lambda x:1+cos(x), 0, 2*pi, rgbcolor=(1,0,0),linestyle=":")
sage: show(P1+P2)
```

Solution: This is the same as before. It's the difference of two areas. Figure out the limits, which are where the curves intersect, i.e., the θ *such that*

$$3\cos(\theta) = 1 + \cos(\theta).$$

Solving, $2\cos(\theta) = 1$, *so* $\cos(\theta) = 1/2$, *hence* $\theta = \pi/3$ *and* $\theta = -\pi/3$. *Thus the area is*

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (3\cos(\theta))^2 - (1+\cos(\theta))^2 d\theta \\ &= \int_{0}^{\pi/3} (3\cos(\theta))^2 - (1+\cos(\theta))^2 d\theta, \end{aligned}$$

since the integrand is an even function. Now, expand this out algebraically and *integrate term-by-term:*

$$\begin{aligned}
\int_0^{\pi/3} (3\cos(\theta))^2 - (1+\cos(\theta))^2 d\theta &= \int_0^{\pi/3} (8\cos^2(\theta) - 2\cos(\theta) - 1) d\theta \\
&= \int_0^{\pi/3} \left(8 \cdot \frac{1}{2}(1+\cos(2\theta)) - 2\cos(\theta) - 1\right) d\theta \\
&= \int_0^{\pi/3} 3 + 4\cos(2\theta) - 2\cos(\theta) d\theta \\
&= \Big[3\theta + 2\sin(2\theta) - 2\sin(\theta)\Big]_0^{\pi/3} \\
&= \pi + 2 \cdot \sqrt{\frac{3}{2}} - 2\sqrt{\frac{3}{2}} - 0 - 2 \cdot 0 - 2 \cdot 0 \\
&= \pi\,.
\end{aligned}$$

Practice 3.2.1. *The area of the shaded region in Figure 3.9.*

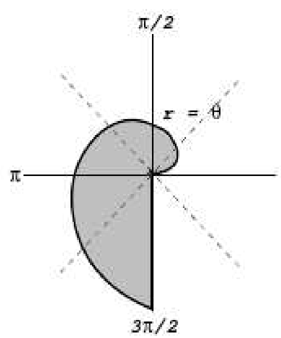

Figure 3.9: Graph of $r = \theta$.

3.3 Complex Numbers

A complex number is an expression of the form $a + bi$, where a and b are real numbers, and $i^2 = -1$. We add and multiply complex numbers as follows:

$$\begin{aligned}
(a+bi) + (c+di) &= (a+c) + (b+d)i \\
(a+bi) \cdot (c+di) &= (ac-bd) + (ad+bc)i
\end{aligned}$$

The *complex conjugate* of a complex number is

$$\overline{a+bi} = a - bi.$$

Note that
$$(a+bi)(\overline{a+bi}) = a^2 + b^2$$
is a real number (has no complex part).

If $c + di \neq 0$, then
$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{1}{c^2+d^2}((ac+bd)+(bc-ad)i).$$

Example 3.3.1. $(1-2i)(8-3i) = 2-19i$ *and* $1/(1+i) = (1-i)/2 = 1/2-(1/2)i$.

Complex numbers are incredibly useful in providing better ways to understand ideas in calculus, and more generally in many applications (e.g., electrical engineering, quantum mechanics, fractals, etc.). For example,

- Every polynomial $f(x)$ **factors** as a product of linear factors $(x-\alpha)$, if we allow the α's in the factorization to be complex numbers. For example,
$$f(x) = x^2 + 1 = (x-i)(x+i).$$
This will provide an easier to use variant of the "partial fractions" integration technique, which we will see later.

- Complex numbers are in **correspondence** with points in the plane via $(x, y) \leftrightarrow x + iy$. Via this correspondence we obtain a way to add and *multiply* points in the plane.

- Similarly, points in **polar coordinates** correspond to complex numbers:
$$(r, \theta) \leftrightarrow r(\cos(\theta) + i\sin(\theta)).$$

- Complex numbers provide a very nice way to remember and **understand trig identities**.

3.3.1 Polar Form

The *polar form* of a complex number $x + iy$ is $r(\cos(\theta) + i\sin(\theta))$ where (r, θ) are any choice of polar coordinates that represent the point (x, y) in rectangular coordinates. Recall that you can find the polar form of a point using that
$$r = \sqrt{x^2+y^2} \quad \text{and} \quad \theta = \tan^{-1}(y/x).$$

NOTE: Historically, the "existence" of complex numbers wasn't generally accepted until people got used to a geometric interpretation of them.

Example 3.3.2. *Find the polar form of* $1 + i$.

Solution. *We have* $r = \sqrt{2}$, *so*
$$1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right).$$

Example 3.3.3. *Find the polar form of $\sqrt{3}-i$.*
Solution. *We have $r=\sqrt{3+1}=2$, so*

$$\sqrt{3}-i = 2\left(\frac{\sqrt{3}}{2}+i\frac{-1}{2}\right) = 2\left(\cos(-\pi/6)+i\sin(-\pi/6)\right)$$

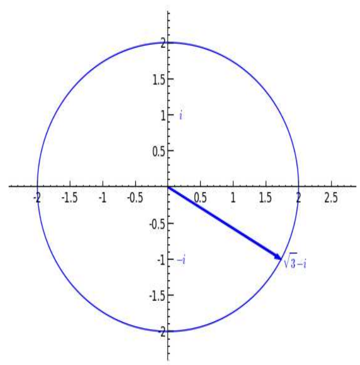

Figure 3.10: Plot of $\sqrt{3}-i$, as a vector.

This was plotted in Sage *using these commands:*

```
sage: P1 = circle((0,0), 2)
sage: P2 = arrow((0,0),(sqrt(3),-1))
sage: P3 = text("$\sqrt{3}-i$",(2,-1))
sage: P4 = text("$-i$",(0.2,-1))
sage: P5 = text("$i$",(0.2,1))
sage: show(P1+P2+P3+P4+P5)
```

Finding the polar form of a complex number is exactly the same problem as finding polar coordinates of a point in rectangular coordinates. The only hard part is figuring out what θ is.

If we write complex numbers in rectangular form, their sum is easy to compute:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

The beauty of polar coordinates is that if we write two complex numbers in polar form, then their *product* is very easy to compute:

$$r_1(\cos(\theta_1)+i\sin(\theta_1))\cdot r_2(\cos(\theta_2)+i\sin(\theta_2)) = (r_1r_2)(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)).$$

The magnitudes multiply and the angles add. The above formula is true because of the double angle identities for sin and cos:

$$
\begin{aligned}
&(\cos(\theta_1) + i\sin(\theta_1)) \cdot (\cos(\theta_2) + i\sin(\theta_2)) \\
&\qquad = (\cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2)) + i(\sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2)) \\
&\qquad = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).
\end{aligned}
$$

For example, the power of a singular complex number in polar form is easy to compute; just power the r and multiply the angle.

Theorem 3.3.1 (De Moivre's)**.** *For any integer n we have*

$$(r(\cos(\theta) + i\sin(\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Example 3.3.4. *Compute $(1+i)^{2006}$.*
Solution. *We have*

$$
\begin{aligned}
(1+i)^{2006} &= (\sqrt{2}\,(\cos(\pi/4) + i\sin(\pi/4)))^{2006} \\
&= \sqrt{2}^{2006}\,(\cos(2006\pi/4) + i\sin(2006\pi/4))) \\
&= 2^{1003}\,(\cos(3\pi/2) + i\sin(3\pi/2))) \\
&= -2^{1003}i
\end{aligned}
$$

To get $\cos(2006\pi/4) = \cos(3\pi/2)$ we use that $2006/4 = 501.5$, so by periodicity of cosine, we have

$$\cos(2006\pi/4) = \cos((501.5)\pi - 250(2\pi)) = \cos(1.5\pi) = \cos(3\pi/2).$$

Here's a quick summary of what we've just learned: Given a point (x, y) in the plane, we can also view it as $x+iy$ or in polar form as $r(\cos(\theta)+i\sin(\theta))$. Polar form is great since it's good for multiplication, powering, and for extracting roots:

$$r_1(\cos(\theta_1)+i\sin(\theta_1))r_2(\cos(\theta_2)+i\sin(\theta_2)) = (r_1r_2)(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)).$$

(If you divide, you subtract the angle.) The point is that the polar form *works better* with multiplication than the rectangular form. For any integer n, we have

$$(r(\cos(\theta) + i\sin(\theta)))^n = r^n(\cos(n\theta) + i\sin(n\theta)).$$

Since we know how to raise a complex number in polar form to the nth power, we can find all numbers with a given power, hence find the nth roots of a complex number.

Proposition 3.3.1 (nth roots)**.** *A complex number $z = r(\cos(\theta) + i\sin(\theta))$ has n distinct nth roots:*

$$r^{1/n}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right),$$

for $k = 0, 1, \ldots, n-1$. Here $r^{1/n}$ is the real positive n-th root of r.

As a double-check, note that by De Moivre, each number listed in the proposition has nth power equal to z.

An application of De Moivre is to computing $\sin(n\theta)$ and $\cos(n\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$. For example,

$$\begin{aligned}\cos(3\theta) + i\sin(3\theta) &= (\cos(\theta) + i\sin(\theta))^3 \\ &= (\cos(\theta)^3 - 3\cos(\theta)\sin(\theta)^2) + i(3\cos(\theta)^2\sin(\theta) - \sin(\theta)^3)\end{aligned}$$

Equate real and imaginary parts to get formulas for $\cos(3\theta)$ and $\sin(3\theta)$. In the next section we will discuss going in the other direction, i.e., writing powers of sin and cos in terms of sin and cosine.

Example 3.3.5. *Find the cube roots of* 2.
Solution. *Write* 2 *in polar form as*

$$2 = 2(\cos(0) + i\sin(0)).$$

Then the three cube roots of 2 *are*

$$2^{1/3}(\cos(2\pi k/3) + i\sin(2\pi k/3)),$$

for $k = 0, 1, 2$. *I.e.,*

$$2^{1/3}, \quad 2^{1/3}(-1/2 + i\sqrt{3}/2), \quad 2^{1/3}(-1/2 - i\sqrt{3}/2).$$

3.4 Complex Exponentials and Trigonometric Identities

Recall that

$$r_1(\cos(\theta_1)+i\sin(\theta_1))r_2(\cos(\theta_2)+i\sin(\theta_2)) = (r_1r_2)(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)). \tag{3.8}$$

The angles add. You've seen something similar before:

$$e^a e^b = a^{a+b}.$$

This connection between exponentiation and (3.8) gives us an idea!

If $z = x + iy$ is a complex number, *define*

$$e^z = e^x(\cos(y) + i\sin(y)).$$

We have just written polar coordinates in another form. It's a shorthand for the polar form of a complex number:

$$r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}.$$

Theorem 3.4.1. *If z_1, z_2 are two complex numbers, then*

$$e^{z_1}e^{z_2} = e^{z_1+z_2}$$

Proof.

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{a_1}(\cos(b_1) + i\sin(b_1)) \cdot e^{a_2}(\cos(b_2) + i\sin(b_2)) \\ &= e^{a_1+a_2}(\cos(b_1+b_2) + i\sin(b_1+b_2)) \\ &= e^{z_1+z_2}. \end{aligned}$$

Here we have just used (3.8). □

The following theorem is amazing, since it involves calculus.

Theorem 3.4.2. *If w is a complex number, then*

$$\frac{d}{dx}e^{wx} = we^{wx},$$

for x real. In fact, this is even true for x a complex variable (but we haven't defined differentiation for complex variables yet).

Proof. Write $w = a + bi$.

$$\begin{aligned} \frac{d}{dx}e^{wx} &= \frac{d}{dx}e^{ax+bix} \\ &= \frac{d}{dx}(e^{ax}(\cos(bx) + i\sin(bx))) \\ &= \frac{d}{dx}(e^{ax}\cos(bx) + ie^{ax}\sin(bx)) \\ &= \frac{d}{dx}(e^{ax}\cos(bx)) + i\frac{d}{dx}(e^{ax}\sin(bx)) \end{aligned}$$

Now we use the product rule to get

$$\begin{aligned} &\frac{d}{dx}(e^{ax}\cos(bx)) + i\frac{d}{dx}(e^{ax}\sin(bx)) \\ &\qquad = ae^{ax}\cos(bx) - be^{ax}\sin(bx) + i(ae^{ax}\sin(bx) + be^{ax}\cos(bx)) \\ &\qquad = e^{ax}(a\cos(bx) - b\sin(bx) + i(a\sin(bx) + b\cos(bx)) \end{aligned}$$

On the other hand,

$$\begin{aligned} we^{wx} &= (a+bi)e^{ax+bxi} \\ &= (a+bi)e^{ax}(\cos(bx) + i\sin(bx)) \\ &= e^{ax}(a+bi)(\cos(bx) + i\sin(bx)) \\ &= e^{ax}((a\cos(bx) - b\sin(bx)) + i(a\sin(bx)) + b\cos(bx)) \end{aligned}$$

Wow!! We did it! □

That Theorem 3.4.2 is true is pretty amazing. It's what really gets complex analysis going.

Example 3.4.1. *Here's another amusing fact (if only for its obfuscating effect):* $1 = -e^{i\pi}$.
Solution. *By definition, have* $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i0 = -1$.

3.4.1 Trigonometry and Complex Exponentials

Amazingly, trig functions can also be expressed back in terms of the complex exponential. Then *everything* involving trig functions can be transformed into something involving the exponential function. This is very surprising.

In order to easily obtain trig identities like $\cos(x)^2 + \sin(x)^2 = 1$, let's write $\cos(x)$ and $\sin(x)$ as complex exponentials. From the definitions we have

$$e^{ix} = \cos(x) + i\sin(x),$$

so

$$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x).$$

Adding these two equations and dividing by 2 yields a formula for $\cos(x)$, and subtracting and dividing by $2i$ gives a formula for $\sin(x)$:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}. \tag{3.9}$$

We can now derive trig identities. For example,

$$\begin{aligned}
\sin(2x) &= \frac{e^{i2x} - e^{-i2x}}{2i} \\
&= \frac{(e^{ix} - e^{-ix})(e^{ix} + e^{-ix})}{2i} \\
&= 2\frac{e^{ix} - e^{-ix}}{2i}\frac{e^{ix} + e^{-ix}}{2} = 2\sin(x)\cos(x).
\end{aligned}$$

I'm unimpressed, given that you can get this much more directly using

$$(\cos(2x) + i\sin(2x)) = (\cos(x) + i\sin(x))^2 = \cos^2(x) - \sin^2(x) + i2\cos(x)\sin(x),$$

and equating imaginary parts. But there are more interesting examples.

Next we verify that (3.9) implies that $\cos(x)^2 + \sin(x)^2 = 1$. We have

$$\begin{aligned}
4(\cos(x)^2 + \sin(x)^2) &= \left(e^{ix} + e^{-ix}\right)^2 + \left(\frac{e^{ix} - e^{-ix}}{i}\right)^2 \\
&= e^{2ix} + 2 + e^{-2ix} - (e^{2ix} - 2 + e^{-2ix}) = 4.
\end{aligned}$$

The equality just appears as a follow-your-nose algebraic calculation.

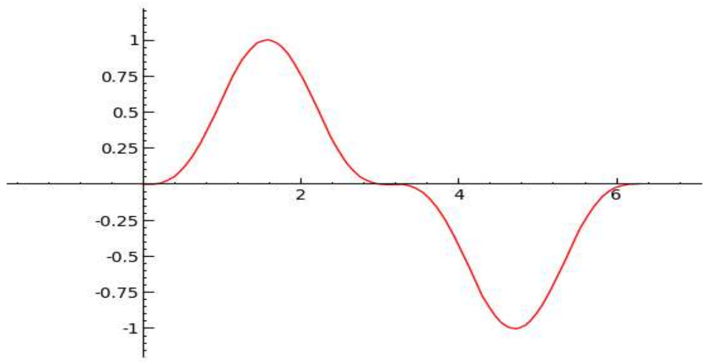

Figure 3.11: Plot of $y = \sin(x)^3$.

Example 3.4.2. *Compute* $\sin(x)^3$ *as a sum of sines and cosines with no powers.*

Solution. *We use (3.9):*

$$\begin{aligned}
\sin(x)^3 &= \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 \\
&= \left(\frac{1}{2i}\right)^3 (e^{ix} - e^{-ix})^3 \\
&= \left(\frac{1}{2i}\right)^3 (e^{ix} - e^{-ix})(e^{ix} - e^{-ix})(e^{ix} - e^{-ix}) \\
&= \left(\frac{1}{2i}\right)^3 (e^{ix} - e^{-ix})(e^{2ix} - 2 + e^{-2ix}) \\
&= \left(\frac{1}{2i}\right)^3 (e^{3ix} - 2e^{ix} + e^{-ix} - e^{ix} + 2e^{-ix} - e^{-3ix}) \\
&= \left(\frac{1}{2i}\right)^3 ((e^{3ix} - e^{-3ix}) - 3(e^{ix} - e^{-ix})) \\
&= -\left(\frac{1}{4}\right)\left[\frac{e^{3ix} - e^{-3ix}}{2i} - 3 \cdot \frac{e^{ix} - e^{-ix}}{2i}\right] \\
&= \frac{3\sin(x) - \sin(3x)}{4}.
\end{aligned}$$

You can also do this in Sage*:*

```
sage: y = sin(x)^3
sage: maxima(y).trigreduce()
(3*sin(x)-sin(3*x))/4
```

3.5 Integrals of Trigonometric Functions

There are an overwhelming number of combinations of trigonometric functions which appear in integrals, but fortunately they fall into a few patterns and most of their integrals can be found using reduction formulas and tables of integrals. This section examines some of the patterns of these combinations and illustrates how some of their integrals can be derived.

$$\int \sin(ax)\sin(bx)\,dx, \quad \int \cos(ax)\cos(bx)\,dx, \quad \int \sin(ax)\cos(bx)\,dx.$$

Products of Sine and Cosine:

All of these integrals are handled by referring to the trigonometric identities for sine and cosine of sums and differences:

$$\begin{aligned}
\sin(A+B) &= \sin(A)\cos(B) + \cos(A)\sin(B)\\
\sin(A-B) &= \sin(A)\cos(B) - \cos(A)\sin(B)\\
\cos(A+B) &= \cos(A)\cos(B) - \sin(A)\sin(B)\\
\cos(A-B) &= \cos(A)\cos(B) + \sin(A)\sin(B).
\end{aligned}$$

By adding or subtracting the appropriate pairs of identities, we can write the various products such as $\sin(ax)\cos(bx)$ as a sum or difference of single sines or cosines. For example, by adding the first two identities we get $2\sin(A)\cos(B) = \sin(A+B) + \sin(A-B)$ so $\sin(A)\cos(B) = 2\sin(A+B) + \sin(A-B)$. Using this last identity, the integral of $\sin(ax)\cos(bx)$ for $a \neq b$ is relatively easy:

$$\begin{aligned}
\int \sin(ax)\cos(bx)\,dx &= \int \frac{1}{2}[\sin((a+b)x) + \sin((a-b)x)]\,dx\\
&= \frac{1}{2}[\frac{-\cos((a-b)x)}{a-b} + \frac{-\cos((a+b)x)}{a+b}] + C.
\end{aligned}$$

The other integrals of products of sine and cosine follow in a similar manner.

For $a \neq b$:

$$\boxed{\begin{aligned}
\int \sin(ax)\sin(bx)\,dx &= \frac{1}{2}[\frac{\sin((a-b)x)}{a-b} - \frac{\sin((a+b)x)}{a+b}] + C,\\
\int \cos(ax)\cos(bx)\,dx &= \frac{1}{2}[\frac{\sin((a-b)x)}{a-b} + \frac{\sin((a+b)x)}{a+b}] + C,\\
\int \sin(ax)\cos(bx)\,dx &= -\frac{1}{2}[\frac{\cos((a-b)x)}{a-b} + \frac{\cos((a+b)x)}{a+b}] + C.
\end{aligned}}$$

This is confirmed by Sage:

```
sage: a,b = var("a,b")
sage: integral(sin(a*x)*cos(b*x),x)
-((b - a)*cos((b + a)*x) + (-b - a)*cos((b - a)*x))/(2*b^2 - 2*a^2)
sage: integral(sin(a*x)*sin(b*x),x)
-((b - a)*sin((b + a)*x) + (-b - a)*sin((b - a)*x))/(2*b^2 - 2*a^2)
sage: integral(cos(a*x)*cos(b*x),x)
((b - a)*sin((b + a)*x) + (b + a)*sin((b - a)*x))/(2*b^2 - 2*a^2)
```

For $a = b$:

$$\boxed{\begin{aligned} \int \sin(ax)^2\,dx &= \frac{x}{2} - \frac{\sin(ax)\cos(ax)}{2a} + C, \\ \int \cos(ax)^2\,dx &= \frac{x}{2} + \frac{\sin(ax)\cos(ax)}{2a} + C, \\ \int \sin(ax)\cos(ax)\,dx &= \frac{\sin(ax)^2}{2a} + C. \end{aligned}}$$

The first and second of these integral formulas follow from the identities $\sin(ax)^2 = \frac{1-\cos(2ax)}{2}$ and $\cos(ax)^2 = \frac{1+\cos(2ax)}{2}$, and the third can be derived by a substitution using the variable to $u = \sin(ax)$. These formulas too are confirmed by Sage:

```
sage: a = var("a")
sage: integral(cos(a*x)^2,x)
(sin(2*a*x) + 2*a*x)/(4*a)
sage: integral(sin(a*x)^2,x)
-(sin(2*a*x) - 2*a*x)/(4*a)
sage: integral(sin(a*x)*cos(a*x),x)
-cos(a*x)^2/(2*a)
```

Remark 3.5.1. *Note that* Sage *tells us that* $\int \sin(ax)\cos(ax)\,dx = -\frac{\cos(ax)^2}{2a} + C$ *but the table tells us that* $\int \sin(ax)\cos(ax)\,dx = \frac{\sin(ax)^2}{2a} + C$. *Aside from the ambiguity in the notation "+C", these are the same since* $\sin(ax)^2 = -\cos(ax) + 1$. *In other words, if you keep in mind that "+C" in one equation is not the same as "+C" in another, these formulas are the same.*

Example 3.5.1. *Compute* $\int \sin^3(x)dx$.
We use trig. identities and compute the integral directly as follows:

$$\begin{aligned}
\int \sin^3(x)dx &= \int \sin^2(x)\sin(x)dx \\
&= \int [1-\cos^2(x)]\sin(x)dx \\
&= -\cos(x) + \frac{1}{3}\cos^3(x) + c \qquad \textit{(substitution } u = \cos(x)\textit{)}
\end{aligned}$$

This idea always works for odd *powers of* $\sin(x)$.

Example 3.5.2. *What about* even *powers?! Compute* $\int \sin^4(x)dx$. *We have*

$$\begin{aligned}
\sin^4(x) &= [\sin^2(x)]^2 \\
&= \left[\frac{1-\cos(2x)}{2}\right]^2 \\
&= \frac{1}{4}\cdot\left[1 - 2\cos(2x) + \cos^2(2x)\right] \\
&= \frac{1}{4}\left[1 - 2\cos(2x) + \frac{1}{2} + \frac{1}{2}\cos(4x)\right]
\end{aligned}$$

Thus

$$\begin{aligned}
\int \sin^4(x)dx &= \int \left[\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right]dx \\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + c.
\end{aligned}$$

Key Trick: *Realize that we should write* $\sin^4(x)$ *as* $(\sin^2(x))^2$. *The rest is straightforward.*

Patterns for $\int \sin(x)^m \cos(x)^n\, dx$: If the exponent of sine is *odd*, we can split off one factor $\sin(x)$ and use the trig identity $\sin(x)^2 = 1-\cos(x)^2$ to rewrite the remaining even power of sine in terms of cosine. Then the change of variable $u = \cos(x)$ makes all of the integrals straightforward. If the exponent of cosine is *odd*, we can split off one factor $\cos(x)$ and use the trig identity $\cos(x)^2 = 1-\sin(x)^2$ to rewrite the remaining even power of sine in terms of cosine. Then the change of variable $u = \sin(x)$ makes all of the integrals straightforward. If both exponents are even, we can use the identities $\sin(x)^2 = \frac{1-\cos(2x)}{2}$ and $\cos(x)^2 = \frac{1+\cos(2x)}{2}$ to rewrite the integral in terms of powers of $\cos(2x)$ and then proceed with integrating even powers of cosine.

Example 3.5.3. *This example illustrates a method for computing integrals of trig functions that doesn't require knowing any trig identities at all or any tricks. It is very tedious though. We compute* $\int \sin^3(x)dx$ *using* complex exponentials. *We have*

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2i}.$$

hence

$$\begin{aligned}
\int \sin^3(x)dx &= \int \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 dx \\
&= -\frac{1}{8i}\int (e^{ix} - e^{-ix})^3 dx \\
&= -\frac{1}{8i}\int (e^{ix} - e^{-ix})(e^{ix} - e^{-ix})(e^{ix} - e^{-ix}) dx \\
&= -\frac{1}{8i}\int (e^{2ix} - 2 + e^{-2ix})(e^{ix} - e^{-ix}) dx \\
&= -\frac{1}{8i}\int e^{3ix} - e^{ix} - 2e^{ix} + 2e^{-ix} + e^{-ix} - e^{-3ix} dx \\
&= -\frac{1}{8i}\int e^{3ix} - e^{-3ix} + 3e^{-ix} - 3e^{ix} dx \\
&= -\frac{1}{8i}\left(\frac{e^{3ix}}{3i} - \frac{e^{-3ix}}{-3i} + \frac{3e^{-ix}}{-i} - \frac{3e^{ix}}{i}\right) + c \\
&= \frac{1}{4}\left(\frac{1}{3}\cos(3x) - 3\cos(x)\right) + c \\
&= \frac{1}{12}\cos(3x) - \frac{3}{4}\cos(x) + C.
\end{aligned}$$

The answer looks totally different, but is in fact the same function.

Example 3.5.4. *The complex exponentials method used in the previous example also works for powers of different trig functions. For instance,*

$$\begin{aligned}
\int \sin^3(x)\cos^2(x)\,dx &= \int (\frac{e^{ix} - e^{-ix}}{2i})^3)(\frac{e^{ix} + e^{-ix}}{2})^2)\,dx \\
&= \int -ie^{5ix}/32 + ie^{3ix}/32 + ie^{ix}/16 - ie^{-ix}/16 - ie^{-3ix}/32 + ie^{-5ix}/32 \\
&= -e^{5ix}/160 + e^{3ix}/96 + e^{ix}/16 + e^{-ix}/16 + e^{-3ix}/96 - e^{-5ix}/160 + C \\
&= (-e^{5ix}/160 - e^{-5ix}/160) + (e^{3ix}/96 + e^{-3ix}/96) + (e^{ix}/16 + e^{-ix}/16) + C \\
&= -\cos(5x)/80 + \cos(3x)/32 + \cos(x)/8 + C.
\end{aligned}$$

Here are some more identities that we'll use in illustrating some tricks below.

$$\boxed{\frac{d}{dx}\tan(x) = \sec^2(x) \qquad (3.10)}$$

and

$$\boxed{\frac{d}{dx}\sec(x) = \sec(x)\tan(x). \qquad (3.11)}$$

Also,

$$\boxed{1 + \tan^2(x) = \sec^2(x). \qquad (3.12)}$$

Example 3.5.5. *Compute* $\int \tan^3(x)dx$. *We have*

$$\begin{aligned}
\int \tan^3(x)dx &= \int \tan(x)\tan^2(x)dx \\
&= \int \tan(x)\left[\sec^2(x) - 1\right] dx \\
&= \int \tan(x)\sec^2(x)dx - \int \tan(x)dx \\
&= \frac{1}{2}\tan^2(x) - \ln|\sec(x)| + c
\end{aligned}$$

Here we used the substitution $u = \tan(x)$, *so* $du = \sec^2(x)dx$, *so*

$$\int \tan(x)\sec^2(x)dx = \int udu = \frac{1}{2}u^2 + c = \frac{1}{2}\tan^2(x) + c.$$

Also, with the substitution $u = \cos(x)$ *and* $du = -\sin(x)dx$ *we get*

$$\int \tan(x)dx = \int \frac{\sin(x)}{\cos(x)}dx = -\int \frac{1}{u}du = -\ln|u| + c = -\ln|\sec(x)| + c.$$

Key trick: *Write* $\tan^3(x)$ *as* $\tan(x)\tan^2(x)$.

Example 3.5.6. *Here's one that combines trig identities with the funnest variant of integration by parts.* Compute $\int \sec^3(x)dx$.
We have

$$\int \sec^3(x)dx = \int \sec(x)\sec^2(x)dx.$$

Let's use integration by parts.

$$\begin{aligned}
u &= \sec(x) & v &= \tan(x) \\
du &= \sec(x)\tan(x)dx & dv &= \sec^2(x)dx
\end{aligned}$$

The above integral becomes

$$\begin{aligned}\int \sec(x)\sec^2(x)dx &= \sec(x)\tan(x) - \int \sec(x)\tan^2(x)dx \\ &= \sec(x)\tan(x) - \int \sec(x)[\sec^2(x)-1]dx \\ &= \sec(x)\tan(x) - \int \sec^3(x) + \int \sec(x)dx \\ &= \sec(x)\tan(x) - \int \sec^3(x) + \ln|\sec(x)+\tan(x)|\end{aligned}$$

This is familiar. Solve for $\int \sec^3(x)$. *We get*

$$\int \sec^3(x)dx = \frac{1}{2}\Big[\sec(x)\tan(x) + \ln|\sec(x)+\tan(x)|\Big] + c$$

3.5.1 Some Remarks on Using Complex-Valued Functions

Consider functions of the form

$$f(x) + ig(x), \tag{3.13}$$

where x is a real variable and f, g are real-valued functions. For example,

$$e^{ix} = \cos(x) + i\sin(x).$$

We observed before that

$$\frac{d}{dx}e^{wx} = we^{wx}$$

hence

$$\int e^{wx}dx = \frac{1}{w}e^{wx} + c.$$

For example, writing it e^{ix} as in (3.13), we have

$$\begin{aligned}\int e^{ix}dx &= \int \cos(x)dx + i\int \sin(x)dx \\ &= \sin(x) - i\cos(x) + c \\ &= -i(\cos(x) + i\sin(x)) + c \\ &= \frac{1}{i}e^{ix}.\end{aligned}$$

Example 3.5.7. *Let's compute* $\int \frac{1}{x+i}dx$. *Wouldn't it be nice if we could just write* $\ln(x+i)+c$*? This is useless for us though, since we haven't even* defined $\ln(x+i)$*! However, we can "rationalize the denominator" by writing*

$$\begin{aligned}
\int \frac{1}{x+i} dx &= \int \frac{1}{x+i} \cdot \frac{x-i}{x-i} dx \\
&= \int \frac{x-i}{x^2+1} dx \\
&= \int \frac{x}{x^2+1} dx - i \int \frac{1}{x^2+1} dx \\
&= \frac{1}{2} \ln|x^2+1| - i \tan^{-1}(x) + c.
\end{aligned}$$

This informs how we would define $\ln(z)$ *for* z *complex (which you'll do if you take a course in complex analysis).*
Key trick: *Get the* i *in the numerator.*

The next example illustrates an alternative to the method of Section 3.5.

Example 3.5.8.

$$\begin{aligned}
\int \sin(5x)\cos(3x) dx &= \int \left(\frac{e^{i5x} - e^{-i5x}}{2i}\right) \cdot \left(\frac{e^{i5x} + e^{-i5x}}{2}\right) dx \\
&= \frac{1}{4i} \int \left(e^{i8x} - e^{-i8x} + e^{i2x} - e^{-i2x}\right) dx + c \\
&= \frac{1}{4i} \left(\frac{e^{i8x}}{8i} + \frac{e^{-i8x}}{8i} + \frac{e^{i2x}}{2i} + \frac{e^{-i2x}}{2i}\right) + c \\
&= -\frac{1}{4} \left[\frac{1}{4} \cos(8x) + \cos(2x)\right] + c
\end{aligned}$$

This is *more tedious than the method in 3.5. But it is* completely straightforward. *You don't need any trig formulas or anything else. You just multiply it out, integrate, etc., and remember that* $i^2 = -1$.

Chapter 4

Integration techniques

4.1 Trigonometric Substitutions

The first homework problem is to compute

$$\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}dx. \tag{4.1}$$

Your first idea might be to do some sort of substitution, e.g., $u = x^2 - 1$, but $du = 2xdx$ is nowhere to be seen and this simply doesn't work. Likewise, integration by parts gets us nowhere. However, a technique called "inverse trig substitutions" and a trig identity easily dispenses with the above integral and several similar ones! Here's the crucial table:

Expression	Inverse Substitution	Relevant Trig Identity
$\sqrt{a^2-x^2}$	$x = a\sin(\theta), -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2(\theta) = \cos^2(\theta)$
$\sqrt{a^2+x^2}$	$x = a\tan(\theta), -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2-a^2}$	$x = a\sec(\theta), 0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2(\theta) - 1 = \tan^2(\theta)$

Inverse substitution works as follows. If we write $x = g(t)$, then

$$\int f(x)dx = \int f(g(t))g'(t)dt.$$

This is *not* the same as substitution. You can just apply inverse substitution to any integral directly—usually you get something even worse, but for the integrals in this section using a substitution can vastly improve the situation.

If g is a $1-1$ function, then you can even use inverse substitution for a definite integral. The limits of integration are obtained as follows.

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))g'(t)dt. \tag{4.2}$$

To help you understand this, note that as t varies from $g^{-1}(a)$ to $g^{-1}(b)$, the function $g(t)$ varies from $a = g(g^{-1}(a)$ to $b = g(g^{-1}(b))$, so f is being integrated over exactly the same values. Note also that (4.2) once again illustrates Leibniz's brilliance in designing the notation for calculus.

Let's give it a shot with (4.1). From the table we use the inverse substitution

$$x = \sec(\theta).$$

We get

$$\begin{aligned}
\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}dx &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec(\theta)}\sqrt{\sec^2(\theta)-1}\sec(\theta)\tan(\theta)d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec(\theta)}\tan(\theta)\sec(\theta)\tan(\theta)d\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^(\theta)d\theta \\
&= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 1+\cos(2\theta)d\theta \\
&= \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} \\
&= \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}
\end{aligned}$$

Wow! That was like magic. This is really an amazing technique. Let's use it again to find the area of an ellipse.

Example 4.1.1. *Consider an ellipse with radii a and b, so it goes through $(0, \pm b)$ and $(\pm a, 0)$. An equation for the part of an ellipse in the first quadrant is*

$$y = b\sqrt{1-\frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2-x^2}.$$

Thus the area of the entire ellipse is

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx.$$

The 4 is because the integral computes 1/4th of the area of the whole ellipse. So we need to compute

$$\int_0^a \sqrt{a^2-x^2}\,dx$$

Obvious substitution with $u = a^2 - x^2$...? nope. Integration by parts...? nope.

Let's try inverse substitution. The table above suggests using $x = a\sin(\theta)$, so $dx = a\cos(\theta)d\theta$. We get

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2(\theta)}d\theta &= a^2 \int_0^{\frac{\pi}{2}} \cos^2(\theta)d\theta && (4.3)\\
&= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} 1 + \cos(2\theta)d\theta && (4.4)\\
&= \frac{a^2}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}} && (4.5)\\
&= \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}. && (4.6)
\end{aligned}$$

Thus the area is

$$4\frac{b}{a}\frac{\pi a^2}{4} = \pi ab.$$

Consistency Check: *If the ellipse is a circle, i.e., $a = b = r$, this is πr^2, which is a well-known formula for the area of a circle.*

Remark 4.1.1. *Trigonometric substitution is useful for functions that involve $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a}$, but* not all at once!. *See the above table for how to do each.*

One other important technique is to use completing the square.

Example 4.1.2. *Compute $\int \sqrt{5 + 4x - x^2}\,dx$. We* complete the square*:*

$$5 + 4x - x^2 = 5 - (x-2)^2 + 4 = 9 - (x-2)^2.$$

Thus

$$\int \sqrt{5 + 4x - x^2}\,dx = \int \sqrt{9 - (x-2)^2}\,dx.$$

We do a usual substitution to get rid of the $x - 2$. Let $u = x - 2$, so $du = dx$. Then

$$\int \sqrt{9 - (x-2)^2}\,dx = \int \sqrt{9 - y^2}\,dy.$$

Now we have an integral that we can do; it's almost identical to the previous example, but with $a = 9$ (and this is an indefinite integral). Let $y = 3\sin(\theta)$, so

$dy = 3\cos(\theta)d\theta$. *Then*

$$\begin{aligned}
\int \sqrt{9-(x-2)^2}\,dx &= \int \sqrt{9-y^2}\,dy \\
&= \int \sqrt{3^2 - 3^2\sin^2(\theta)}3\cos(\theta)d\theta \\
&= 9\int \cos^2(\theta)\,d\theta \\
&= \frac{9}{2}\int 1+\cos(2\theta)d\theta \\
&= \frac{9}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C
\end{aligned}$$

Of course, we must transform *back into a function in x, and that's a little tricky. Use that*

$$x-2 = y = 3\sin(\theta),$$

so that

$$\theta = \sin^{-1}\left(\frac{x-2}{3}\right).$$

$$\begin{aligned}
&\int \sqrt{9-(x-2)^2}\,dx = \cdots \\
&\qquad = \frac{9}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C \\
&\qquad = \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \sin(\theta)\cos(\theta)\right] + C \\
&\qquad = \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \left(\frac{x-2}{3}\right)\cdot\left(\frac{\sqrt{9-(x-2)^2}}{3}\right)\right] + C.
\end{aligned}$$

Here we use that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. *Also, to compute* $\cos(\sin^{-1}\left(\frac{x-2}{3}\right))$, *we draw a right triangle with side lengths $x-2$ and $\sqrt{9-(x-2)^2}$, and hypotenuse* 3.

Example 4.1.3. *Compute*

$$\int \frac{1}{\sqrt{t^2-6t+13}}dt$$

To compute this, we complete the square, etc.

$$\int \frac{1}{\sqrt{t^2-6t+13}}dt = \int \frac{1}{\sqrt{(t-3)^2+4}}dt$$

(You may want to visualize a triangle with sides 2 and $t-3$ and hypotenuse $\sqrt{(t-3)^2+4}$.) Then

$$
\begin{aligned}
t-3 &= 2\tan(\theta) \\
\sqrt{(t-3)^2+4} &= 2\sec(\theta) = \frac{2}{\cos(\theta)} \\
dt &= 2\sec^2(\theta)d\theta
\end{aligned}
$$

Back to the integral, we have

$$
\begin{aligned}
\int \frac{1}{\sqrt{(t-3)^2+4}}dt &= \int \frac{2\sec^2(\theta)}{2\sec(\theta)}d\theta \\
&= \int \sec(\theta)d\theta \\
&= \ln|\sec(\theta)+\tan(\theta)| + C \\
&= \ln\left|\sqrt{(t-3)^2+4}2 + \frac{t-3}{2}\right| + C.
\end{aligned}
$$

4.2 Integration by Parts

The product rule is that

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x).$$

Integrating both sides leads to a new fundamental technique for integration:

$$f(x)g(x) = \int f(x)g'(x)dx + \int g(x)f'(x)dx. \tag{4.7}$$

Now rewrite (4.7) as

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx.$$

Shorthand notation:

$$
\begin{aligned}
u &= f(x) & du &= f'(x)dx \\
v &= g(x) & dv &= g'(x)dx
\end{aligned}
$$

Then have

$$\int udv = uv - \int vdu.$$

So what! But what's the big deal? Integration by parts is a fundamental technique of integration. It is also a key step in the proof of many theorems in calculus.

Example 4.2.1. $\int x\cos(x)dx$.

$$u = x \qquad v = \sin(x)$$
$$du = dx \qquad dv = \cos(x)dx$$

We get

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = x\sin(x) + \cos(x) + c.$$

"Did this do anything for us?" Indeed, it did.

Wait a minute—how did we know to pick $u = x$ and $v = \sin(x)$? We could have picked them other way around and still written down true statements. Let's try that:

$$u = \cos(x) \qquad v = \frac{1}{2}x^2$$
$$du = -\sin(x)dx \qquad dv = xdx$$
$$\int x\cos(x)dx = \frac{1}{2}x\cos(x) + \int \frac{1}{2}x^2\sin(x)dx.$$

Did this help!? NO. Integrating $x^2\sin(x)$ is harder than integrating $x\cos(x)$. This formula is completely correct, but is hampered by being useless in this case. So how do *you pick them?*

> *Choose the u so that when you differentiate it you get something* simpler*; when you pick dv, try to choose something whose antiderivative is* simpler.

Sometimes you have to try more than once. But with a good eraser nobody will know that it took you two tries.

Question If integration by parts once is good, then sometimes twice is even better? *Yes, in some examples (see Example 4.2.4). But in the above example, you just undo what you did and basically end up where you started, or you get something even worse.*

Example 4.2.2. *Compute* $\int_0^{\frac{1}{2}} \sin^{-1}(x)dx$. *Two points:*

1. *It's a definite integral.*
2. *There is only one function; would you think to do integration by parts? But it is a product; it just doesn't look like it at first glance.*

Your choice is made for you, since we'd be back where we started *if we put $dv = \sin^{-1}(x)dx$.*

$$u = \sin^{-1}(x) \qquad v = x$$
$$du = \frac{1}{\sqrt{1-x^2}} \qquad dv = dx$$

We get
$$\int_0^{\frac{1}{2}} \sin^{-1}(x)dx = \left[x\sin^{-1}(x)\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}dx.$$
Now we use substitution with $w = 1-x^2$, $dw = -2xdx$, hence $xdx = -\frac{1}{2}dw$.

$$\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int w^{-\frac{1}{2}}dw = -w^{\frac{1}{2}} + c = -\sqrt{1-x^2} + c.$$

Hence

$$\int_0^{\frac{1}{2}} \sin^{-1}(x)dx = \left[x\sin^{-1}(x)\right]_0^{\frac{1}{2}} + \left[\sqrt{1-x^2}\right]_0^{\frac{1}{2}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

But shouldn't we change the limits because we did a substitution? *(No, since we computed the indefinite integral and put it back; this time we did the other option.)*

Is there another way to do this? *I don't know. But for any integral, there might be several different techniques. If you can think of any other way to guess an antiderivative, do it; you can always differentiate as a check.*

Note: Integration by parts is tailored toward doing indefinite integrals.

Example 4.2.3. *This example illustrates how to use integration by parts twice. We compute*
$$\int x^2 e^{-2x}dx$$

$$u = x^2 \qquad v = -\frac{1}{2}e^{-2x}$$
$$du = 2xdx \qquad dv = e^{-2x}dx$$

We have
$$\int x^2e^{-2x}dx = -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}dx.$$
Did this help? It helped, but it did not *finish the integral off. However, we can deal with the remaining integral, again using integration by parts. If you do it twice, you* what to keep going in the same direction. *Do not switch your choice, or you'll undo what you just did.*

$$u = x \qquad v = -\frac{1}{2}e^{-2x}$$
$$du = dx \qquad dv = e^{-2x}dx$$

$$\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c.$$

Now putting this above, we have

$$\int x^2e^{-2x}dx = -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c = -\frac{1}{4}e^{-2x}(2x^2+2x+1) + c.$$

Do you think you might have to do integration by parts three times? What if it were $\int x^3 e^{-2x} dx$? Grrr – you'd have to do it three times.

Example 4.2.4. Compute $\displaystyle\int e^x \cos(x) dx$. *Which should be u and which should be v? Taking the derivatives of each type of function does not change the type. As a practical matter, it doesn't matter. Which would you* prefer *to find the antiderivative of? (Both choices work, as long as you keep going in the same direction when you do the second step.)*

$$u = \cos(x) \qquad v = e^x$$
$$du = -\sin(x) dx \qquad dv = e^x dx$$

We get

$$\int e^x \cos(x) dx = e^x \cos(x) + \int e^x \sin(x) dx.$$

We have to do it again. This time we choose (going in the same direction*):*

$$u = \sin(x) \qquad v = e^x$$
$$du = \cos(x) dx \qquad dv = e^x dx$$

We get

$$\int e^x \cos(x) dx = e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x) dx.$$

Did we get anywhere? Yes! No! First impression: all this work, and we're back where we started from! Yuck. Clearly we don't want to integrate by parts yet again. **BUT.** *Notice the* minus *sign in front of $\int e^x \cos(x) dx$; You can add the integral to both sides and get*

$$2 \int e^x \cos(dx) = e^x \cos(x) + e^x \sin(x) + c.$$

Hence

$$\int e^x \cos(dx) = \frac{1}{2} e^x (\cos(x) + \sin(x)) + c.$$

4.2.1 More General Uses of Integration By Parts

The Integration By Parts Formula is also used to derive many of the entries in the Table of Integrals. For some integrands such as $x^n \ln(x)$, the result is simply a function, an antiderivative of the integrand. For some integrands such as $\sin(x)^n$, the result is a reduction formula, a formula which still contains an integral, but the new integrand is the sine function raised to a smaller power, $\sin(x)^{n-2}$. By repeatedly applying the reduction formula, we can evaluate the integral of sine raised to any positive integer power.

Practice 4.2.1. *Let $n \neq -1$ be an integer. Evaluate $\int x^n \ln(x)\, dx$ using $u = \ln(x)$ and $dv = x^n dx$.*

Practice 4.2.2. *Let $n \neq -1$ be an integer. Apply integration by parts to $\int x^n e^x\, dx$ using $u = x^n$ and $dv = e^x dx$.*

4.3 Factoring Polynomials

How do you compute something like

$$\int \frac{x^2+2}{(x-1)(x+2)(x+3)} dx?$$

So far you have no method for doing this. The trick (which is called partial fraction decomposition), is to write

$$\int \frac{x^2+2}{x^3+4x^2+x-6} dx = \int \frac{1}{4(x-1)} - \frac{2}{x+2} + \frac{11}{4(x+3)} dx \tag{4.8}$$

The integral on the right is then easy to do (the answer involves ln's).

But *how on earth* do you right the rational function on the left hand side as a sum of the nice terms of the right hand side? Doing this is called "partial fraction decomposition", and it is a fundamental idea in mathematics. It relies on our ability to factor polynomials and solve linear equations. As a first hint, notice that

$$x^3 + 4x^2 + x - 6 = (x-1) \cdot (x+2) \cdot (x+3),$$

so the denominators in the decomposition correspond to the factors of the denominator.

Before describing the secret behind (4.8), we'll discuss some background about how polynomials and rational functions work.

Theorem 4.3.1 (Fundamental Theorem of Algebra)**.** *If $f(x) = a_n x^n + \cdots a_1 x + a_0$ is a polynomial, then there are complex numbers $c, \alpha_1, \ldots \alpha_n$ such that*

$$f(x) = c(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n).$$

Example 4.3.1. *For example,*

$$3x^2 + 2x - 1 = 3 \cdot \left(x - \frac{1}{3}\right) \cdot (x+1).$$

And

$$(x^2+1) = (x+i)^2 \cdot (x-i)^2.$$

If $f(x)$ is a polynomial, the roots α of f correspond to the factors of f. Thus if

$$f(x) = c(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n),$$

then $f(\alpha_i) = 0$ for each i (and nowhere else).

Definition 4.3.1 (Multiplicity of Zero)**.** *The multiplicity of a zero α of $f(x)$ is the number of times that $(x-\alpha)$ appears as a factor of f.*

For example, if $f(x) = 7(x-2)^{99} \cdot (x+17)^5 \cdot (x-\pi)^2$, then 2 is a zero with multiplicity 99, π is a zero with multiplicity 2, and -1 is a "zero multiplicity 0".

Definition 4.3.2 (Rational Function)**.** *A rational function is a quotient*

$$f(x) = \frac{g(x)}{h(x)},$$

where $g(x)$ and $h(x)$ are polynomials.

For example,

$$f(x) = \frac{x^{10}}{(x-i)^2(x+\pi)(x-3)^3} \tag{4.9}$$

is a rational function.

Definition 4.3.3 (Pole)**.** *A pole of a rational function $f(x)$ is a complex number α such that $|f(x)|$ is unbounded as $x \to \alpha$.*

For example, for (4.9) the poles are at i, π, and 3. They have multiplicity 2, 1, and 3, respectively.

4.4 Partial Fractions

Rational functions (polynomials divided by polynomials) and their integrals are important in mathematics and applications, but if you look through a table of integral formulas, you will find very few formulas for their integrals. Partly that is because the general formulas are rather complicated and have many special cases, and partly it is because they can all be reduced to just a few cases using the algebraic technique discussed in this section, Partial Fraction Decomposition. In algebra you learned to add rational functions to get a single rational function. Partial Fraction Decomposition is a technique for reversing that procedure to "decompose" a single rational function into a sum of simpler rational functions. Then the integral of the single rational function can be evaluated as the sum of the integrals of the simpler functions.

Example 4.4.1. *Use the algebraic decomposition $\frac{17x35}{2x^2-5x} = \frac{7}{x} + \frac{3}{2x-5}$ to evaluate $\int \frac{17x35}{2x^2-5x}\,dx$.*

Solution: The decomposition allows us to exchange the original integral for two much easier ones:

$$\begin{aligned}\int \frac{17x35}{2x^2-5x}\,dx &= \int \frac{7}{x}\,dx + \int \frac{3}{2x-5}\,dx \\ &= 7\ln|x| + \frac{3}{2}\ln|2x-5| + C.\end{aligned}$$

When Sage *computes this integral, it implicitly assumes that $x > 0$:*

```
sage: integral((17*x-35)/(2*x^2-5*x),x)
3*log(2*x - 5)/2 + 7*log(x)
```

Note that Sage *also leaves off the constant of integration, but this is more of an abbreviation than a matter of precision.*

Practice 4.4.1. *Use the algebraic decomposition* $\frac{7x-11}{3x^2-8x-3} = \frac{4}{3x+1} + \frac{1}{x-3}$ *to evaluate* $\int \frac{7x-11}{3x^2-8x-3}\,dx$.

The Example illustrates how to use a "decomposed" fraction with integrals, but it does not show how to achieve the decomposition. The algebraic basis for the Partial Fraction Decomposition technique is that every polynomial can be factored into a product of linear factors $ax + b$ and irreducible quadratic factors $ax^2 + bx + c$ (with $b^2 - 4ac < 0$). These factors may not be easy to find, and they will typically be more complicated than the examples in this section, but every polynomial has such factors. Before we apply the Partial Fraction Decomposition technique, the fraction must have the following form.

(a) (the degree of the numerator) < (degree of the denominator);

(b) the denominator has been factored into a product of linear factors and irreducible quadratic factors.

If assumption (a) is not true, we can use polynomial division until we get a remainder which has a smaller degree than the denominator. If assumption (b) is not true, we simply cannot use the Partial Fraction Decomposition technique.

Distinct Linear Factors

If the denominator can be factored into a product of distinct linear factors, then the original fraction can be written as the sum of fractions of the form $\frac{\text{number}}{\text{linear factor}}$. Our job is to find the values of the numbers in the numerators, and that typically requires solving a system of equations.

Example 4.4.2. *Find constants A and B such that* $\frac{17x35}{(2x-5)x} = \frac{A}{x} + \frac{B}{2x-5}$.

Solution: First, note the roots of the denominator are $\{0, 5/2\}$. *Cross multiply:*

$$17x35 = A(2x - 5) + Bx.$$

Now "kill B and solve for A using the first root $x = 0$*:*

$$-35 = 17 \cdot 035 = A(2 \cdot 0 - 5) + B \cdot 0 = -5A.$$

This gives $A = 7$. *Next, "kill A and solve for B using the second root* $x = 5/2$*:*

$$17 \cdot 5/235 = A(2 \cdot 5/2 - 5) + B5/2 = 5B/2.$$

This gives $B = 3$, *so* $\frac{17x35}{2x^2-5x} = \frac{7}{x} + \frac{3}{2x-5}$.

In Sage *this is very easy:*

```
sage: f = (17*x-35)/(2*x^2-5*x)
sage: f.partial_fraction()
3/(2*x - 5) + 7/x
```

Practice 4.4.2. *Find values of A and B so* $\frac{6x-7}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$.

In general, there is one unknown coefficient for each distinct linear factor of the denominator. However, if the number of distinct linear factors is large, we would need to solve a large system of equations for the unknowns.

Practice 4.4.3. *Using partial fractions, solve* $\int \frac{2x^2+7x+9}{x(x+1)(x+3)}$.

Other possible cases are listed as follows. (Of course, a rational function can involve more than one case as well.)

- **Distinct Irreducible Quadratic Factors**

  If the factored denominator includes a distinct irreducible quadratic factor, then the Partial Fraction Decomposition sum contains a fraction of the form of a linear polynomial with unknown coefficients divided by the irreducible quadratic factor:

  $$\frac{ax+b}{cx^2+dx+e}.$$

- **Repeated Factors**

  If the factored denominator contains a linear factor raised to a power (greater than one), then we need to start the decomposition with several terms. There should be one term with one unknown coefficient for each power of the linear factor. For example,

  $$\frac{ax+b}{(cx+d)^s}.$$

Here is the general procedure:
Partial fraction decomposition of $N(x)/D(x)$: Let $N(x)$ be a polynomial of lower degree than another polynomial $D(x)$.
1. Factor $D(x)$ into irreducible factors having real coefficients. Now $D(x)$ is a product of distinct terms of the form $(ax+b)^r$ or $(ax^2+bx+c)^s$, for some integers $r > 0$, $s > 0$. For each term $(ax+b)^r$ the partial fraction decomposition of $N(x)/D(x)$ contains a sum of terms of the form

$$\frac{A_1}{(ax+b)} + \cdots + \frac{A_r}{(ax+b)^r}$$

for some constants A_i, and for each term $(ax^2+bx+c)^r$ the PFD of $N(x)/D(x)$ contains a sum of terms of the form

$$\frac{B_1x+C_1}{(ax^2+bx+c)} + \cdots + \frac{B_sx+C_s}{(ax^2+bx+c)^s}.$$

for some constants B_i, C_j. $\frac{N(x)}{D(x)}$ is the sum of all these simpler rational functions.

2. Now you have an expression for $\frac{N(x)}{D(x)}$ which is a sum of simpler rational functions. The next step is to solve for these constants A's, B's, C's occurring in the numerators. Cross multiply both sides by $D(x)$ and expand out the resulting polynomial identity for $N(x)$ in terms of the A's, B's, C's. Equating coefficients of powers of x on both sides gives rise to a linear system of equations for the A's, B's, C's which you can solve.

Practice 4.4.4. *Logistic Growth: The growth rate of many different populations depends not only on the number of individuals (leading to exponential growth) but also on a "carrying capacity" of the environment. If x is the population at time t and the growth rate of x is proportional to the product of the population and the carrying capacity M minus the population, then the growth rate is described by the differential equation*

$$\frac{dx}{dt} = kx(M - x),$$

where k and M are constants for a given species in a given environment.

Let $k = 1$ and $M = 100$, and assume the initial population is $x(0) = 5$.

(a) Solve the differential equation $\frac{dx}{dt} = kx(M - x)$, for x.

(b) Graph the population $x(t)$ for $0 \leq t \leq 20$.

(c) When will the population be 20? 50? 90? 100?

(d) What is the population after a "long" time? (Find the limit, as t becomes arbitrarily large, of x.)

(e) Explain the shape of the graph in (a) in terms of a population of bacteria.

(f) When is the growth rate largest? (Maximize dx/dt.)

(g) What is the population when the growth rate is largest?

Practice 4.4.5. *Chemical Reaction: In some chemical reactions, a new material X is formed from materials A and B, and the rate at which X forms is proportional to the product of the amount of A and the amount of B remaining in the solution. Let x represent the amount of material X present at time t, and assume that the reaction begins with a grams of A, b grams of B, and no material X ($x(0) = 0$). Then the rate of formation of material X can be described by the differential equation*

$$\frac{dx}{dt} = k(a - x)(b - x).$$

Solve the differential equation for x if $k = 1$ and the reaction begins with (i) 7 grams of A and 5 grams of B, and (ii) 6 grams of A and 6 grams of B.

4.5 Integration of Rational Functions Using Partial Fractions

Our goal today is to compute integrals of the form

$$\int \frac{P(x)}{Q(x)} dx$$

by decomposing $f = \frac{P(x)}{Q(x)}$. This is called partial fraction expansion.

Theorem 4.5.1 (Fundamental Theorem of Algebra over the Real Numbers)**.** *A real polynomial of degree $n \geq 1$ can be factored as a constant times a product of linear factors $x - a$ and irreducible quadratic factors $x^2 + bx + c$.*

Note that $x^2 + bx + c = (x - \alpha)(x - \bar{\alpha})$, where $\alpha = z + iw$, $\bar{\alpha} = z - iw$ are complex conjugates.

Types of rational functions $f(x) = \frac{P(x)}{Q(x)}$. To do a partial fraction expansion, first make sure $\deg(P(x)) < \deg(Q(x))$ using long division. Then there are four possible situation, each of increasing generality (and difficulty):

1. $Q(x)$ is a product of distinct linear factors;
2. $Q(x)$ is a product of linear factors, some of which are repeated;
3. $Q(x)$ is a product of distinct irreducible quadratic factors, along with linear factors some of which may be repeated; and,
4. $Q(x)$ is has repeated irreducible quadratic factors, along with possibly some linear factors which may be repeated.

The general partial fraction expansion theorem is beyond the scope of this course. However, you might find the following special case and its proof interesting.

Theorem 4.5.2. *Suppose p, q_1 and q_2 are polynomials that are relatively prime (have no factor in common). Then there exists polynomials α_1 and α_2 such that*

$$\frac{p}{q_1 q_2} = \frac{\alpha_1}{q_1} + \frac{\alpha_2}{q_2}.$$

Proof. Since q_1 and q_2 are relatively prime, using the Euclidean algorithm (long division), we can find polynomials s_1 and s_2 such that

$$1 = s_1 q_1 + s_2 q_2.$$

Dividing both sides by $q_1 q_2$ and multiplying by p yields

$$\frac{p}{q_1 q_2} = \frac{\alpha_1}{q_1} + \frac{\alpha_2}{q_2},$$

which completes the proof. □

Example 4.5.1. *Compute*

$$\int \frac{x^3 - 4x - 10}{x^2 - x - 6} dx.$$

First do long division. Get quotient of $x + 1$ *and remainder of* $3x - 4$. *This means that*

$$\frac{x^3 - 4x - 10}{x^2 - x - 6} = x + 1 + \frac{3x - 4}{x^2 - x - 6}.$$

Since we have distinct linear factors, we know that we can write

$$f(x) = \frac{3x - 4}{x^2 - x - 6} = \frac{A}{x - 3} + \frac{B}{x + 2},$$

for real numbers A, B. *A clever way to find* A, B *is to substitute appropriate values in, as follows. We have*

$$f(x)(x - 3) = \frac{3x - 4}{x + 2} = A + B \cdot \frac{x - 3}{x + 2}.$$

Setting $x = 3$ *on both sides we have (taking a limit):*

$$A = f(3) = \frac{3 \cdot 3 - 4}{3 + 2} = \frac{5}{5} = 1.$$

Likewise, we have

$$B = f(-2) = \frac{3 \cdot (-2) - 4}{-2 - 3} = 2.$$

Thus

$$\begin{aligned} \int \frac{x^3 - 4x - 10}{x^2 - x - 6} dx &= \int x + 1 + \frac{1}{x - 3} + \frac{2}{x + 2} \\ &= \frac{x^2 + 2x}{2} + 2 \log|x + 2| + \log|x - 3| + C. \end{aligned}$$

Example 4.5.2. *Compute the partial fraction expansion of* $\frac{x^2}{(x-3)(x+2)^2}$. *By the partial fraction theorem, there are constants* A, B, C *such that*

$$\frac{x^2}{(x - 3)(x + 2)^2} = \frac{A}{x - 3} + \frac{B}{x + 2} + \frac{C}{(x + 2)^2}.$$

Note that there's no possible way this could work without the $(x+2)^2$ *term, since otherwise the common denominator would be* $(x - 3)(x + 2)$. *We have*

$$\begin{aligned} A &= [f(x)(x - 3)]_{x=3} = \frac{x^2}{(x + 2)^2}|_{x=3} = \frac{9}{25}, \\ C &= \left[f(x)(x + 2)^2\right]_{x=-2} = -\frac{4}{5}. \end{aligned}$$

This method will not get us B! For example,

$$f(x)(x+2) = \frac{x^2}{(x-3)(x+2)} = A \cdot \frac{x+2}{x-3} + B + \frac{C}{x+2}.$$

While true this is useless.

Instead, we use that we know A and C, and evaluate at another value of x, say 0.

$$f(0) = 0 = \frac{\frac{9}{25}}{-3} + \frac{B}{2} + \frac{-\frac{4}{5}}{(2)^2},$$

so $B = \frac{16}{25}$. *Thus finally,*

$$\begin{aligned} \int \frac{x^2}{(x-3)(x+2)^2} &= \int \frac{\frac{9}{25}}{x-3} + \frac{\frac{16}{25}}{x+2} + \frac{-\frac{4}{5}}{(x+2)^2}. \\ &= \frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{\frac{4}{5}}{x+2} + \textit{constant}. \end{aligned}$$

Example 4.5.3. *Let's compute* $\int \frac{1}{x^3+1}dx$. *Notice that* $x+1$ *is a factor, since* -1 *is a root. We have*

$$x^3 + 1 = (x+1)\left(x^2 - x + 1\right).$$

There exist constants A, B, C *such that*

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.$$

Then

$$A = f(x)(x+1)|_{x=-1} = \frac{1}{3}.$$

You could find B, C *by factoring the quadratic over the complex numbers and getting complex number answers. Instead, we evaluate x at a couple of values. For example, at* $x = 0$ *we get*

$$f(0) = 1 = \frac{1}{3} + \frac{C}{1},$$

so $C = \frac{2}{3}$. *Next, use* $x = 1$ *to get B.*

$$\begin{aligned} f(1) = \frac{1}{1^3+1} &= \frac{\frac{1}{3}}{(1)+1} + \frac{B(1)+\frac{2}{3}}{(1)^2-(1)+1} \\ \frac{1}{2} &= \frac{1}{6} + B + \frac{2}{3}, \end{aligned}$$

so

$$B = \frac{3}{6} - \frac{1}{6} - \frac{4}{6} = -\frac{1}{3}.$$

Finally,

$$\begin{aligned}\int \frac{1}{x^3+1}dx &= \int \frac{\frac{1}{3}}{x+1} - \frac{\frac{1}{3}x}{x^2-x-1} + \frac{\frac{2}{3}}{x^2-x-1}dx \\ &= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{x-2}{x^2-x+1}dx\end{aligned}$$

It remains to compute

$$\int \frac{x-2}{x^2-x+1}dx.$$

First, complete the square to get

$$x^2-x+1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4}.$$

Let $u = (x-\frac{1}{2})$, *so* $du = dx$ *and* $x = u + \frac{1}{2}$. *Then*

$$\begin{aligned}\int \frac{u-\frac{3}{2}}{u^2+\frac{3}{4}}du &= \int \frac{udu}{u^2+\frac{3}{4}} - \frac{3}{2}\int \frac{1}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}du \\ &= \frac{1}{2}\ln\left|u^2+\frac{3}{4}\right| - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + c \\ &= \frac{1}{2}\ln\left|x^2-x+1\right| - \sqrt{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c\end{aligned}$$

Finally, we put it all together and get

$$\begin{aligned}\int \frac{1}{x^3+1}dx &= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{x-2}{x^2-x+1}dx \\ &= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left|x^2-x+1\right| + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + c\end{aligned}$$

Problem: Compute $\int \cos^2(x)e^{-3x}dx$ using complex exponentials. The answer is

$$-\frac{1}{6}e^{-3x} + \frac{1}{13}e^{-3x}\sin(2x) - \frac{3}{26}e^{-3x}\cos(2x) + c.$$

Here's how to get it.

$$\begin{aligned}\int \cos^2(x)e^{-3x}dx &= \int \frac{e^{2ix}+2+e^{-2ix}}{4}e^{-3x}dx \\ &= \frac{1}{4}\left[\frac{e^{(2i-3)x}}{2i-3} - \frac{2}{3}e^{-3x} + \frac{e^{(-2i-3)x}}{-2i-3}\right] + c \\ &= -\frac{1}{6}e^{-3x} + \frac{e^{-3x}}{4}\left[\frac{e^{2ix}}{2i-3} - \frac{e^{-2ix}}{2i+3}\right] + c\end{aligned}$$

Simplify the inside part requires some imagination:

$$\begin{aligned}\frac{e^{2ix}}{2i-3} - \frac{e^{-2ix}}{2i+3} &= \frac{1}{13}(-2ie^{2ix} - 3e^{2ix} + 2ie^{-2ix} - 3e^{-2ix}) \\ &= \frac{1}{13}\left(4\sin(2x) - 6\cos(2x)\right)\end{aligned}$$

4.6 Improper Integrals

Example 4.6.1. *Make sense of* $\int_0^\infty e^{-x}dx$. *The integrals*

$$\int_0^t e^{-x}dx$$

make sense for each real number t. *So consider*

$$\lim_{t\to\infty}\int_0^t e^{-x}dx = \lim_{t\to\infty}[-e^{-x}]_0^t = 1.$$

Geometrically the area under the whole curve is the limit of the areas for finite values of t.

Example 4.6.2. *Consider* $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$.

Problem: The denominator of the integrand tends to 0 *as* x *approaches the upper endpoint. Define*

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{1-x^2}}dx &= \lim_{t\to 1^-}\int_0^t \frac{1}{\sqrt{1-x^2}}dx \\ &= \lim_{t\to 1^-}\left(\sin^{-1}(t) - \sin^{-1}(0)\right) = \sin^{-1}(1) = \frac{\pi}{2}\end{aligned}$$

Here $t \to 1^-$ *means the limit as* t *tends to* 1 from the left.

Example 4.6.3. *There can be multiple points at which the integral is improper. For example, consider*

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx.$$

A crucial point is that we take the limit for the left and right endpoints independently. We use the point 0 *(for convenience only!) to break the integral in* *half.*

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx \\
&= \lim_{s\to-\infty} \int_{s}^{0} \frac{1}{1+x^2} dx + \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2} dx \\
&= \lim_{s\to-\infty} (\tan^{-1}(0) - \tan^{-1}(s)) + \lim_{t\to\infty} (\tan^{-1}(t) - \tan^{-1}(0)) \\
&= \lim_{s\to-\infty} (-\tan^{-1}(s)) + \lim_{t\to\infty} (\tan^{-1}(t)) \\
&= -\frac{-\pi}{2} + \frac{\pi}{2} = \pi.
\end{aligned}$$

Example 4.6.4. *Consider $\int_{-\infty}^{\infty} x dx$. Notice that*

$$\int_{-\infty}^{\infty} x dx = \lim_{s\to-\infty} \int_{s}^{0} x dx + \lim_{t\to\infty} \int_{0}^{t} x dx.$$

This diverges since each factor diverges independently. But notice that

$$\lim_{t\to\infty} \int_{-t}^{t} x dx = 0.$$

This is not *what $\int_{-\infty}^{\infty} x dx$ means (in this course – in a later course it could be interpreted this way)! This illustrates the importance of treating each bad point separately (since Example 4.6.3) doesn't.*

Example 4.6.5. *Consider $\int_{-1}^{1} \frac{1}{\sqrt[3]{x}} dx$. We have*

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{\sqrt[3]{x}} dx &= \lim_{s\to0^-} \int_{-1}^{s} x^{-\frac{1}{3}} dx \ + \ \lim_{t\to0^+} \int_{t}^{1} x^{-\frac{1}{3}} dx \\
&= \lim_{s\to0^-} \left(\frac{3}{2} s^{\frac{2}{3}} - \frac{3}{2}\right) + \lim_{t\to0^+} \left(\frac{3}{2} - \frac{3}{2} t^{\frac{2}{3}}\right) = 0.
\end{aligned}$$

This illustrates how to be careful and break the function up into two pieces when there is a discontinuity.

Example 4.6.6. *Compute $\int_{-1}^{3} \frac{1}{x-2} dx$. A few weeks ago you might have done this:*

$$\int_{-1}^{3} \frac{1}{x-2} dx = [\ln|x-2|]_{-1}^{3} = \ln(3) - \ln(1) \qquad \textit{(totally wrong!)}$$

This is not valid because the function we are integrating has a pole at $x = 2$. The integral is improper, and is only defined if both the following limits exists:

$$\lim_{t\to2^-} \int_{-1}^{t} \frac{1}{x-2} dx \qquad \textit{and} \qquad \lim_{t\to2^+} \int_{t}^{3} \frac{1}{x-2} dx.$$

However, the limits diverge, e.g.,

$$\lim_{t\to 2^+}\int_t^3 \frac{1}{x-2}dx = \lim_{t\to 2^+}(\ln|1| - \ln|t-2|) = -\lim_{t\to 2^+}\ln|t-2| = -\infty.$$

Thus $\int_{-1}^{3}\frac{1}{x-2}dx$ *is divergent.*

4.6.1 Convergence, Divergence, and Comparison

In this section we discuss using comparison to determine if an improper integrals converges or diverges. Recall that if f and g are continuous functions on an interval $[a,b]$ and $g(x) \le f(x)$, then

$$\int_a^b g(x)dx \le \int_a^b f(x)dx.$$

This observation can be *incredibly useful* in determining whether or not an improper integral converges.

Not only does this technique help in determining whether integrals converge, but it also gives you some information about their values, which is often much easier to obtain than computing the exact integral.

Theorem 4.6.1 (Comparison Theorem (special case))**.** *Let f and g be continuous functions with $0 \le g(x) \le f(x)$ for $x \ge a$.*

1. *If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.*
2. *If $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)dx$ diverges.*

Proof. Since $g(x) \ge 0$ for all x, the function

$$G(t) = \int_a^t g(x)dx$$

is a non-decreasing function. If $\int_a^\infty f(x)dx$ converges to some value B, then for any $t \ge a$ we have

$$G(t) = \int_a^t g(x)dx \le \int_a^t f(x)dx \le B.$$

Thus in this case $G(t)$ is a non-decreasing function bounded above, hence the limit $\lim_{t\to\infty} G(t)$ exists. This proves the first statement.

Likewise, the function

$$F(t) = \int_a^t f(x)dx$$

is also a non-decreasing function. If $\int_a^\infty g(x)dx$ diverges then the function $G(t)$ defined above is still non-decreasing and $\lim_{t\to\infty} G(t)$ does not exist, so $G(t)$ is not bounded. Since $g(x) \le f(x)$ we have $G(t) \le F(t)$ for all $\ge a$, hence $F(t)$ is also unbounded, which proves the second statement. □

The theorem is very intuitive if you think about areas under a graph. "If the bigger integral converges then so does the smaller one, and if the smaller one diverges so does the bigger ones."

Example 4.6.7. *Does $\int_0^\infty \frac{\cos^2(x)}{1+x^2}dx$ converge? Answer: YES.*
Since $0 \leq \cos^2(x) \leq 1$, we really do have

$$0 \leq \frac{\cos^2(x)}{1+x^2} \leq \frac{1}{1+x^2}.$$

Thus

$$\int_0^\infty \frac{1}{1+x^2}dx = \lim_{t\to\infty} \tan^{-1}(t) = \frac{\pi}{2},$$

so $\int_0^\infty \frac{\cos^2(x)}{1+x^2}dx$ converges.

But why did we use $\frac{1}{1+x^2}$? It's a guess that turned out to work. You could have used something else, e.g., $\frac{c}{x^2}$ for some constant c. This is an illustration of how in mathematics sometimes you have to use your imagination or guess and see what happens. Don't get anxious—instead, relax, take a deep breath and explore.

For example, alternatively we could have done the following:

$$\int_1^\infty \frac{\cos^2(x)}{1+x^2}dx \leq \int_1^\infty \frac{1}{x^2}dx = 1,$$

and this works just as well, since $\int_0^1 \frac{\cos^2(x)}{1+x^2}dx$ converges (as $\frac{\cos^2(x)}{1+x^2}$ is continuous).

Example 4.6.8. *Consider $\int_0^\infty \frac{1}{x+e^{-2x}}dx$. Does it converge or diverge? For large values of x, the term e^{-2x} very quickly goes to 0, so we expect this to diverge, since $\int_1^\infty \frac{1}{x}dx$ diverges. For $x \geq 0$, we have $e^{-2x} \leq 1$, so for all x we have*

$$\frac{1}{x+e^{-2x}} \geq \frac{1}{x+1} \qquad \textit{(verify by cross multiplying).}$$

But

$$\int_1^\infty \frac{1}{x+1}dx = \lim_{t\to\infty} [\ln(x+1)]_1^t = \infty$$

Thus $\int_0^\infty \frac{1}{x+e^{-2x}}dx$ must also diverge.

Note that there is a natural analogue of Theorem 4.6.1 for integrals of functions that "blow up" at a point, but we will not state it formally.

Example 4.6.9. *Consider*

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}dx = \lim_{t\to 0^+} \int_t^1 \frac{e^{-x}}{\sqrt{x}}dx.$$

We have

$$\frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}.$$

(Coming up with this comparison might take some work, imagination, and trial and error.) We have

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}dx \leq \int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+} 2\sqrt{1} - 2\sqrt{t} = 2.$$

thus $\int_0^1 \frac{e^{-x}}{\sqrt{x}}dx$ *converges, even though we haven't figured out its value. We just know that it is* ≤ 2*. (In fact, it is* $1.493648265\ldots$*.)*

What if we found a function that is bigger than $\frac{e^{-x}}{\sqrt{x}}$ *and its integral diverges?? So what! This does nothing for you. Bzzzt. Try again.*

Example 4.6.10. *Consider the integral*

$$\int_0^1 \frac{e^{-x}}{x}dx.$$

This is an improper integral since $f(x) = \frac{e^{-x}}{x}$ *has a pole at* $x = 0$*. Does it converge? NO.*
On the interval $[0, 1]$ *we have* $e^{-x} \geq e^{-1}$*. Thus*

$$\begin{aligned}\lim_{t\to 0^+}\int_t^1 \frac{e^{-x}}{x}dx &\geq \lim_{t\to 0^+}\int_t^1 \frac{e^{-1}}{x}dx \\ &= e^{-1}\cdot \lim_{t\to 0^+}\int_t^1 \frac{1}{x}dx \\ &= e^{-1}\cdot \lim_{t\to 0^+} \ln(1) - \ln(t) = +\infty\end{aligned}$$

Thus $\int_0^1 \frac{e^{-x}}{x}dx$ *diverges.*

Chapter 5

Sequences and Series

Our main goal in this chapter is to gain a working knowledge of power series and Taylor series of function with just enough discussion of the details of convergence to get by.

5.1 Sequences

What is

$$\lim_{n\to\infty} \frac{1}{2^n}?$$

You may have encountered sequences long ago in earlier courses and they seemed very difficult. You know much more mathematics now, so they will probably seem easier. On the other hand, we're going to go very quickly.

A sequence is an ordered list of numbers. These numbers may be real, complex, etc., etc., but in this book we will focus entirely on sequences of real numbers. For example,

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \ldots, \frac{1}{2^n}, \ldots$$

Since the sequence is ordered, we can view it as a function with domain the natural numbers $= 1, 2, 3, \ldots$.

Definition 5.1.1 (Sequence)**.** *A sequence $\{a_n\}$ is a function $a : \mathbb{N} \to \mathbb{R}$ that takes a natural number n to $a_n = a(n)$. The number a_n is the nth term.*

For example,

$$a(n) = a_n = \frac{1}{2^n},$$

which we write as $\{\frac{1}{2^n}\}$. Here's another example:

$$(b_n)_{n=1}^{\infty} = \left(\frac{n}{n+1}\right)_{n=1}^{\infty} = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$$

Example 5.1.1. *The Fibonacci sequence* $(F_n)_{n=1}^{\infty}$ *is defined recursively as follows:*

$$F_1 = 1,\ F_2 = 1,\ F_n = F_{n-2} + F_{n-1} \quad \text{for } n \geq 3.$$

Let's return to the sequence $\left(\frac{1}{2^n}\right)_{n=1}^{\infty}$. We write $\lim_{n\to\infty} \frac{1}{2^n} = 0$, since the terms get arbitrarily small.

Definition 5.1.2 (Limit of sequence)**.** *If* $(a_n)_{n=1}^{\infty}$ *is a sequence then that sequence converges to* L*, written* $\lim_{n\to\infty} a_n = L$*, if* a_n *gets arbitrarily close to* L *as* n *get sufficiently large.* Secret rigorous definition: *For every* $\varepsilon > 0$ *there exists* B *such that for* $n \geq B$ *we have* $|a_n - L| < \varepsilon$.

This is exactly like what we did in the previous course when we considered limits of functions. If $f(x)$ is a function, the meaning of $\lim_{x\to\infty} f(x) = L$ is essentially the same. In fact, we have the following fact.

Proposition 5.1.1. *If* f *is a function with* $\lim_{x\to\infty} f(x) = L$ *and* $(a_n)_{n=1}^{\infty}$ *is the sequence given by* $a_n = f(n)$*, then* $\lim_{n\to\infty} a_n = L$.

As a corollary, note that this implies that all the facts about limits that you know from functions also apply to sequences!

Example 5.1.2.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{x\to\infty} \frac{x}{x+1} = 1$$

Example 5.1.3. *The converse of Proposition 5.1.1 is false* in general*, i.e., knowing the limit of the sequence converges doesn't imply that the limit of the function converges. We have* $\lim_{n\to\infty} \cos(2\pi n) = 1$*, but* $\lim_{x\to\infty} \cos(2\pi x)$ *diverges. The converse is OK if the limit involving the function converges.*

Example 5.1.4. *Compute* $\displaystyle\lim_{n\to\infty} \frac{n^3 + n + 5}{17n^3 - 2006n + 15}$. Answer: $\frac{1}{17}$.

5.2 Series

What is

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots?$$

What is

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots?$$

What is

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots?$$

Consider the following sequence of partial sums:

$$a_N = \sum_{n=1}^{N} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^N}.$$

Can we compute

$$\sum_{n=1}^{\infty} \frac{1}{2^n}?$$

These partial sums look as follows:

$$a_1 = \frac{1}{2}, \qquad a_2 = \frac{3}{4}, \qquad a_{10} = \frac{1023}{1024}, \qquad a_{20} = \frac{1048575}{1048576}$$

It looks very likely that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, if it makes any sense. But does it?

In a moment we will *define*

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{2^n} = \lim_{N\to\infty} a_N.$$

A little later we will show that $a_N = \frac{2^N-1}{2^N}$, hence indeed $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.

Definition 5.2.1 (Sum of series)**.** *If $(a_n)_{n=1}^{\infty}$ is a sequence, then the sum of the series is*

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} s_N$$

provided the limit exists. Otherwise we say that $\sum_{n=1}^{\infty} a_n$ diverges.

Example 5.2.1 (Geometric series)**.** *Consider the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ for $a \neq 0$. Then*

$$s_N = \sum_{n=1}^{N} ar^{n-1} = \frac{a(1-r^N)}{1-r}.$$

To see this, multiply both sides by $1-r$ and notice that all the terms in the middle cancel out. For what values of r does $\lim_{N\to\infty} \frac{a(1-r^N)}{1-r}$ converge? If $|r| < 1$, then $\lim_{N\to\infty} r^N = 0$ and

$$\lim_{N\to\infty} \frac{a(1-r^N)}{1-r} = \frac{a}{1-r}.$$

If $|r| > 1$, then $\lim_{N\to\infty} r^N$ diverges, so $\sum_{n=1}^{\infty} ar^{n-1}$ diverges. If $r = \pm 1$, it's clear since $a \neq 0$ that the series also diverges (since the partial sums are $s_N = \pm Na$).

For example, if $a = 1$ and $r = \frac{1}{2}$, we get

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{1}{1-\frac{1}{2}},$$

as claimed earlier.

5.3 The Integral and Comparison Tests

What is $\sum_{n=1}^{\infty} \frac{1}{n^2}$? What is $\sum_{n=1}^{\infty} \frac{1}{n}$?

Recall that Section 5.2 began by asking for the sum of several series. We found the first two sums (which were geometric series) by finding an exact formula for the sum s_N of the first N terms. The third series was

$$A = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots. \tag{5.1}$$

It is difficult to find a nice formula for the sum of the first n terms of this series (i.e., I don't know how to do it).

Remark 5.3.1. *Since I'm a number theorist, I can't help but make some further remarks about sums of the form (5.1). In general, for any $s > 1$ one can consider the sum*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The number A that we are interested in above is thus $\zeta(2)$. The function $\zeta(s)$ is called the Riemann zeta function. There is a natural (but complicated) way of extending $\zeta(s)$ to a (differentiable) function on all complex numbers with a pole at $s = 1$. The Riemann Hypothesis asserts that if s is a complex number and $\zeta(s) = 0$ then either s is an even negative integer or $s = \frac{1}{2} + bi$ for some real number b. This is probably the *most famous unsolved problems in mathematics (e.g., it's one of the Clay Math Institute million dollar prize problems). Another famous open problem is to show that $\zeta(3)$ is not a root of any polynomial with integer coefficients (it is a theorem of Apeéry that $zeta(3)$ is not a fraction).*

The function $\zeta(s)$ is incredibly important in mathematics because it governs the properties of prime numbers. The Euler product representation of $\zeta(s)$ gives a hint as to why this is the case:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{primes\ p} \left(\frac{1}{1 - p^{-s}} \right).$$

To see that this product equality holds when s is real with $\mathrm{Re}(s) > 1$, *use Example 5.2.1 with $r = p^{-s}$ and $a = 1$ above. We have*

$$\frac{1}{1 - p^{-s}} = 1 + p^{-s} + p^{-2s} + \cdots.$$

Thus

$$\begin{aligned}
\prod_{primes\ p} \left(\frac{1}{1-p^{-s}}\right) &= \prod_{primes\ p} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots\right) \cdot \left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots\right) \cdots \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^s},
\end{aligned}$$

where the last line uses the distributive law and that integers factor uniquely as a product of primes.

of $|\zeta(s)|$ for complex s.

This section is how to leverage what you've learned so far in this book to say something about sums that are hard (or even "impossibly difficult") to evaluate exactly. For example, notice (by considering a graph of a step function) that if $f(x) = 1/x^2$, then for positive integer t we have

$$\sum_{n=1}^{t} \frac{1}{n^2} \leq \frac{1}{1^2} + \int_1^t \frac{1}{x^2} dx.$$

Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \frac{1}{1^2} &+ \int_1^{\infty} \frac{1}{x^2} dx \\
&= 1 + \lim_{t\to\infty} \int_1^t \frac{1}{x^2} dx \\
&= 1 + \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t \\
&= 1 + \lim_{t\to\infty} \left[-\frac{1}{t} + \frac{1}{1}\right] = 2
\end{aligned}$$

We conclude that $\sum_{n=1}^{\infty}$ converges, since the sequence of partial sums is getting bigger and bigger and is always ≤ 2. And of course we also know something about $\sum_{n=1}^{\infty} \frac{1}{n^2}$ even though we do not know the exact value: $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 2$. Using a computer we find that

t	$\sum_{n=1}^{t} \frac{1}{n^2}$
1	1
2	$\frac{5}{4} = 1.25$
5	$\frac{5269}{3600} = 1.4636\overline{1}$
10	$\frac{1968329}{1270080} = 1.54976773117$
100	1.63498390018
1000	1.64393456668
10000	1.64483407185
100000	1.6449240669

The table is consistent with the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to a number ≤ 2. In fact Euler was the first to compute $\sum_{n=1}^{\infty}$ exactly; he found that the exact value is

$$\frac{\pi^2}{6} = 1.644934066848226436472415166646025189218949901206798437735557\ldots$$

There are many proofs of this fact, but they don't belong in this book; you can find them on the internet, and are likely to see one if you take more math classes.

We next consider the *harmonic series*

$$\sum_{n=1}^{\infty} \frac{1}{n}. \tag{5.2}$$

Does it converge? Again by inspecting a graph and viewing an infinite sum as the area under a step function, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{1}{x} dx \\ &= \lim_{t\to\infty} [\ln(x)]_1^t \\ &= \lim_{t\to\infty} \ln(t) - 0 = +\infty.\end{aligned}$$

Thus the infinite sum (5.2) must also diverge.

We formalize the above two examples as a general test for convergence or divergence of an infinite sum.

Theorem 5.3.1 (Integral Test and Bound)**.** *Suppose $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$ for integers $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the integral $\int_1^{\infty} f(x)dx$ converges. More generally, for any positive integer k,*

$$\int_k^{\infty} f(x)dx \quad \leq \quad \sum_{n=k}^{\infty} a_n \quad \leq \quad a_k + \int_k^{\infty} f(x)dx. \tag{5.3}$$

The proposition means that you can determine convergence of an infinite series by determining convergence of a corresponding integral. Thus you can apply the powerful tools you know already for integrals to understanding infinite sums. Also, you can use integration along with computation of the first few terms of a series to approximate a series very precisely.

Remark 5.3.2. *Sometimes the first few terms of a series are "funny" or the series doesn't even start at $n = 1$, e.g.,*

$$\sum_{n=4}^{\infty} \frac{1}{(n-3)^3}.$$

In this case use (5.3) with any specific $k > 1$.

Proposition 5.3.1 (Comparison Test)**.** *Suppose $\sum a_n$ and $\sum b_n$ are two series with positive terms. If $\sum b_n$ converges and $a_n \leq b_n$ for all n. then $\sum a_n$ converges. Likewise, if $\sum b_n$ diverges and $a_n \geq b_n$ for all n. then $\sum a_n$ must also diverge.*

Example 5.3.1. Does $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge? *No. We have*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{t\to\infty} (2\sqrt{t} - 2\sqrt{1}) = +\infty$$

Example 5.3.2. Does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge? *Let's apply the comparison test: we have $\frac{1}{n^2+1} < \frac{1}{n^2}$ for every n, so*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Alternatively, we can use the integral test, which also gives as a bonus an upper and lower bound on the sum. Let $f(x) = 1/(1+x^2)$. We have

$$\begin{aligned} \int_1^{\infty} \frac{1}{1+x^2} dx &= \lim_{t\to\infty} \int_1^t \frac{1}{1+x^2} dx \\ &= \lim_{t\to\infty} \tan^{-1}(t) - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

Thus the sum converges. Moreover, taking $k = 1$ in Theorem 5.3.1 we have

$$\frac{\pi}{4} \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \frac{1}{2} + \frac{\pi}{4}.$$

the actual sum is $1.07\ldots$, which is much different than $\sum \frac{1}{n^2} = 1.64\ldots$.

We could prove the following proposition using methods similar to those illustrated in the examples above.

Proposition 5.3.2. *The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.*

5.3.1 Estimating the Sum of a Series

Suppose $\sum a_n$ is a convergent sequence of positive integers. Let

$$R_m = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{m} a_n = \sum_{n=m+1}^{\infty} a_m$$

which is the error if you approximate $\sum a_n$ using the first n terms. From Theorem 5.3.1 we get the following.

Proposition 5.3.3 (Remainder Bound)**.** *Suppose f is a continuous, positive, decreasing function on $[m, \infty)$ and $\sum a_n$ is convergent. Then*

$$\int_{m+1}^{\infty} f(x)dx \leq R_m \leq \int_{m}^{\infty} f(x)dx.$$

Proof. In Theorem 5.3.1 set $k = m + 1$. That gives

$$\int_{m+1}^{\infty} f(x)dx \quad \leq \quad \sum_{n=m+1}^{\infty} a_n \quad \leq \quad a_{m+1} + \int_{m+1}^{\infty} f(x)dx.$$

But

$$a_{m+1} + \int_{m+1}^{\infty} f(x)dx \leq \int_{m}^{\infty} f(x)dx$$

since f is decreasing and $f(m+1) = a_{m+1}$. $\square$

Example 5.3.3. *Estimate $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$ using the first 10 terms of the series. We have*

$$\sum_{n=1}^{10} = \frac{19164113947}{16003008000} = 1.197531985674193\ldots$$

The proposition above with $m = 10$ tells us that

$$0.00413223140495867\ldots = \int_{11}^{\infty} \frac{1}{x^3} dx \leq \zeta(3) - \sum_{n=1}^{10} \leq \int_{10}^{\infty} \frac{1}{x^3} dx = \frac{1}{2 \cdot 10^2} = \frac{1}{200} = 0.005.$$

In fact,

$$\zeta(3) = 1.202056903159594285399738161511449990\ldots$$

and we have

$$\zeta(3) - \sum_{n=1}^{10} = 0.0045249174854010\ldots,$$

so the integral error bound was really good in this case.

Example 5.3.4. *Determine if $\sum_{n=1}^{\infty} \frac{2006}{117n^2+41n+3}$ converges or diverges. Answer: It converges, since*

$$\frac{2006}{117n^2 + 41n + 3} \leq \frac{2006}{117n^2} = \frac{2006}{117} \cdot \frac{1}{n^2},$$

and $\sum \frac{1}{n^2}$ converges.

5.4 Tests for Convergence

5.4.1 The Comparison Test

Theorem 5.4.1 (The Comparison Test)**.** *Suppose $\sum a_n$ and $\sum b_n$ are series with all a_n and b_n positive and $a_n \leq b_n$ for each n.*

1. *If $\sum b_n$ converges, then so does $\sum a_n$.*
2. *If $\sum a_n$ diverges, then so does $\sum b_n$.*

Proof Sketch. The condition of the theorem implies that for any k,

$$\sum_{n=1}^{k} a_n \leq \sum_{n=1}^{k} b_n,$$

from which each claim follows. □

Example 5.4.1. *Consider the series $\sum_{n=1}^{\infty} \frac{7}{3n^2+2n}$. For each n we have*

$$\frac{7}{3n^2+2n} \leq \frac{7}{3} \cdot \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, Theorem 5.4.1 implies that $\sum_{n=1}^{\infty} \frac{7}{3n^2+2n}$ also converges.

Example 5.4.2. *Consider the series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$. It diverges since for each $n \geq 3$ we have*

$$\frac{\ln(n)}{n} \geq \frac{1}{n},$$

and $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges.

5.4.2 Absolute and Conditional Convergence

Definition 5.4.1 (Converges Absolutely)**.** *We say that $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges.*

For example,

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

converges, but does *not* converge absolutely (it converges "conditionally", though we will not explain why in this book).

5.4.3 The Ratio Test

Recall that $\sum_{n=1}^{\infty} a_n$ is a geometric series if and only if $a_n = ar^{n-1}$ for some fixed a and r. Here we call r the *common ratio.* Notice that the ratio of any two successive terms is r:

$$\frac{a_{n+1}}{a_n} = \frac{ar^n}{ar^{n-1}} = r.$$

Moreover, we have $\sum_{n=1}^{\infty} ar^{n-1}$ converges (to $\frac{a}{1-r}$) if and only if $|r| < 1$ (and, of course it diverges if $|r| \geq 1$).

Example 5.4.3. *For example,* $\sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^{n-1}$ *converges to* $\frac{3}{1-\frac{2}{3}} = 9$. *However,* $\sum_{n=1}^{\infty} 3\left(\frac{3}{2}\right)^{n-1}$ *diverges.*

Theorem 5.4.2 (Ratio Test)**.** *Consider a sum* $\sum_{n=1}^{\infty} a_n$. *Then*

1. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent.*

2. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

3. *If* $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$ *then we may conclude nothing from this!*

Proof. We will only prove 1. Assume that we have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$. Let $r = \frac{L+1}{2}$, and notice that $L < r < 1$ (since $0 \leq L < 1$, so $1 \leq L+1 < 2$, so $1/2 \leq r < 1$, and also $r - L = (L+1)/2 - L = (1-L)/2 > 0$).

Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, there is an N such that for all $n > N$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| < r, \qquad \text{so} \qquad |a_{n+1}| < |a_n| \cdot r.$$

Then we have

$$\sum_{n=N+1}^{\infty} |a_n| < |a_{N+1}| \cdot \sum_{n=0}^{\infty} r^n.$$

Here the common ratio for the second one is $r < 1$, hence thus the right-hand series converges, so the left-hand series converges. □

Example 5.4.4. *Consider* $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$. *The ratio of successive terms is*

$$\left|\frac{\frac{(-10)^{n+1}}{(n+1)!}}{\frac{(-10)^n}{n!}}\right| = \frac{10^{n+1}}{(n+1)n!} \cdot \frac{n!}{10^n} = \frac{10}{n+1} \to 0 < 1.$$

Thus this series converges absolutely. *Note, the minus sign is missing above since in the ratio test we take the limit of the absolute values.*

Example 5.4.5. *Consider* $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$. *We have*

$$\left| \frac{\frac{(n+1)^{n+1}}{3\cdot(27)^{n+1}}}{\frac{n^n}{3^{1+3n}}} \right| = \frac{(n+1)(n+1)^n}{27\cdot 27^n} \cdot \frac{27^n}{n^n} = \frac{n+1}{27} \cdot \left(\frac{n+1}{n}\right)^n \to +\infty$$

Thus our series diverges. (Note here that we use that $\left(\frac{n+1}{n}\right)^n \to e$.*)*

Example 5.4.6. *Let's apply the ratio test to* $\sum_{n=1}^{\infty} \frac{1}{n}$. *We have*

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \frac{1}{n+1} \cdot \frac{n}{1} = \frac{n}{n+1} \to 1.$$

This tells us nothing. If this happens... do something else! E.g., in this case, use the integral test.

5.4.4 The Root Test

Since e and ln are inverses, we have $x = e^{\ln(x)}$. This implies the very useful fact that

$$x^a = e^{\ln(x^a)} = e^{a\ln(x)}.$$

As a sample application, notice that for any nonzero c,

$$\lim_{n\to\infty} c^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\log(c)} = e^0 = 1.$$

Similarly,

$$\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} e^{\frac{1}{n}\log(n)} = e^0 = 1,$$

where we've used that $\lim_{n\to\infty} \frac{\log(n)}{n} = 0$, which we could prove using L'Hopital's rule.

Theorem 5.4.3 (Root Test)**.** *Consider the sum* $\sum_{n=1}^{\infty} a_n$.

1. *If* $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*
2. *If* $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
3. *If* $L = 1$, *then we may conclude nothing from this!*

Proof. We apply the comparison test (Theorem 5.4.1). First suppose $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L < 1$. Then there is a N such that for $n \geq N$ we have $|a_n|^{\frac{1}{n}} < k < 1$. Thus for such n we have $|a_n| < k^n < 1$. The geometric series $\sum_{i=N}^{\infty} k^i$ converges, so $\sum_{i=N}^{\infty} |a_n|$ also does, by Theorem 5.4.1. If $|a_n|^{\frac{1}{n}} > 1$ for $n \geq N$, then we see that $\sum_{i=N}^{\infty} |a_n|$ diverges by comparing with $\sum_{i=N}^{\infty} 1$. $\square$

Example 5.4.7. *Let's apply the root test to*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{r}\sum_{n=1}^{\infty} r^n.$$

We have

$$\lim_{n\to\infty} |r^n|^{\frac{1}{n}} = |r|.$$

Thus the root test tells us exactly what we already know about convergence of the geometry series (except when $|r| = 1$).

Example 5.4.8. *The sum $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ is a candidate for the root test. We have*

$$\lim_{n\to\infty}\left|\left(\frac{n^2+1}{2n^2+1}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^2+1}{2n^2+1} = \lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}} = \frac{1}{2}.$$

Thus the series converges.

Example 5.4.9. *The sum $\sum_{n=1}^{\infty}\left(\frac{2n^2+1}{n^2+1}\right)^n$ is a candidate for the root test. We have*

$$\lim_{n\to\infty}\left|\left(\frac{2n^2+1}{n^2+1}\right)^n\right|^{\frac{1}{n}} = \lim_{n\to\infty}\frac{2n^2+1}{n^2+1} = \lim_{n\to\infty}\frac{2+\frac{1}{n^2}}{1+\frac{1}{n^2}} = 2,$$

hence the series diverges!

Example 5.4.10. *Consider $\sum_{n=1}^{\infty}\frac{1}{n}$. We have*

$$\lim_{n\to\infty}\left|\frac{1}{n}\right|^{\frac{1}{n}} = 1,$$

so we conclude nothing!

Example 5.4.11. *Consider $\sum_{n=1}^{\infty}\frac{n^n}{3\cdot(27^n)}$. To apply the root test, we compute*

$$\lim_{n\to\infty}\left|\frac{n^n}{3\cdot(27^n)}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{3}\right)^{\frac{1}{n}}\cdot\frac{n}{27} = +\infty.$$

Again, the limit diverges, as in Example 5.4.5.

5.5 Power Series

Recall that a *polynomial* is a function of the form

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k.$$

$$\boxed{\text{Polynomials are easy!!!}}$$

They are easy to integrate, differentiate, etc.:

$$\frac{d}{dx}\left(\sum_{n=0}^{k} c_n x^n\right) = \sum_{n=1}^{k} n c_n x^{n-1}$$

$$\int \sum_{n=0}^{k} c_n x^n dx = C + \sum_{n=0}^{k} c_n \frac{x^{n+1}}{n+1}.$$

Definition 5.5.1 (Power Series)**.** *A power series is a series of the form*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots,$$

where x is a variable and the c_n are coefficients.

A power series is a function of x for those x for which it converges.

Example 5.5.1. *Consider*

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots.$$

When $|x| < 1$, i.e., $-1 < x < 1$, we have

$$f(x) = \frac{1}{1-x}.$$

But what good could this possibly be? Why is writing the simple function $\frac{1}{1-x}$ as the complicated series $\sum_{n=0}^{\infty} x^n$ of any value?

1. Power series are *relatively easy to work with.* They are "almost" polynomials. E.g.,

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \cdots = \sum_{m=0}^{\infty} (m+1) x^m,$$

   where in the last step we "re-indexed" the series. Power series are only "almost" polynomials, since they don't stop; they can go on forever. More precisely, a power series is a limit of polynomials. But in many cases we can treat them like a polynomial. On the other hand, notice that

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} = \sum_{m=0}^{\infty} (m+1) x^m.$$

2. For many functions, a power series is the *best explicit representation available.*

   Example 5.5.2. *Consider $J_0(x)$, the Bessel function of order 0. It arises as a solution to the differential equation $x^2y'' + xy' + x^2y = 0$, and has the following power series expansion:*

   $$\begin{aligned} J_0(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \\ &= 1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \frac{1}{147456}x^8 - \frac{1}{14745600}x^{10} + \cdots . \end{aligned}$$

   This series is nice since it converges for all x (one can prove this using the ratio test). It is also one of the most explicit forms of $J_0(x)$.

5.5.1 Shift the Origin

It is often useful to shift the origin of a power series, i.e., consider a power series expanded about a different point.

Definition 5.5.2. *The series*

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a power series centered at $x = a$, or "a power series about $x = a$".

Example 5.5.3. *Consider*

$$\begin{aligned} \sum_{n=0}^{\infty}(x-3)^n &= 1 + (x-3) + (x-3)^2 + \cdots \\ &= \frac{1}{1-(x-3)} \qquad \text{equality valid when } |x-3| < 1 \\ &= \frac{1}{4-x} \end{aligned}$$

Here conceptually we are treating 3 like we treated 0 before.

Power series can be written in different ways, which have different advantages and disadvantages. For example,

$$\begin{aligned} \frac{1}{4-x} &= \frac{1}{4} \cdot \frac{1}{1-x/4} \\ &= \frac{1}{4} \cdot \sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n \qquad \text{converges for all } |x| < 4. \end{aligned}$$

Notice that the second series converges for $|x| < 4$, whereas the first converges only for $|x-3| < 1$, which isn't nearly as good.

5.5.2 Convergence of Power Series

Theorem 5.5.1. *Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are* exactly *three possibilities:*

1. *The series converges only when $x = a$.*
2. *The series converges for all x.*
3. *There is an $R > 0$ (called the "radius of convergence") such that $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $|x-a| < R$ and diverges for $|x-a| > R$.*

Example 5.5.4. *For the power series $\sum_{n=0}^{\infty} x^n$, the radius R of convergence is 1.*

Definition 5.5.3 (Radius of Convergence)**.** *As mentioned in the theorem, R is called the radius of convergence.*

If the series converges only at $x = a$, we say $R = 0$, and if the series converges everywhere we say that $R = \infty$.

The *interval of convergence* is the set of x for which the series converges. It will be one of the following:

$$(a-R, a+R), \qquad [a-R, a+R), \qquad (a-R, a+R], \qquad [a-R, a+R]$$

The point being that the statement of the theorem only asserts something about convergence of the series on the open interval $(a-R, a+R)$. What happens at the endpoints of the interval is not specified by the theorem; you can only figure it out by looking explicitly at a given series.

Theorem 5.5.2. *If $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ is differentiable on $(a-R, a+R)$, and*

1. $f'(x) = \displaystyle\sum_{n=1}^{\infty} n \cdot c_n(x-a)^{n-1}$
2. $\displaystyle\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$,

and both the derivative and integral have the same radius of convergence as f.

Example 5.5.5. *Find a power series representation for $f(x) = \tan^{-1}(x)$. Notice that*

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-1)^n x^{2n},$$

which has radius of convergence $R = 1$, since the above series is valid when $|-x^2| < 1$, i.e., $|x| < 1$. Next integrating, we find that

$$f(x) = c + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1},$$

for some constant c*. To find the constant, compute* $c = f(0) = \tan^{-1}(0) = 0$*. We conclude that*

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

Example 5.5.6. *We will see later that the function* $f(x) = e^{-x^2}$ *has power series*

$$e^{-x^2} = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots .$$

Hence

$$\int e^{-x^2}dx = c + x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 + \cdots .$$

This despite the fact that the antiderivative of e^{-x^2} *is not an elementary function.*

5.6 Taylor Series

Example 5.6.1. *Suppose we have a degree-*3 *(cubic) polynomial* p *and we know that* $p(0) = 4$*,* $p'(0) = 3$*,* $p''(0) = 4$*, and* $p'''(0) = 6$*. Can we determine* p*? Answer: Yes! We have*

$$\begin{aligned} p(x) &= a + bx + cx^2 + dx^3 \\ p'(x) &= b + 2cx + 3dx^2 \\ p''(x) &= 2c + 6dx \\ p'''(x) &= 6d \end{aligned}$$

From what we mentioned above, we have:

$$\begin{aligned} a &= p(0) = 4 \\ b &= p'(0) = 3 \\ c &= \frac{p''(0)}{2} = 2 \\ d &= \frac{p'''(0)}{6} = 1 \end{aligned}$$

Thus

$$p(x) = 4 + 3x + 2x^2 + x^3.$$

Amazingly, we can use the idea of Example 5.6.1 to compute power series expansions of functions. E.g., we will show below that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Convergent series are determined by the values of their derivatives.

Consider a general power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

We have

$$\begin{aligned} c_0 &= f(a) \\ c_1 &= f'(a) \\ c_2 &= \frac{f''(a)}{2} \\ \cdots & \\ c_n &= \frac{f^{(n)}(a)}{n!}, \end{aligned}$$

where for the last equality we use that

$$f^{(n)}(x) = n!c_n + (x-a)(\cdots + \cdots)$$

Remark 5.6.1. *The definition of* $0!$ *is* 1 *(it's the empty product). The empty sum is* 0 *and the empty product is* 1*.*

Theorem 5.6.1 (Taylor Series)**.** *If* $f(x)$ *is a function that equals a power series centered about* a*, then that power series expansion is*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots \end{aligned}$$

Remark 5.6.2. *WARNING: There are functions that have all derivatives defined, but do not equal their Taylor expansion. E.g.,* $f(x) = e^{-1/x^2}$ *for* $x \neq 0$ *and* $f(0) = 0$*. It's Taylor expansion is the* 0 *series (which converges everywhere), but it is not the* 0 *function.*

Definition 5.6.1 (Maclaurin Series)**.** *A Maclaurin series is just a Taylor series with* $a = 0$*. I will not use the term "Maclaurin series" ever again (it's common in textbooks).*

Example 5.6.2. *Find the Taylor series for* $f(x) = e^x$ *about* $a = 0$*. We have* $f^{(n)}(x) = e^x$*. Thus* $f^{(n)}(0) = 1$ *for all* n*. Hence*

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

What is the radius of convergence? Use the ratio test:

$$\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}x^{n+1}}{\frac{1}{n!}x^n}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!}|x|$$
$$= \lim_{n\to\infty}\frac{|x|}{n+1} = 0, \qquad \textit{for any fixed } x.$$

Thus the radius of convergence is ∞*.*

Example 5.6.3. *Find the Taylor series of* $f(x) = \sin(x)$ *about* $x = \frac{\pi}{2}$*.*[1] *We have*

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}\left(\frac{\pi}{2}\right)}{n!}\left(x-\frac{\pi}{2}\right)^n.$$

To do this we have to puzzle out a pattern*:*

$$\begin{aligned} f(x) &= \sin(x) \\ f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f'''(x) &= -\cos(x) \\ f^{(4)}(x) &= \sin(x) \end{aligned}$$

First notice how the signs behave. For $n = 2m$ *even,*

$$f^{(n)}(x) = f^{(2m)}(x) = (-1)^{n/2}\sin(x)$$

and for $n = 2m+1$ *odd,*

$$f^{(n)}(x) = f^{(2m+1)}(x) = (-1)^m\cos(x) = (-1)^{(n-1)/2}\cos(x)$$

For $n = 2m$ *even we have*

$$f^{(n)}(\pi/2) = f^{(2m)}\left(\frac{\pi}{2}\right) = (-1)^m.$$

and for $n = 2m+1$ *odd we have*

$$f^{(n)}(\pi/2) = f^{(2m+1)}\left(\frac{\pi}{2}\right) = (-1)^m\cos(\pi/2) = 0.$$

Finally,

$$\begin{aligned} \sin(x) &= \sum_{n=0}^{\infty}\frac{f^{(n)}(\pi/2)}{n!}(x-\pi/2)^n \\ &= \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(x-\frac{\pi}{2}\right)^{2m}. \end{aligned}$$

[1]Evidently this expansion was first found in India by Madhava of Sangamagrama (1350-1425).

Next we use the ratio test to compute the radius of convergence. We have

$$\lim_{m\to\infty} \frac{\left|\dfrac{(-1)^{m+1}}{(2(m+1))!}\left(x-\dfrac{\pi}{2}\right)^{2(m+1)}\right|}{\left|\dfrac{(-1)^m}{(2m)!}\left(x-\dfrac{\pi}{2}\right)^{2m}\right|} = \lim_{m\to\infty}\frac{(2m)!}{(2m+2)!}\left(x-\frac{\pi}{2}\right)^2$$

$$= \lim_{m\to\infty}\frac{\left(x-\frac{\pi}{2}\right)^2}{(2m+2)(2m+1)}$$

which converges for each x. Hence $R=\infty$.

Example 5.6.4. *Find the Taylor series for* $\cos(x)$ *about $a = 0$. We have* $\cos(x) = \sin\left(x+\frac{\pi}{2}\right)$. *Thus from Example 5.6.3 (with infinite radius of convergence) and that the Taylor expansion is unique, we have*

$$\begin{aligned}\cos(x) &= \sin\left(x+\frac{\pi}{2}\right)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x+\frac{\pi}{2}-\frac{\pi}{2}\right)^{2n}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\end{aligned}$$

5.7 Applications of Taylor Series

This section is about an example in the theory of relativity. Let m be the (relativistic) mass of an object and m_0 be the mass at rest (rest mass) of the object. Let v be the velocity of the object relative to the observer, and let c be the speed of light. These three quantities are related as follows:

$$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} \qquad \text{(relativistic) mass}$$

The total energy of the object is mc^2:

$$\boxed{E = mc^2.}$$

In relativity we define the kinetic energy to be

$$K = mc^2 - m_0c^2. \tag{5.4}$$

What? Isn't the kinetic energy $\frac{1}{2}m_0v^2$?

Notice that

$$mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1 - \frac{v^2}{c^2}}} - m_0c^2 = m_0c^2\left[\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} - 1\right].$$

Let

$$f(x) = (1 - x)^{-\frac{1}{2}} - 1$$

Let's compute the Taylor series of f. We have

$$\begin{aligned}
f(x) &= (1 - x)^{-\frac{1}{2}} - 1 \\
f'(x) &= \frac{1}{2}(1 - x)^{-\frac{3}{2}} \\
f''(x) &= \frac{1}{2} \cdot \frac{3}{2}(1 - x)^{-\frac{5}{2}} \\
f^{(n)}(x) &= \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n}(1 - x)^{-\frac{2n+1}{2}}.
\end{aligned}$$

Thus

$$f^{(n)}(0) = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n}.$$

Hence

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\
&= \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n \cdot n!}x^n \\
&= \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \cdots
\end{aligned}$$

We now use this to analyze the kinetic energy (5.4):

$$\begin{aligned} mc^2 - m_0c^2 &= m_0c^2 \cdot f\left(\frac{v^2}{c^2}\right) \\ &= m_0c^2 \cdot \left(\frac{1}{2} \cdot \frac{v^2}{c^2} + \frac{3}{8} \cdot \frac{v^2}{c^2} + \cdots\right) \\ &= \frac{1}{2}m_0v^2 + m_0c^2 \cdot \left(\frac{3}{8}\frac{v^2}{c^2} + \cdots\right) \end{aligned}$$

And we can ignore the higher order terms if $\frac{v^2}{c^2}$ is small. But how small is "small" enough, given that $\frac{v^2}{c^2}$ appears in an infinite sum?

5.7.1 Estimation of Taylor Series

Suppose

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Write

$$R_N(x) := f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We call

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

the Nth degree *Taylor polynomial*. Notice that

$$\lim_{N\to\infty} T_N(x) = f(x)$$

if and only if

$$\lim_{N\to\infty} R_N(x) = 0.$$

We would like to estimate $f(x)$ with $T_N(x)$. We need an estimate for $R_N(x)$.

Theorem 5.7.1 (Taylor's theorem)**.** *If $|f^{(N+1)}(x)| \leq M$ for $|x-a| \leq d$, then*

$$|R_N(x)| \leq \frac{M}{(N+1)!}|x-a|^{N+1} \qquad \text{for } |x-a| \leq d.$$

For example, if $N = 0$, this says that

$$|R(x)| = |f(x) - f(a)| \leq M|x-a|,$$

i.e.,

$$\left|\frac{f(x)-f(a)}{x-a}\right| \leq M,$$

which should look familiar from a previous class (Mean Value Theorem).

Applications:

1. One can use Theorem 5.7.1 to prove that functions converge to their Taylor series.

2. Returning to the relativity example above, we apply Taylor's theorem with $N = 1$ and $a = 0$. With $x = -v^2/c^2$ and M any number such that $|f''(x)| \leq M$, we have
$$|R_1(x)| \leq \frac{M}{2}x^2.$$
For example, if we assume that $|v| \leq 100m/s$ we use
$$|f''(x)| \leq \frac{3}{2}(1 - 100^2/c^2)^{-5/2} = M.$$
Using $c = 3 \times 10^8 m/s$, we get
$$|R_1(x)| \leq 4.17 \cdot 10^{-10} \cdot m_0.$$
Thus for $v \leq 100m/s \sim 225$mph, then the error in throwing away relativistic factors is $10^{-10}m_0$. This is like 200 feet out of the distance to the sun (93 million miles). So relativistic and Newtonian kinetic energies are almost the same for reasonable speeds.

Chapter 6

Some Differential Equations

This chapter is an introduction to differential equations, a major field in applied and theoretical mathematics and a very useful one for engineers, scientists, and others who study changing phenomena. The physical laws of motion and heat and electricity can be written as differential equations. The growth of a population, the changing gene frequencies in that population, and the spread of a disease can be described by differential equations. Economic and social models use differential equations, and the earliest examples of "chaos" came from studying differential equations used for modeling atmospheric behavior. Some scientists even say that the main purpose of a calculus course should be to teach people to understand and solve differential equations.

Differential Equations

Algebraic equations contain constants and variables, and the solutions of an algebraic equation are typically numbers. For example, $x = 3$ and $x = -2$ are solutions of the algebraic equation $x^2 = x + 6$. Differential equations contain derivatives or differentials of functions. Solutions of differential equations are functions. The differential equation $y' = 3x^2$ has infinitely many solutions, and two of those solutions are the functions $y = x^3 + 2$ and $y = x^3 - 4$. You have already solved lots of differential equations: every time you found an antiderivative of a function $f(x)$, you solved the differential equation $y' = f(x)$ to get a solution y. You have also used differential equations in applications. Areas, volumes, work, and motion problems all involved integration and finding antiderivatives so they all used differential equations. The differential equation $y' = f(x)$, however, is just the beginning. Other applications generate different differential equations.

Checking Solutions of Differential Equations

Whether a differential equation is easy or difficult to solve, it is important to be able to check that a possible solution really satisfies the differential equation. A possible solution of an algebraic equation can be checked by putting the solution into the equation to see if the result is true: $x = 3$ is a solution of $5x + 1 = 16$ since $5(3) + 1 = 16$ is true. Similarly, a solution of a differential equation can be checked by substituting the function and the appropriate derivatives into the

equation to see if the result is true: $y = x^2$ is a solution of $xy' = 2y$ since $y' = 2x$ and $x(2x) = 2(x^2)$ is true.
2

Example 6.0.1. *For every value of C, the function $y = Cx^2$ is a solution of $xy' = 2y$. Find the value of C so that $y(5) = 50$.*

Solution: Substituting the initial condition $x = 5$ and $y = 50$ into the solution $y = Cx^2$, we have that $50 = C5^2$ so $C = 50/25 = 2$. The function $y = 2x^2$ satisfies both the differential equation and the initial condition.

6.1 Separable Equations

A *separable differential equation* is a first order differential equation that can be written in the form

$$\frac{dy}{dx} = \frac{f(x)}{h(y)}.$$

These can be solved by integration, by noting that

$$h(y)dy = f(x)dx,$$

hence

$$\int h(y)dy = \int f(x)dx + C.$$

This latter equation defines y implicitly as a function of x (we have added a "$+C$" onto one side just to emphasize that you only need one constant of integration for the solution), and in some cases it is possible to explicitly solve for y as a function of x.

6.2 Logistic Equation

The logistics equation is a differential equation that models population growth. Often in practice a differential equation models some physical situation, and you should "read it" as doing so.

Exponential growth:

$$\frac{1}{P}\frac{dP}{dt} = k.$$

This says that the "relative (percentage) growth rate" is constant. As we saw before, the solutions are

$$P_{(t)} = P_0 \cdot e^{kt}.$$

Note that this model only works for a little while. In everyday life the growth couldn't actually continue at this rate indefinitely. This exponential growth model ignores limitations on resources, disease, etc. Perhaps there is a better model?

Over time we expect the growth rate should level off, i.e., decrease to 0. What about

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{K}\right), \tag{6.1}$$

where K is some large constant called the *carrying capacity*, which is much bigger than $P = P(t)$ at time 0. The carrying capacity is the maximum population that the environment can support. Note that if $P > K$, then $dP/dt < 0$ so the population declines. The differential equation (6.1) is called the logistic model (or logistic differential equation). There are, of course, other models one could use, e.g., the Gompertz equation.

First question: are there any *equilibrium solutions* to (6.1), i.e., solutions with $dP/dt = 0$, i.e., constant solutions? In order that $dP/dt = 0$ then $0 = k\left(1 - \frac{P}{K}\right)$, so the two equilibrium solutions are $P(t) = 0$ and $P(t) = K$.

The logistic differential equation (6.1) is separable, so you can separate the variables with one variable on one side of the equality and one on the other. This means we can easily solve the equation by integrating. We rewrite the equation as

$$\frac{dP}{dt} = -\frac{k}{K}P(P-K).$$

Now separate:

$$\frac{K dP}{P(P-K)} = -k \cdot dt,$$

and integrate both sides

$$\int \frac{K dP}{P(P-K)} = \int -k \cdot dt = -kt + C.$$

On the left side we get

$$\int \frac{K dP}{P(P-K)} = \int \left(\frac{1}{P-K} - \frac{1}{P}\right) dP = \ln|P-K| - \ln|P| + *$$

Thus

$$\ln|K-P| - \ln|P| = -kt + c,$$

so

$$\ln|(K-P)/P| = -kt + c.$$

Now exponentiate both sides:

$$(K-P)/P = e^{-kt+c} = Ae^{-kt}, \qquad \text{where } A = e^c.$$

Thus

$$K = P(1 + Ae^{-kt}),$$

so

$$P(t) = \frac{K}{1 + Ae^{-kt}}.$$

Note that $A = 0$ also makes sense and gives an equilibrium solution. In general we have $\lim_{t\to\infty} P(t) = K$. In any particular case we can determine A as a function of $P_0 = P(0)$ by using that

$$P(0) = \frac{K}{1+A} \qquad \text{so} \qquad A = \frac{K}{P_0} - 1 = \frac{K - P_0}{P_0}.$$

Chapter 7

Appendix: Integral tables

Trigonometry

Trigonometric Functions

T1. $$\sin^2 x + \cos^2 x = 1$$

T2. $$\tan^2 x + 1 = \sec^2 x$$

T3. $$\cot^2 x + 1 = \csc^2 x$$

T4. $$\sin(x \pm y) = \sin x \, \cos y \pm \cos x \, \sin y$$

T5. $$\cos(x \pm y) = \cos x \, \cos y \mp \sin x \, \sin y$$

T6. $$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \, \tan y}$$

T7. $$\tan(\frac{x}{2}) = \frac{\sin x}{1 + \cos x}$$

T8. $$\sin(2x) = 2 \sin x \, \cos x$$

T9. $$\cos(2x) = \cos^2 x - \sin^2 x$$

T10. $$\sin^2 x = {}^1\!/_2(1 - \cos(2x))$$

T11. $$\cos^2 x = {}^1\!/_2(1 + \cos(2x))$$

T12. $$\sin x \, \sin y = {}^1\!/_2\big(\cos(x - y) - \cos(x + y)\big)$$

T13. $$\cos x \, \cos y = {}^1\!/_2\big(\cos(x - y) + \cos(x + y)\big)$$

T14. $$\sin x \, \cos y = {}^1\!/_2\big(\sin(x - y) + \sin(x + y)\big)$$

T15. $$c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \, \sin(\omega t + \phi),$$

$$\text{where } A = \sqrt{c_1^2 + c_2^2}, \quad \phi = 2 \arctan \frac{c_1}{c_2 + A}$$

Hyperbolic Functions

T16. $$\cosh x = \frac{e^x + e^{-x}}{2}$$

T17. $$\sinh x = \frac{e^x - e^{-x}}{2}$$

T18. $$\cosh^2 x - \sinh^2 x = 1$$

T19. $$\tanh^2 x + \operatorname{sech}^2 x = 1$$

T20. $$\coth^2 x - \operatorname{csch}^2 x = 1$$

T21. $$\sinh(x \pm y) = \sinh x \, \cosh y \pm \cosh x \, \sinh y$$

T22. $$\cosh(x \pm y) = \cosh x \, \cosh y \pm \sinh x \, \sinh y$$

T23. $$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \, \tanh y}$$

T24. $$\sinh(2x) = 2 \sinh x \, \cosh x$$

T25. $$\cosh(2x) = \cosh^2 x + \sinh^2 x$$

T26. $$\sinh x \, \sinh y = {}^1\!/\!_2\big(\cosh(x+y) - \cosh(x-y)\big)$$

T27. $$\cosh x \, \cosh y = {}^1\!/\!_2\big(\cosh(x+y) + \cosh(x-y)\big)$$

T28. $$\sinh x \, \cosh y = {}^1\!/\!_2\big(\sinh(x+y) + \sinh(x-y)\big)$$

Power Series

P1. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad -\infty < x < \infty$$

P2. $$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \quad -\infty < x < \infty$$

P3. $$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \quad -\infty < x < \infty$$

P4. $$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

P5. $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \quad -1 < x < 1$$

P6. $$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \quad -\infty < x < \infty$$

P7. $$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \quad -\infty < x < \infty$$

P8. $$\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \cdots, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

P9. $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

P10. Taylor Series with remainder:

$$\begin{aligned} f(x) &= \sum_{n=0}^{N} \tfrac{f^{(n)}(a)}{n!}(x-a)^n + R_{N+1}(x), \quad \text{where} \\ R_{N+1}(x) &= \tfrac{f^{(N+1)}(\xi)}{(N+1)!}(x-a)^{N+1} \qquad \text{for some } \xi \text{ between } a \text{ and } x. \end{aligned}$$

Table of Integrals

A constant of integration should be added to each formula. The letters a, b, m, and n denote constants; u and v denote functions of an independent variable such as x.

Standard Integrals

I1. $$\int u^n\,du = \frac{u^{n+1}}{n+1}, \quad n \neq -1$$

I2. $$\int \frac{du}{u} = \ln|u|$$

I3. $$\int e^u\,du = e^u$$

I4. $$\int a^u\,du = \frac{a^u}{\ln a}, \quad a > 0$$

I5. $$\int \cos u\,du = \sin u$$

I6. $$\int \sin u\,du = -\cos u$$

I7. $$\int \sec^2 u\,du = \tan u$$

I8. $$\int \csc^2 u\,du = -\cot u$$

I9. $$\int \sec u \tan u \, du = \sec u$$

I10. $$\int \csc u \cot u \, du = -\csc u$$

I11. $$\int \tan u \, du = -\ln|\cos u|$$

I12. $$\int \cot u \, du = \ln|\sin u|$$

I13. $$\int \sec u \, du = \ln|\sec u + \tan u|$$

I14. $$\int \csc u \, du = \ln|\csc u - \cot u|$$

I15. $$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan\left(\frac{u}{a}\right)$$

I16. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right)$$

I17. $$\int u \, dv = uv - \int v \, du$$

Integrals involving $au + b$

I18. $$\int (au+b)^n du = \frac{(au+b)^{n+1}}{(n+1)a}, \quad n \neq -1$$

I19. $$\int \frac{du}{au+b} = \frac{1}{a} \ln|au+b|$$

I20. $$\int \frac{u \, du}{au+b} = \frac{u}{a} - \frac{b}{a^2} \ln|au+b|$$

I21. $$\int \frac{u \, du}{(au+b)^2} = \frac{b}{a^2(au+b)} + \frac{1}{a^2} \ln|au+b|$$

I22. $$\int \frac{du}{u(au+b)} = \frac{1}{b} \ln\left|\frac{u}{au+b}\right|$$

I23. $$\int u\sqrt{au+b}\,du = \frac{2(3au-2b)}{15a^2}(au+b)^{3/2}$$

I24. $$\int \frac{u\,du}{\sqrt{au+b}} = \frac{2(au-2b)}{3a^2}\sqrt{au+b}$$

I25. $$\int u^2\sqrt{au+b}\,du = \frac{2}{105a^3}\left(8b^2-12abu+15a^2u^2\right)(au+b)^{3/2}$$

I26. $$\int \frac{u^2\,du}{\sqrt{au+b}} = \frac{2}{15a^3}\left(8b^2-4abu+3a^2u^2\right)\sqrt{au+b}$$

Integrals involving $u^2 \pm a^2$

I27. $$\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|$$

I28. $$\int \frac{u\,du}{u^2\pm a^2} = {}^1\!/_2\ln\left|u^2\pm a^2\right|$$

I29. $$\int \frac{u^2\,du}{u^2-a^2} = u+\frac{a}{2}\ln\left|\frac{u-a}{u+a}\right|$$

I30. $$\int \frac{u^2\,du}{u^2+a^2} = u-a\arctan\left(\frac{u}{a}\right)$$

I31. $$\int \frac{du}{u(u^2\pm a^2)} = \pm\frac{1}{2a^2}\ln\left|\frac{u^2}{u^2\pm a^2}\right|$$

Integrals involving $\sqrt{u^2 \pm a^2}$

I32. $$\int \frac{u\,du}{\sqrt{u^2\pm a^2}} = \sqrt{u^2\pm a^2}$$

I33. $$\int u\sqrt{u^2\pm a^2}\,du = \frac{1}{3}\left(u^2\pm a^2\right)^{3/2}$$

I34. $$\int \frac{du}{\sqrt{u^2\pm a^2}} = \ln\left|u+\sqrt{u^2\pm a^2}\right|$$

I35. $$\int \frac{u^2\,du}{\sqrt{u^2\pm a^2}} = \frac{u}{2}\sqrt{u^2\pm a^2}\mp\frac{a^2}{2}\ln\left|u+\sqrt{u^2\pm a^2}\right|$$

I36. $$\int \frac{du}{u\sqrt{u^2+a^2}} = \frac{1}{a}\ln\left|\frac{u}{a+\sqrt{u^2+a^2}}\right|$$

I37. $$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{u}{a}\right)$$

I38. $$\int \frac{du}{u^2\sqrt{u^2\pm a^2}} = \mp\frac{\sqrt{u^2\pm a^2}}{a^2u}$$

I39. $$\int \sqrt{u^2\pm a^2}\,du = \frac{u}{2}\sqrt{u^2\pm a^2} \pm \frac{a^2}{2}\ln\left|u+\sqrt{u^2\pm a^2}\right|$$

I40. $$\int u^2\sqrt{u^2\pm a^2}\,du = \frac{u}{4}\left(u^2\pm a^2\right)^{3/2} \mp \frac{a^2u}{8}\sqrt{u^2\pm a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2\pm a^2}\right|$$

I41. $$\int \frac{\sqrt{u^2+a^2}}{u}\,du = \sqrt{u^2+a^2} - a\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right|$$

I42. $$\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\operatorname{arcsec}\left(\frac{u}{a}\right)$$

I43. $$\int \frac{\sqrt{u^2\pm a^2}}{u^2}\,du = -\frac{\sqrt{u^2\pm a^2}}{u} + \ln\left|u+\sqrt{u^2\pm a^2}\right|$$

Integrals involving $\sqrt{a^2-u^2}$

I44. $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\left(\frac{u}{a}\right)$$

I45. $$\int \frac{u\,du}{\sqrt{a^2-u^2}} = -\sqrt{a^2-u^2}$$

I46. $$\int u\sqrt{a^2-u^2}\,du = -\frac{1}{3}\left(a^2-u^2\right)^{3/2}$$

I47. $$\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|$$

I48. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|$$

I49. $$\int u^2\sqrt{a^2-u^2}\,du = -\frac{u}{4}\left(a^2-u^2\right)^{3/2} + \frac{a^2u}{8}\sqrt{a^2-u^2} + \frac{a^4}{8}\arcsin\left(\frac{u}{a}\right)$$

I50. $$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{\sqrt{a^2-u^2}}{u} - \arcsin\left(\frac{u}{a}\right)$$

I51. $$\int \frac{u^2\,du}{\sqrt{a^2-u^2}} = -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\left(\frac{u}{a}\right)$$

I52. $$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{\sqrt{a^2-u^2}}{a^2u}$$

Integrals involving trigonometric functions

I53. $$\int \sin^2(au)\,du = \frac{u}{2} - \frac{\sin(2au)}{4a}$$

I54. $$\int \cos^2(au)\,du = \frac{u}{2} + \frac{\sin(2au)}{4a}$$

I55. $$\int \sin^3(au)\,du = \frac{1}{a}\left(\frac{\cos^3(au)}{3} - \cos(au)\right)$$

I56. $$\int \cos^3(au)\,du = \frac{1}{a}\left(\sin(au) - \frac{\sin^3(au)}{3}\right)$$

I57. $$\int \sin^2(au)\,\cos^2(au)\,du = \frac{u}{8} - \frac{1}{32a}\sin(4au)$$

I58. $$\int \tan^2(au)\,du = \frac{1}{a}\tan(au) - u$$

I59. $$\int \cot^2(au)\,du = -\frac{1}{a}\cot(au) - u$$

I60. $$\int \sec^3(au)\,du = \frac{1}{2a}\sec(au)\tan(au) + \frac{1}{2a}\ln|\,\sec(au) + \tan(au)\,|$$

I61. $$\int \csc^3(au)\,du = -\frac{1}{2a}\csc(au)\cot(au) + \frac{1}{2a}\ln|\,\csc(au) - \cot(au)\,|$$

I62. $$\int u\sin(au)\,du = \frac{1}{a^2}\left(\sin(au) - au\cos(au)\right)$$

I63. $$\int u\cos(au)\,du = \frac{1}{a^2}\left(\cos(au) + au\sin(au)\right)$$

I64. $$\int u^2 \sin(au)\, du = \frac{1}{a^3}\left(2au\sin(au) - (\,a^2u^2 - 2\,)\cos(au)\right)$$

I65. $$\int u^2 \cos(au)\, du = \frac{1}{a^3}\left(2au\cos(au) + (\,a^2u^2 - 2\,)\sin(au)\right)$$

I66. $$\int \sin(au)\, \sin(bu)\, du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)}, \qquad a^2 \neq b^2$$

I67. $$\int \cos(au)\, \cos(bu)\, du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)}, \qquad a^2 \neq b^2$$

I68. $$\int \sin(au)\, \cos(bu)\, du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)}, \qquad a^2 \neq b^2$$

I69. $$\int \sin^n u\, du = -\frac{1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\, du$$

Integrals involving hyperbolic functions

I70. $$\int \sinh(au)\, du = \frac{1}{a}\cosh(au)$$

I71. $$\int \sinh^2(au)\, du = \frac{1}{4a}\sinh(2au) - \frac{u}{2}$$

I72. $$\int \cosh(au)\, du = \frac{1}{a}\sinh(au)$$

I73. $$\int \cosh^2(au)\, du = \frac{u}{2} + \frac{1}{4a}\sinh(2au)$$

I74. $$\int \sinh(au)\, \cosh(bu)\, du = \frac{\cosh\left((a+b)\,u\right)}{2(a+b)} + \frac{\cosh\left((a-b)\,u\right)}{2(a-b)}$$

I75. $$\int \sinh(au)\, \cosh(au)\, du = \frac{1}{4a}\cosh(2au)$$

I76. $$\int \tanh u\, du = \ln(\cosh u)$$

I77. $$\int \operatorname{sech} u\, du = \arctan(\sinh u) = 2\arctan\left(e^u\right)$$

Integrals involving exponential functions

I78. $$\int ue^{au}\,du = \frac{e^{au}}{a^2}(au-1)$$

I79. $$\int u^2e^{au}\,du = \frac{e^{au}}{a^3}\left(a^2u^2-2au+2\right)$$

I80. $$\int u^ne^{au}\,du = \frac{1}{a}u^ne^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$$

I81. $$\int e^{au}\sin(bu)\,du = \frac{e^{au}}{a^2+b^2}\left(a\sin(bu)-b\cos(bu)\right)$$

I82. $$\int e^{au}\cos(bu)\,du = \frac{e^{au}}{a^2+b^2}\left(a\cos(bu)+b\sin(bu)\right)$$

Integrals involving inverse trigonometric functions

I83. $$\int \arcsin\left(\frac{u}{a}\right)\,du = u\arcsin\left(\frac{u}{a}\right)+\sqrt{a^2-u^2}$$

I84. $$\int \arccos\left(\frac{u}{a}\right)\,du = u\arccos\left(\frac{u}{a}\right)-\sqrt{a^2-u^2}$$

I85. $$\int \arctan\left(\frac{u}{a}\right)\,du = u\arctan\left(\frac{u}{a}\right)-\frac{a}{2}\ln\left(a^2+u^2\right)$$

Integrals involving inverse hyperbolic functions

I86. $$\int \operatorname{arcsinh}\left(\frac{u}{a}\right)\,du = u\operatorname{arcsinh}\left(\frac{u}{a}\right)-\sqrt{u^2+a^2}$$

I87.

$$\begin{aligned}\int \operatorname{arccosh}\left(\frac{u}{a}\right)\,du &= u\operatorname{arccosh}\left(\tfrac{u}{a}\right)-\sqrt{u^2-a^2} \qquad \operatorname{arccosh}\left(\tfrac{u}{a}\right)>0;\\ &= u\operatorname{arccosh}\left(\tfrac{u}{a}\right)+\sqrt{u^2-a^2} \qquad \operatorname{arccosh}\left(\tfrac{u}{a}\right)<0.\end{aligned}$$

I88. $$\int \operatorname{arctanh}\left(\frac{u}{a}\right)\, du = u \operatorname{arctanh}\left(\frac{u}{a}\right) + \frac{a}{2}\ln\left(a^2 - u^2\right)$$

Integrals involving logarithm functions

I89. $$\int \ln u \, du = u(\ln u - 1)$$

I90. $$\int u^n \ln u \, du = u^{n+1}\left[\frac{\ln u}{n+1} - \frac{1}{(n+1)^2}\right], \qquad n \neq -1$$

Wallis' Formulas

I91.

$$\begin{aligned}\int_0^{\pi/2} \sin^m x \, dx &= \int_0^{\pi/2} \cos^m x \, dx \\ &= \tfrac{(m-1)(m-3)\ldots(2 \text{ or } 1)}{m(m-2)\ldots(3 \text{ or } 2)} k,\end{aligned}$$

where $k = 1$ if m is odd and $k = \pi/2$ if m is even.

I92. $$\int_0^{\pi/2} \sin^m x \cos^n x \, dx =$$

$$\frac{(m-1)(m-3)\ldots(2 \,\text{or}\, 1)(n-1)(n-3)\ldots(2 \,\text{or}\, 1)}{(m+n)(m+n-2)\ldots(2 \,\text{or}\, 1)} k,$$

where $k = \pi/2$ if both m and n are even and $k = 1$ otherwise.

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